A double pendulum swing experiment: In search of a better bat

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Experimental results on the large-amplitude motion of a double pendulum are presented, with emphasis on the first half cycle. The initial part of the swing is reproducible and is of interest in modeling various human movement activities such as running, throwing, kicking, and the swing of a bat or racquet. Beyond this time, the motion is chaotic. The forces and torques acting on each pendulum segment are analyzed to explain its motion. The results show how a "perfect" bat could be designed where all the kinetic energy from the player's arms is transferred to the ball and none is retained in the arms or the bat after the impact.
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I. INTRODUCTION

A double pendulum is a well-known example of chaotic behavior.[1,2] The first half cycle is quite predictable, a fact that is well known by golfers and baseball players. Williams,[3] Daish,[4] and Jorgensen[5,6] pioneered the use of a double pendulum to model the swing of a golf club, and others have used the model to describe the action of upper or lower limb segments in activities such as throwing,[7–9] running,[10] and kicking.[9,11] The model has been used to predict the effects of applying a positive or negative wrist torque during a golf swing, and the theoretical advantages and disadvantages of doing so have been well documented.[12–14] Nevertheless, it is difficult to find any measurements of the swing of a mechanical double pendulum, or of the wrist torque used in a golf swing, or in any other activity where the wrist plays an important dynamic role.

Measurements of the swing of an implement in a sporting environment are complicated by the fact that the motion is usually three dimensional and not entirely reproducible. This paper describes an experiment using a simple mechanical double pendulum and a comparison of the measured behavior with theoretical predictions. For the latter purpose, the equations of motion for a double pendulum are derived in terms of the forces acting on each segment. Daish[4] and Jorgensen[5] indicated that a double pendulum is too complicated to solve in this manner, and instead used Lagrange's equations. In fact, it is simpler and more useful for the present purposes to derive the relations directly from Newtonian mechanics.

Daish[4] provided a simple qualitative description of the motion of a double pendulum and showed the strong similarity between a manually operated mechanical pendulum and an actual golf swing. The main objective of this paper is to provide a more quantitative description of a double pendulum, both in terms of experimental results and calculations of the relevant forces and torques. The results provide an answer to the following question. Can a bat, club, or racquet be designed such that all of the energy generated by the player is transferred to the striking implement and all of the energy in the implement is transferred to the ball, without any energy being retained in the arms or in the implement after the impact? The answer is yes, but the rules of each game would need to be changed to allow the use of a heavier ball.

II. EQUATIONS OF MOTION

The double pendulum shown in Fig. 1 consists of an upper segment or arm AB of mass M_1 and length L_1 pivoted at a fixed point A, and a lower segment or rod BC of mass M_2 connected to the arm at B by a frictionless hinge. The center of mass of the arm is denoted by G_1 and is located a distance h_1 from A. The center of mass of the rod, G_2, is located at a distance h_2 from point B. The arm is inclined at angle θ to the vertical and rotates clockwise in a vertical plane at angular velocity $\omega_1=-d\theta/dt$. The minus sign is introduced because θ decreases when the pendulum is released from rest. The rod is inclined at angle ϕ to the vertical and rotates clockwise at angular velocity $\omega_2=-d\phi/dt$. Point B rotates in a circular arc about A at speed $L_1\omega_1$, while G_2 rotates in a circular arc with respect to B at speed $h_2\omega_2$.

The x, y coordinates of G_2, with respect to an origin at A, are ($L_1 \sin\theta + h_2 \sin,\phi, -L_1\cos\theta - h_2\cos\phi$). If V is the velocity of G_2, then the components of V are given by

$$V_x=\frac{dx}{dt}=-L_1\omega_1\cos\theta-h_2\omega_2\cos\phi, \tag{1a}$$

$$V_y=\frac{dy}{dt}=-L_1\omega_1\sin\theta-h_2\omega_2\sin\phi. \tag{1b}$$

Upper-case M and V are used to denote the arm and rod (or bat) mass and velocity, while lower-case m and v are used to denote ball mass and velocity. The arm exerts a force on the rod at B with components F_x and F_y given by

$$F_x=M_2\frac{dV_x}{dt}=-M_2\left[L_1\cos\theta\frac{d\omega_1}{dt}+L_1\omega_1^2\sin\theta + h_2\cos\phi\frac{d\omega_2}{dt}+h_2\omega_2^2\sin\phi\right], \tag{2a}$$

$$F_y-M_2g=M_2\frac{dV_y}{dt}=-M_2\left[L_1\sin\theta\frac{d\omega_1}{dt}-L_1\omega_1^2\cos\theta + h_2\sin\phi\frac{d\omega_2}{dt}-h_2\omega_2^2\cos\phi\right]. \tag{2b}$$

To model the swing of a club or bat, we assume that a torque C_1 is applied to the arm and a torque C_2 is applied to the rod, both torques arising from equal and opposite muscle and joint reaction forces at the respective joints. The rod exerts

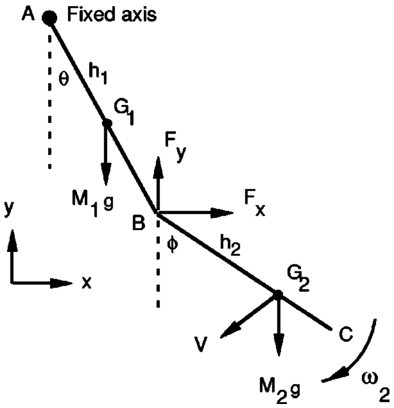

Fig. 1. Forces acting on a double pendulum pivoted at point A and hinged at point B. The upper and lower segments could represent two arm segments, two leg segments, or an arm and a striking implement.

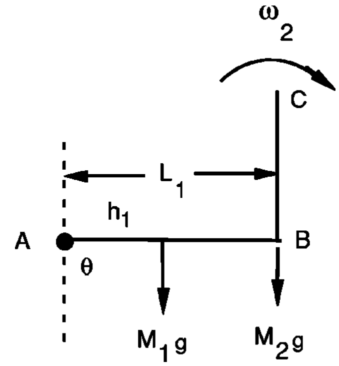

Fig. 2. Initial position of double pendulum for the experiments described in this paper.

an equal and opposite torque $-C_2$ on the arm. The rod also exerts a clockwise torque on the arm, about axis A, given by $F_xL_1\cos\theta+F_yL_1\sin\theta$. The net torque acting on the arm is given by

$$C_1-C_2+M_1gh_1\sin\theta+F_xL_1\cos\theta+F_yL_1\sin\theta=I_1\frac{d\omega_1}{dt},\tag{3}$$

where I_1 is the moment of inertia of the arm about the axis at A. The torque on the rod about an axis through its center of mass is given by

$$C_2+F_xh_2\cos\phi+F_yh_2\sin\phi=I_{\text{c.m.}}\frac{d\omega_2}{dt},\tag{4}$$

where $I_{\text{c.m.}}$ is the moment of inertia of the rod about an axis through its center of mass. If we substitute Eq. (2) for F_x and F_y, we find that

$$C_1-C_2=A\frac{d\omega_1}{dt}+B\cos\beta\frac{d\omega_2}{dt}+B\omega_2^2\sin\beta$$
$$-(M_1h_1+M_2L_1)g\sin\theta,\tag{5a}$$

$$C_2=I_2\frac{d\omega_2}{dt}+B\cos\beta\frac{d\omega_1}{dt}-B\omega_1^2\sin\beta-M_2h_2g\sin\phi,\tag{5b}$$

where $\beta=\phi-\theta$ is the angle between the arm and the rod, $A=I_1+M_2L_1^2$, $B=M_2h_2L_1$, and $I_2=I_{\text{c.m.}}+M_2h_2^2$ is the moment of inertia of the rod about point B. Equation (5) is identical to that derived in Refs. 4 and 5 when account is made of the differences in the defined angles and the simplifying assumption made in Ref. 4 that gravity can be neglected in a high-speed golf swing.

Equation (5) provides little insight into the physical nature of the various terms or the outcomes arising from a given set of initial conditions. To simplify matters, consider the situation shown in Fig. 2 where the arm is horizontal and the rod is vertical. In this situation $\theta=90°$, $\phi=180°$, and $\beta=90°$. If we further assume that $C_1=C_2=0$, then Eqs. (5a) and (5b) reduce to

$$M_1gh_1-F_yL_1=I_1\frac{d\omega_1}{dt},\tag{6a}$$

$$F_xh_2=-I_{\text{c.m.}}\frac{d\omega_2}{dt},\tag{6b}$$

where $M_2g-F_y=M_2L_1d\omega_1/dt+M_2h_2\omega_2^2$ and $F_x=M_2h_2d\omega_2/dt-M_2L_1\omega_1^2$. The terms containing ω_1^2 and ω_2^2 are due to the centripetal forces acting on the rod in the $-x$ and $-y$ directions, respectively. The force terms $L_1d\omega_1/dt$ and $h_2d\omega_2/dt$ arise from the linear acceleration of G_2 in the $-y$ and $+x$ directions, respectively. F_x could therefore be positive, zero, or negative, depending on the relative magnitudes of $d\omega_2/dt$ and ω_1^2.

The situation shown in Fig. 2 corresponds to the initial position of the double pendulum in the experiment described in the following. In that position, $\omega_1=\omega_2=0$ at $t=0$, in which case we find from Eq. (6) that the initial angular accelerations of the arm and the rod are given by

$$\frac{d\omega_1}{dt}=(M_1h_1+M_2L_1)g/A,\tag{7a}$$

$$\frac{d\omega_2}{dt}=0,\tag{7b}$$

when $C_1=C_2=0$. If released from rest, the arm rotates clockwise due to the gravitational forces on the arm and the rod, while the rod drops vertically. Because the rod has no initial angular acceleration, it effectively acts as a point mass at the end of the arm. The total moment of inertia of the arm plus the rod in Eq. (7a) is therefore given by $I_1+M_2L_1^2=A$.

A significant torque on the rod develops only after the arm has rotated through an angle of about 15°. By the time the rod has rotated through an angle of 180° into a golf ball striking position, the arm may have rotated well past the vertical position. In practice, the initial stage of the swing of a club, bat, or racquet is controlled by using the wrist to maintain a constant "wrist-cock" angle β of about 90° between the arm and the implement in the hand. After the arm (and the implement) has rotated through an angle of about 45°, the torque on the implement arising from the centripetal force is large enough to swing it out from the arm without any assistance from the wrist. As a result, the arm and the implement may then be approximately in line at the instant the implement makes contact with a ball.

The effect of the wrist was simulated in the present experiment by means of a mechanical stop that prevented β from

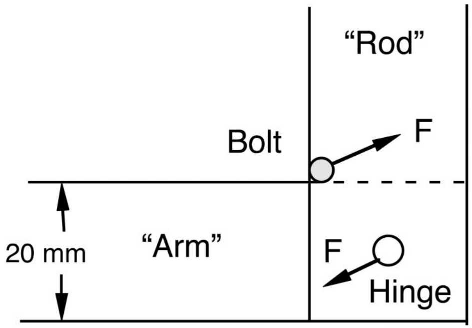

Fig. 3. Stop mechanism used to simulate the wrist cock action used by baseball, golf, and tennis players.

exceeding 90°. With the stop inserted, the whole pendulum rotates initially as a rigid body about the pivot point with a moment of inertia equal to $A+I_2$ and with $\omega_1=\omega_2$. The moment of inertia in this case is easily calculated from the parallel axis theorem. Under these conditions, we find from Eq. (5) that

$$(A+I_2)\frac{d\omega_2}{dt}=(M_1h_1+M_2L_1)g\sin\theta+M_2h_2g\cos\theta, \qquad (8)$$

when $C_1=0$, $\beta=90°$, and $\omega_1=\omega_2$. To a good approximation, $d\omega_2/dt$ remains constant while θ decreases from 90° to $\approx 70°$, in which case Eq. (5b) has a simple analytical solution. As the pendulum rotates, C_2 decreases in time and becomes zero when θ is about 75°, in the manner shown later in Figs. 8 and 10.

An interesting question is whether the behavior of the pendulum is sensitive to the initial conditions. Without the stop in place, the rod drops vertically for the conditions described previously. However, if the rod starts from a position a few degrees away from the vertical, it will tend to fall forward or backward. The subsequent motion of the pendulum might therefore be quite sensitive to the initial alignment of the rod. Indeed, the authors of Refs. 1 and 2 found that sensitivity to initial conditions is the primary signature of the chaotic behavior of a double pendulum, a fact that is now well known regarding chaos in general. Consequently, a preliminary experiment was conducted to determine the reproducibility of the pendulum swing during its first half cycle.

III. EXPERIMENTAL RESULTS

A double pendulum was constructed from two 0.30 m lengths of aluminum bars, each 20 mm wide, with clearance holes at each end to provide simple pivot and hinge joints using bolts and loose fitting nuts. The upper bar or arm was 6 mm thick and had a mass of 109.3 g, while the lower bar or rod was 1.5 mm thick and had a mass of 25.0 g. Measurements were taken with and without a mechanical stop attached to the top end of the rod. The stop (Fig. 3) consisted of a small bolt passing through the edge of the rod and was used to prevent the rod from rotating more than 90° backward with respect to the arm. Contact between the bolt and the edge of the arm resulted in a force F on the rod. An equal and opposite reaction force is exerted at the hinge joint if there is no acceleration of the rod, resulting in a clockwise torque C_2 on the rod and a counterclockwise torque C_2 on the arm. This pair of forces exerts no net force, and it exerts a torque that does not depend on the location of the rotation axis. Under dynamic loading conditions, the two forces shown in Fig. 3 may not be exactly equal and opposite, but the net result of the two forces will be equivalent to a couple plus a net force at the hinge joint. Similar circumstances arise in human wrist and elbow joints where tendons pull on bone at points close to the joint rotation axis. The combined effect of the tendon and joint reaction forces is equivalent to a couple plus a net accelerating force acting on the joint.

Table I. Parameters of the upper arm and lower rod. The units of the moment of inertia are kg m^2.

Upper arm	Rod 1	Rod 2
$L_1=0.30$ m	$L_2=0.300$ m	0.300 m
$h_1=0.15$ m	$h_2=0.150$ m	0.231 m
$M_1=109.3$ g	$M_2=25.0$ g	76.3 g
$I_1=0.003\,28$	$I_2=0.000\,75$	0.004 49
	$I_{\text{c.m.}}=1.875\times10^{-4}$	4.295×10^{-4}

Measurements also were made with a 51.3 g mass attached at a point 30 mm from the bottom end of the lower bar to increase its moment of inertia. The resulting parameters of each bar are shown in Table I, where rod 2 refers to the 25 g bar with the 51.3 g mass attached.

Each swing was commenced by releasing the pendulum from rest in the orientation shown in Fig. 2. Video clips of each swing, at 25 frames/s, were transferred to a computer for analysis using Videopoint software that allowed the (x,y) coordinates of selected points to be digitized. Slightly different x and y scale calibration factors for the video image were needed to ensure that the length of each arm remained constant, regardless of its orientation. The data were then processed to determine the angular position of each segment at 0.04 s intervals. The angular velocity of each segment was calculated by dividing the angular displacement between each frame by 0.04.

The trajectory of point C (Figs. 1 and 2) for five nominally identical pendulum swings is shown in Fig. 4(a), using the lower bar without the added mass and without a stop. The pendulum was released by hand after aligning point C with a marker to ensure that it was released from the same point for each swing, within about 1 mm in the x and y directions. Twenty such swings were recorded, but only five typical results are shown in Fig. 4 for clarity. The trajectory of point B is not shown because point B moves in a circular arc of radius 0.30 m about the pivot point A. Under no conditions did the lower segment come to a stop at the bottom of its swing, as it apparently did in trial 3 of Ref. 1. In all cases, the lower segment reached its maximum speed at a time when point C was close to its lowest position [near $x=0$, $y=-0.6$ in Fig. 4(a)].

The results in Fig. 4 show clearly that the trajectory of the swing is reproducible up to a certain point, beyond which its subsequent motion depends sensitively on the initial conditions. If the region in Fig. 4(a) is expanded close to the starting point, it is seen that the initial part of each trajectory correlates closely with the subsequent behavior. For example, the dashed curve in Fig. 4(a) is the outermost curve both at the start and the end of the trajectory. Similarly, the innermost curve shows the same behavior. It appears from Fig. 4(a) that the behavior of the pendulum is most sensitive in the region just before the lower and upper segments come

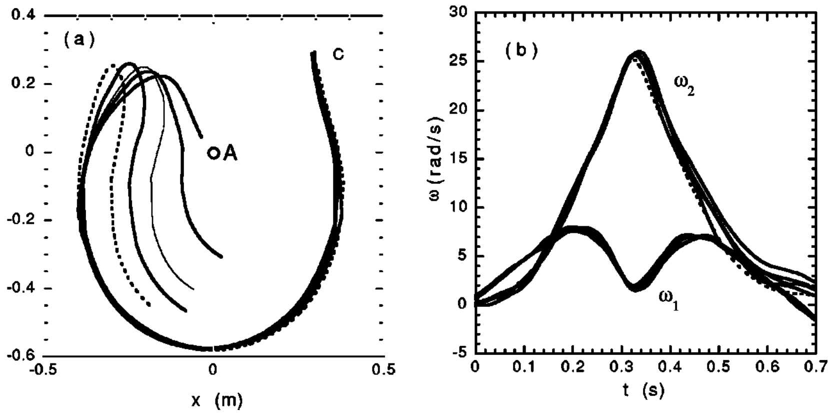

Fig. 4. (a) Trajectories of point C for five swings, all nominally the same, without a stop and without the additional mass on the lower segment. (b) ω_1 and ω_2 vs t for the same five swings.

to rest. However, it can be seen from Fig. 4(b) that significant differences in ω_1 and ω_2 are observed soon after the lower segment reaches its maximum speed (near the bottom of its swing).

Our primary interest is the initial swing of a double pendulum, up to the point where the lower segment reaches its maximum speed and shortly afterward. This part of the swing is quite reproducible. The chaotic motion that develops after this point is of no interest to golf, baseball, or tennis players and will not be discussed in this paper.

Angular velocity results for individual swings are shown in Fig. 5 for the four configurations of the pendulum, with or without the stop inserted, and with or without the additional 51.3 g mass. Numerical solutions of Eq. (5) are shown for comparison in Fig. 6. The agreement is very good, apart from some minor discrepancies that could probably be eliminated by improvements in experimental technique such as improved bearing joints, a higher speed camera, a lower speed pendulum, better camera lens, and improved data analysis. In that respect, the simplest way of slowing a physical pendulum is to pivot it at a point near its center of mass.

Some features of interest that are apparent in Figs. 5 and 6 are the following:

(a) $\omega_1 = \omega_2$ during the first 0.2 s after release of the pendulum when the stop is used. Without the stop, the rod falls vertically without significant rotation early in time.

(b) Regardless of whether a stop is used or not, the angular speed of the upper arm decreases to a minimum as the lower rod approaches its maximum angular velocity, implying that angular momentum is transferred from

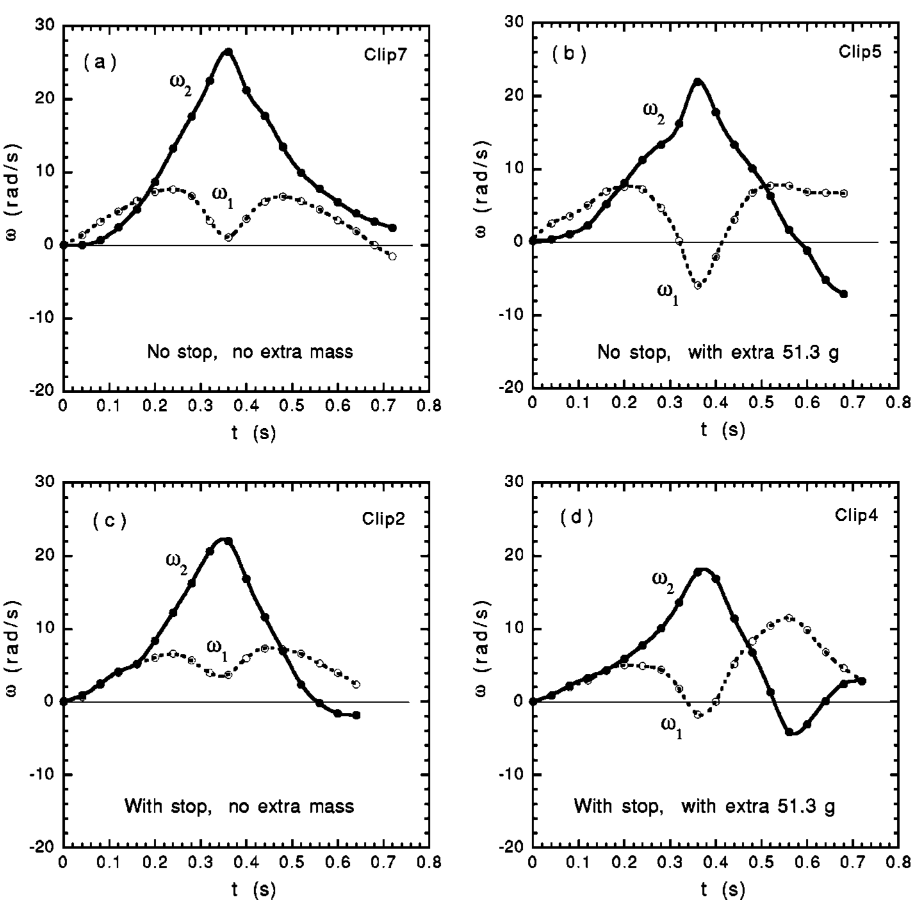

Fig. 5. Experimental results, with cubic spline fits to the experimental data points. ω_1 is the angular velocity of the upper segment which comes to rest in (a) and which reverses direction in (b) and (d) while the lower segment is rotating at angular velocity ω_2.

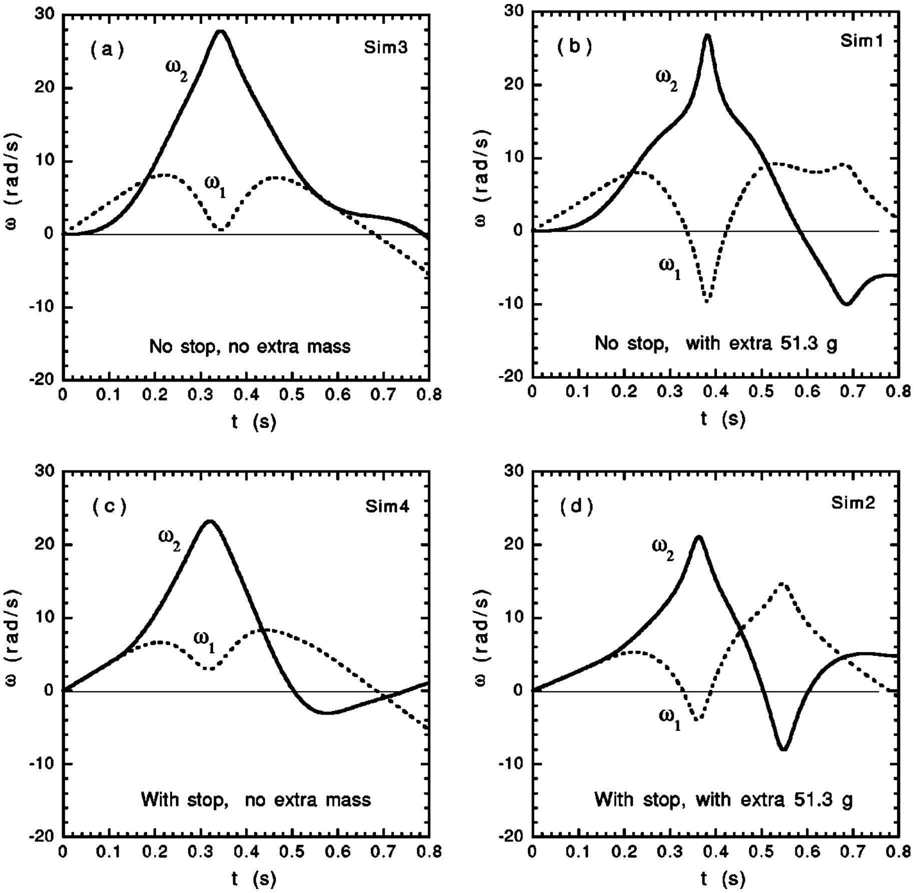

Fig. 6. Solutions of Eq. (5) for the parameters shown in Table I. The agreement with the experimental results in Fig. 5 is very good.

the upper to the lower segment. This transfer is a feature that is common to most upper and lower body segment movements in activities such as walking, kicking, throwing, and bat or club swinging. In Sec. IV it is shown that momentum transfer arises primarily as a result of the centripetal force on the rod and the equal and opposite force on the arm.

(c) The peak value of ω_2 decreases when a stop is used or when the additional 51.3 g mass is added to the lower segment (despite the increase in initial potential energy with the added mass). The time at which ω_2 is a maximum decreases slightly when a stop is used and increases when the 51.3 g mass is added to the lower segment.

(d) Each segment of the pendulum (without the added mass) has a small amplitude quarter period of oscillation $T/4=0.224$ s when mounted as a physical pendulum of length 0.3 m. The observed first quarter period of the double pendulum is surprisingly close to that of a physical pendulum with a uniform mass distribution and length 0.6 m (where $T/4=0.317$ s). Despite the nonlinear nature of the equations describing a double pendulum, the quarter period of oscillation can therefore be estimated to a good approximation in terms of a single pendulum having the same overall length as the double pendulum.

(e) Additional mass on the lower segment acts to increase the coupling between the two segments, with the result that the upper segment comes to rest and then reverses direction for a short time while the lower segment continues to rotate clockwise. An obvious implication is that the coupling between the two segments would decrease if the mass of the lower segment were reduced. For example, if $M_2=0$, there would be no coupling at all. If $M_1\rightarrow 0$, the coupling remains strong, as is easily demonstrated by supporting the lower segment by a string.

If the pendulum were used to hit a ball located 0.6 m below the pivot point A, the only successful swing would be that shown in Fig. 5(a) or 6(a). The worst swing would be that shown in Fig. 5(d) or 6(d), because the lowest position reached by point C is 65 mm above the ball. The swing in Fig. 6(c) misses the ball by 5 mm assuming the ball is a point mass. A perfect swing in a sporting context therefore requires that the arm and the rod should be closely in line and that ω_2 should be a maximum at the moment of impact. This result is best achieved in the present experiment without the stop. Maximum impact speed results when $\omega_1=0$ at impact because all the initial potential energy is then transferred to the rod and none is retained as kinetic energy of the arm. The angular velocity results shown in Fig. 6 are consistent with the fact that the total (kinetic plus potential) energy remains constant during each swing.

IV. TORQUE ON LOWER SEGMENT

Calculations of the torque components acting on the lower segment were made to determine the primary cause of its rotation. A guess might be that the lower segment rotates for a reason similar to that responsible for the rotation of a single pendulum. That is, the gravitational force acting through the

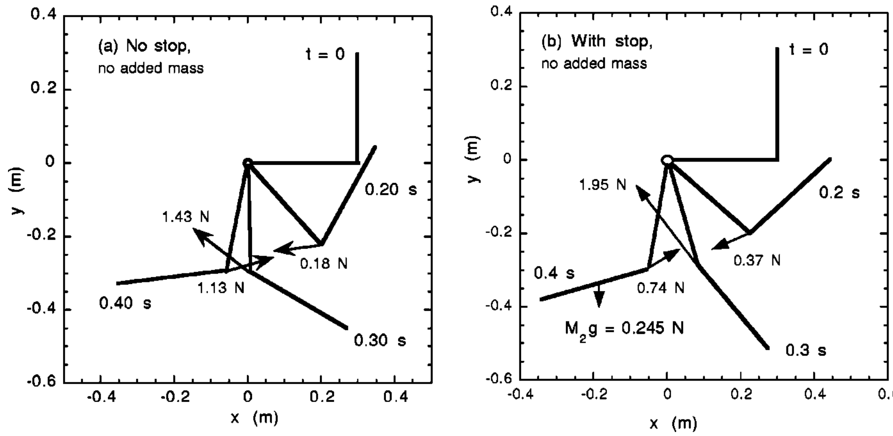

Fig. 7. Force of upper arm on lower rod. A relatively small gravitational force M_2g also acts on the rod, as shown in (b) at $t=0.40$ s. The force on the lower rod is directed approximately parallel to the rod and perpendicular to the motion of its center of mass, thereby providing a relatively large centripetal force and torque on the rod.

center of mass of the lower segment exerts a torque about the hinge point B, and hence the segment rotates. Because point B rotates simultaneously about the upper support A, the motion of the lower segment is more complicated than the motion of a single pendulum. This complication could result in an increase or a decrease in the torque on the lower segment. However, the torque acting about an accelerating axis is related simply to the angular acceleration about that axis only if the axis passes through the center of mass of a body. Consequently, the torques responsible for the rotation of the lower segment are best determined with respect to its center of mass. In that case, the gravitational force on the lower segment makes no direct contribution to the torque about the center of mass. However, the gravitational force M_2g makes an indirect contribution to the torque about the center of mass via the reaction force at hinge joint B, as described in Sec. V. An additional reason to focus our attention on the rotation about the center of mass is that the gravitational force is almost negligible when swinging a club, bat, or racquet at high speed.

Figure 7 shows the magnitude and direction of the force acting on the rod due to the upper arm for the conditions shown in Figs. 6(a) and 6(c). This force has components F_x and F_y given by Eq. (2). Apart from a short interval near the start of each swing, the force is directed approximately along the same line as the rod but at a slight angle. As a result, a torque is exerted on the rod about an axis through its center of mass, in addition to the torque exerted by the torque C_2. The force is directed primarily in a direction perpendicular to the path of the rod center of mass and can be attributed mainly to the centripetal force, as described in the following. An equal and opposite force acts on the arm in such a way that ω_1 decreases as ω_2 increases (and vice versa).

It is instructive to resolve the total force on the rod into components $F_\parallel$ and $F_\perp$ acting in directions parallel and perpendicular to the velocity vector V. These components are given by

$$F_\parallel=\frac{F_xV_x+(F_y-M_2g)V_y}{V}=M_2\frac{dV}{dt}, \tag{9a}$$

$$F_\perp=\frac{(F_y-M_2g)V_x-F_xV_y}{V}=M_2\frac{V^2}{R}, \tag{9b}$$

where R is the instantaneous radius of curvature of the path followed by the center of mass. The parallel component acts to alter the magnitude of V while the perpendicular component provides the centripetal force. The components $F_\parallel$ and $F_\perp$ include contributions from the weight M_2g, given, respectively, by $M_2g\cos\gamma$ and $M_2g\sin\gamma$, where $\gamma=\tan^{-1}(V_x/V_y)$ is the angle between the velocity vector V and the vertical. If these components are subtracted from Eqs. (9a) and (9b), respectively, we can determine the contributions of the upper arm to $F_\parallel$ and $F_\perp$, which we denote by $F_{A\parallel}$ and $F_{A\perp}$. The clockwise torques about the center of mass due to $F_{A\parallel}$ and $F_{A\perp}$ are then given by

$$\tau_\parallel=F_{A\parallel}h_2\sin(\phi-\gamma), \tag{10a}$$

$$\tau_\perp=F_{A\perp}h_2\cos(\phi-\gamma), \tag{10b}$$

in which case Eq. (4) can be expressed in the form

$$C_2+\tau_\parallel+\tau_\perp=I_{\mathrm{c.m.}}\frac{d\omega_2}{dt}. \tag{11}$$

The three contributions in Eq. (11) are shown in Fig. 8. Apart from a short period at the beginning of the swing, the primary torque on the rod arises from the centripetal force on the rod. The $\tau_\perp$ torque is negative at the beginning of the swing, as is the $\tau_\parallel$ torque, a result that is due to the effect of the stop. Because the entire pendulum initially rotates as a rigid body, the rod segment rotates outward and downward. The arm segment must therefore exert a force on the rod with $F_x>0$ to accelerate the rod outward, even though the center of curvature is at the pivot point A, and the centripetal force

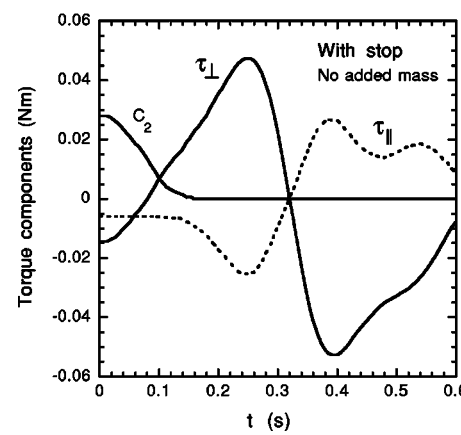

Fig. 8. Components of the torque on the lower rod about an axis through its center of mass.

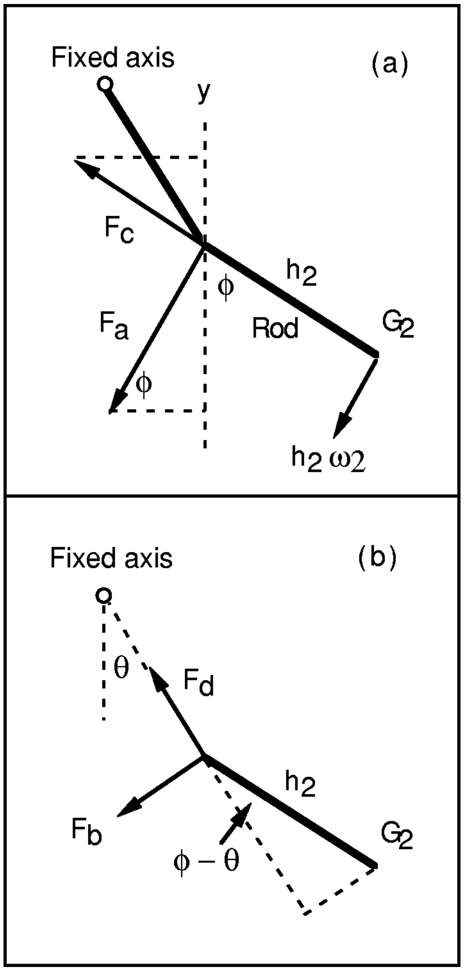

Fig. 9. Four of the five force components acting at the hinge joint on the rod segment.

initially has a component in the negative x direction. As the rod picks up speed, the centripetal force increases, F_x decreases and then reverses direction, and C_2 decreases to zero.

V. A CLOSER LOOK AT THE TORQUES

The net force acting on each segment of a double pendulum can be resolved into x and y components or into components perpendicular and parallel to the velocity vector. Equation (2) indicates that the net force acting on the rod at point B can be represented as the sum of five easily recognizable components. Each component can be regarded as an inertial force Ma, where M is the mass of the segment and a is the corresponding component of the acceleration of its center of mass. Each component acts on the rod along a line through point B, and an equal and opposite force is exerted on the arm. Each component also contributes to the total torque on each segment. Four of the five inertial force components acting on the rod are shown in Fig. 9 and can be described as follows:

(1) G_2 rotates at speed $h_2\omega_2$ with respect to point B. The linear acceleration of G_2 requires a force $F_a = M_2h_2d\omega_2/dt$ acting in a direction perpendicular to the rod, with components $-F_a\cos\phi$ and $-F_a\sin\phi$ in the x and y directions, respectively. F_a exerts a torque $\tau_a = -F_ah_2$ about G_2 (in a counterclockwise direction).

(2) Point B rotates at the speed $L_1\omega_1$. The linear acceleration of B requires a force $F_b = M_2L_1d\omega_1/dt$ on the rod acting in a direction perpendicular to the arm, with components $-F_b\cos\theta$ and $-F_b\sin\theta$ in the x and y directions, respectively. F_b exerts a torque $\tau_b = -F_bh_2\cos(\phi-\theta)$ about G_2 (in a counterclockwise direction).

(3) $F_c = M_2h_2\omega_2^2$ represents the centripetal force on the rod arising from the rotation of G_2 about point B. F_c is directed along the rod toward point B and has components $-F_c\sin\phi$ and $F_c\cos\phi$ in the x and y directions. F_c exerts no torque about G_2.

(4) $F_d = M_2L_1\omega_1^2$ represents the centripetal force on the rod arising from the rotation of point B at speed $L_1\omega_1$. F_d is directed along the arm toward the fixed point A and has components $-F_d\sin\theta$ and $F_d\cos\theta$ in the x and y directions. F_d exerts a torque $\tau_d = F_dh_2\sin(\phi-\theta)$ about G_2 (in a clockwise direction).

(5) The fifth component acting at point B is the reaction force M_2g, which acts vertically upward and exerts a clockwise torque $\tau_g = M_2gh_2\sin\phi$ about G_2. This force is the least obvious force component, but it is the only component at point B when the arm and rod are both at rest and both vertical.

It is readily verified that the summation of the x and y components of the five forces leads to Eq. (2), while the summation of the respective torques gives the right-hand side of Eq. (5b). It is easy to verify that the five force components, acting in opposite directions on the arm at point B, exert a torque about pivot point A that is described by the right-hand side of Eq. (5a).

The interaction of the two segments is nonlinear and not easy to interpret, but there are some features that can be readily described in terms of the various force components. For example, suppose that a baseball player pushes an initially stationary bat forward by applying a force on the handle in a direction perpendicular to its long axis, without applying a simultaneous wrist torque. The bat center of mass will accelerate forward, but the bat also will rotate about an axis within the bat in such a way that the handle moves forward while the tip moves backward. In practice, a batter also applies a wrist torque at the beginning of a swing so that the handle and tip will both move forward. Once the bat has reached a certain speed, the batter can reduce the wrist torque to zero because sufficient torque will be supplied by the centripetal force F_d directed along the arms of the batter. After an additional short delay the batter can then reduce the rotation speed of his arms, in which case a force is applied to the handle in a direction perpendicular to the arms and in the backward direction. The bat center of mass will decelerate, but the resulting torque will swing the tip around at high speed, in a manner that can be described in terms of the second force component F_b and its associated torque. A more difficult question is whether a reduction in arm speed is a deliberate act on the part of the batter or whether it is a natural effect arising from the angular acceleration of the bat. A measurement of the torque C_1 would assist in answering this question. In the experiments described previously, the

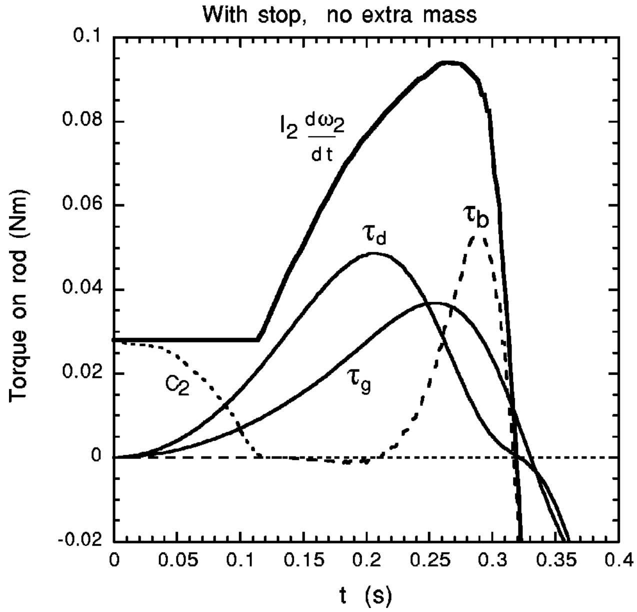

Fig. 10. Torque components acting on the lower rod for the conditions shown in Fig. 6(c).

reduction in ω_1 was a natural effect, and the pendulum had no choice in the matter because C_1 was zero.

The swing of a bat can therefore be regarded as a three-stage process in which the dominant torques on the bat are due to C_2, F_d, and F_b, applied in an ordered sequence. The relative magnitudes of the various torque components on the lower rod for the conditions shown in Fig. 6(c) are shown in Fig. 10. In the case of a mechanical double pendulum, the gravitational torque τ_g also plays an important role during the swing.

VI. ENERGY TRANSFER FROM ARM TO BALL

It was shown that the maximum speed of the rod results if the pendulum arm comes to a temporary stop at the instant when the rod and the arm are both vertical. In that case, all the initial potential energy of the system ends up as kinetic energy in the rod. If the rod were used to strike a ball at that instant, then all of the energy in the rod could be transferred to the ball, provided the ball has sufficient mass and no vibrational energy is retained by the rod. If the struck ball is initially at rest, the most efficient transfer of energy from the rod to the ball results when the effective mass of the rod is equal to the mass of the ball. The effective mass of a striking implement, M_e, is equal to its actual mass, M, for an impact at its center of mass, but is less than the actual mass at other impact points.[15,16] In terms of an equivalent point mass collision, M_e/M for a racquet is typically about one half for an impact near the first vibrational node of the racquet, the "sweet spot" of the racquet.[15,16] Closer to the tip of a racquet, M_e/M is typically about 1/3 or 1/4. For a baseball bat, M_e/M is about 0.8 near the sweet spot and about 0.4 near the tip of the bat. However, a baseball is not at rest when it is struck, in which case the mass of the ball needed to stop the bat is less than M_e (see the Appendix).

A simple experiment was conducted using the pendulum setup in Fig. 5(a) to impact a small superball suspended by a string. By adjusting the location of the ball, it was found that the pendulum could be brought to a complete stop on impact with the ball, thereby creating a perfect bat and ball system in which all the energy in the arm and the rod was transferred to the ball.

An interesting question is why such a result does not occur when a golfer swings a club, a batter swings a bat, or a tennis player swings a racquet. In these situations, the player's arm or arms are observed to slow down just prior to impact, but they do not come to a complete stop, even for an instant. The upper arm and torso may come to a complete stop, but the forearm continues to rotate at high speed during the impact and after the impact is over, as does the implement in the hand. There are obvious inefficiencies here, whereby useful kinetic energy that could be transferred to the ball is wasted when it is retained in the arms and in the striking implement.

There are two main reasons why energy is retained in a striking implement and in the arms of a golfer, batter, or tennis player. One has to do with the large difference in mass between the ball and the arms of the person striking the ball. The other is that the driving force is supplied primarily by the arms and torso and not by gravity alone. A fundamental problem when striking a ball is that the ball mass is typically much smaller than the mass of a person's arms. A person can project a ball by throwing it or by hitting it directly with one or both hands, but the energy transfer efficiency is low due to the energy retained in the arm. The primary function of a striking implement of intermediate mass is to improve the energy transfer efficiency.[17] It is only when the implement has the same effective mass as the ball that one can expect 100% energy transfer efficiency from the implement to the ball. In most cases of interest in sports, however, the effective mass of a striking implement is larger than the mass of the ball, in which case the striking implement retains some of its kinetic energy after impact with the ball. The mass of a striking implement is approximately equal to the geometric mean of the mass of the ball and the arms. For example, a 340 g tennis racquet is 5.9 times heavier than a 57 g tennis ball and a 2000 g arm is 5.9 times heavier than a 340 g racquet. Similarly, a 930 g baseball bat is 6.4 times heavier than a 144 g baseball, and two 3000 g arms are 6.4 times heavier than the bat. Forearm mass is quoted in the appendix of most biomechanics textbooks[18]) and is typically about 1.9% of total body mass. The mass of one upper arm is about 3.2% of total body mass.

In Fig. 5(a), the upper beam (109.3 g) was 4.4 times heavier than the lower beam (25.0 g), so the proportions are about right to simulate a bat or racquet swing. Nevertheless, had I used a mass ratio of about 6 rather 4.4, the upper arm would not have come to a stop because the coupling between each segment would then be weaker. Figure 11 shows the result of adding the 51.3 g mass to the center of the upper beam (total mass 161 g) rather than the lower beam to give a mass ratio of 6.4. Regardless of whether the stop is used or not, ω_1 decreases but does not drop to zero as ω_2 increases to its maximum value. We can deduce that the arms of a golfer, tennis, or baseball player do not come to a temporary halt just prior to striking a ball because the coupling between the arms and the swung implement is relatively weak. The actual coupling will depend on the relative moments of inertia as well as the relative masses of the various segments. In situations where both arms are used to swing an implement, the coupling also will depend on the fact that four arm segments are involved, not just a single arm as in the present experiment. Further experiments would be required to sepa-

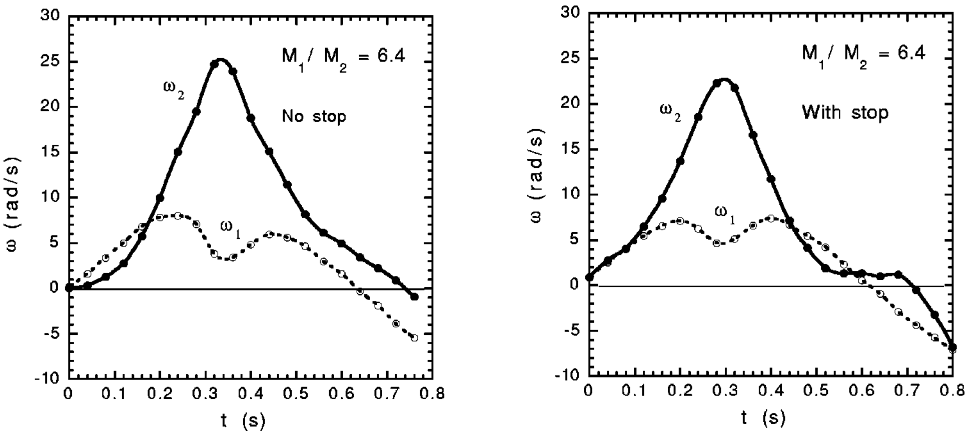

Fig. 11. Simulated swing of a baseball bat with $M_1/M_2 = 6.4$. In this case the coupling between the segments is too weak to bring the upper segment to rest.

rately determine the influence of mass and moment of inertia on the degree of coupling, possibly using a mechanical pendulum but preferably with human subjects. In that respect, data such as that obtained in Ref. 19 on swing speed versus bat moment of inertia are of interest, but need to be augmented by estimates of arm mass and arm speed.

In sporting situations, the primary torque acting on the arms does not arise from the gravitational force on the arms but from the torque C_1 acting at the shoulders (and elbow). Equation (5) shows that the primary torque applied to the upper arm has the form $C_1 + M_1 g h_1 \sin\theta$. In golf, the torque arising from the weight force $M_1 g$ decreases to zero as the arm rotates into a vertical position. A baseball bat is swung more or less in a horizontal plane, in which case the gravitational torque plays an even smaller role. In a tennis serve, the gravitational torque also is very small and opposes rotation of the arm. Jorgensen[5,6] found that a golf swing can be modeled by assuming that C_1 remains constant throughout the swing and that C_1 is typically much larger than the gravitational torque. In this case, the arm will accelerate well ahead of the lower segment unless the wrists are used to provide a positive torque on the lower segment during the early stages of the swing.[13] A wrist torque is not required later in the swing because the centripetal force required to rotate a bat, club, or racquet at high speed can be almost as large as the weight of the player. The result of applying a positive wrist torque early in time, coupled with a longer backswing in the case of a golf club, is that the upper and lower segments are more closely in line at a time when the lower segment is rotating at maximum speed. An additional effect of the positive wrist torque, shown in Figs. 4, 5, and 8, is a smaller decrease in the angular velocity of the arm as the lower segment approaches maximum speed. This effect also contributes to the fact that a player's arms do not come to a temporary halt during the swing of a sporting implement.

VII. CONCLUSIONS

The motion of a double pendulum is much more complicated than a single pendulum, exhibiting chaotic behavior at large amplitude after the first half cycle. During the first half cycle the motion is reproducible and serves as an excellent model for examining the motion of upper and lower segments of the body and the swing of various implements used in sport. The results of experiments conducted with a simple mechanical double pendulum agreed well with theoretical predictions and demonstrated the significance of the relative masses of the two segments in determining the mutual coupling between segments and the effects of the wrist torque used to start the swing of an implement. The application of a positive wrist torque early in the swing is needed to prevent the arm rotating too far ahead of the implement, but it reduces the maximum possible swing speed of the implement. Conversely, Jorgensen[5] found that a negative wrist torque applied immediately after the initial positive torque would not only delay uncocking of the wrists, but also would act to increase club head speed.

In principle, an implement and ball could be chosen to transfer all of the energy from the arm to the ball, but balls used in golf, tennis, baseball, softball, and cricket are too light to allow for this possibility. The original players of these sports opted for a light ball that would travel at high speed and light implements that could be swung rapidly, rather than choosing a heavy ball and a heavy implement to maximize energy transfer from the arm to the ball. As a result, players find that their arms and implements "follow through" after contacting the ball rather than coming to a sudden halt or bouncing backward. Nevertheless, it should be possible in principle for a player to choose a swing style or an implement of appropriate mass and moment of inertia that will optimize the use of the energy available in the player's arm in order to channel the largest possible fraction of that energy into the ball.

In this paper it was assumed that the mass of a player's arm is fixed and the mass of the implement and ball could both be varied to maximize the energy coupled to the ball. A more practical but more difficult question is how to design and swing a bat to maximize the energy coupled to the ball, when the arm mass and ball mass are both fixed. The answer will depend on experimental data yet to be obtained that establishes the relations among bat speed, arm speed, arm mass, bat mass, and bat moment of inertia.

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APPENDIX: ENERGY TRANSFER FROM BAT TO BALL

The collision between a bat of mass M and a ball of mass m is represented in Fig. 12. Before the collision, the bat

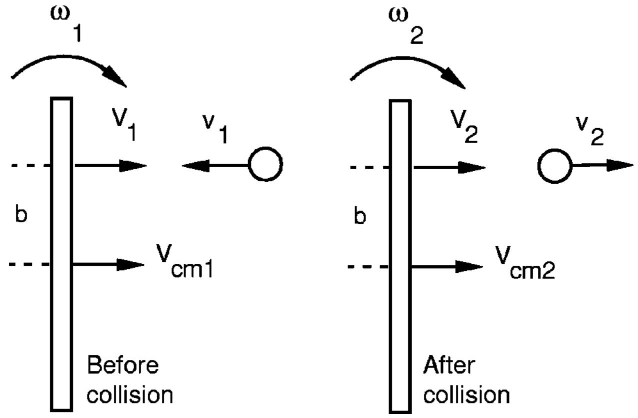

Fig. 12. Generic bat and ball collision. The bat could equally well be a racquet or a club. If all the energy in the bat is transferred to the ball, then $V_{\mathrm{c.m.},2} = \omega_2 = 0$.

rotates at angular velocity ω_1, its center of mass translates at speed $V_{\mathrm{c.m.},1}$ and the impact point on the bat translates at speed V_1. The impact point is a distance b from the bat center of mass, and hence $V_1 = V_{\mathrm{c.m.},1} + b\omega_1$. The corresponding parameters after the collision are denoted by the subscript 2, so that $V_2 = V_{\mathrm{c.m.},2} + b\omega_2$. The ball is incident at speed v_1 and exits along the same path as the incident ball at speed v_2. The essential features of the collision can be determined by considering a free, rigid bat. The fact that a bat is hand held adds to the complexity of the problem,[20] but if the ball is struck at a point near the sweet spot, the impact force at the handle end of the bat can largely be ignored because the bat rotates during and after the collision about an axis near the end of the handle. Conservation of linear and angular momentum is described by the relations

$$MV_{\mathrm{c.m.},1} - mv_1 = MV_{\mathrm{c.m.},2} + mv_2, \tag{A1}$$

and

$$I_{\mathrm{c.m.}}\omega_1 - mv_1 b = I_{\mathrm{c.m.}}\omega_2 + mv_2 b, \tag{A2}$$

where $I_{\mathrm{c.m.}}$ is the moment of inertia of the bat about an axis through its center of mass.

The energy loss in the bat and ball is most conveniently described in terms of the coefficient of restitution, e, defined by

$$e = \frac{v_2 - V_2}{v_1 + V_1} = \frac{v_2 - V_{\mathrm{c.m.},2} - b\omega_2}{v_1 + V_{\mathrm{c.m.},1} + b\omega_1}. \tag{A3}$$

If all the energy of the bat is transferred to the ball, then $V_{\mathrm{c.m.},2} = \omega_2 = 0$, and hence

$$m = \frac{M_e}{[e + (1+e)v_1/V_1]}, \tag{A4}$$

where $M_e = MI_{\mathrm{c.m.}}/(I_{\mathrm{c.m.}} + Mb^2)$ is the effective mass of the bat. The collision is equivalent to that of a point mass M_e at speed V_1 colliding head-on with the ball. The collision of a golf club or tennis racquet with the ball corresponds to $v_1 = 0$, and hence the club or racquet comes to rest if $m = M_e/e$. The kinetic energy of the ball is a maximum when $m = M_e$, but the energy transferred from the club or racquet is maximized when $m = M_e/e$. In baseball, the ball is incident at finite speed, and hence the mass of the ball required to stop the bat is less than M_e/e by an amount that depends on the ratio of v_1 to V_1. For example, if we assume that $v_1 = V_1$, $e = 0.5$, and $M_e = 0.7$ kg, then $m = 0.35$ kg. A conventional baseball has a mass of about 0.14 kg. For the same bat parameters ($e = 0.5$ and $M_e = 0.7$) Eq. (A4) indicates that a bat swung at low speed toward a high speed incoming ball will come to rest even with a conventional ball, provided that $v_1/V_1 \approx 3.0$.

[a]Electronic mail: cross@physics.usyd.edu.au

[1]T. Shinbrot, C. Grebogi, J. Wisdom, and J. A. Yorke, "Chaos in a double pendulum," Am. J. Phys. **60**, 491–499 (1992).

[2]R. B. Levien and S. M. Tan, "Double pendulum: An experiment in chaos," Am. J. Phys. **61**, 1038–1044 (1993).

[3]D. Williams, "The dynamics of the golf swing," Q. J. Mech. Appl. Math. **20**, 247–264 (1967).

[4]C. B. Daish, *The Physics of Ball Games* (English Universities Press, London, 1972).

[5]T. Jorgensen, "On the dynamics of the swing of a golf club," Am. J. Phys. **38**, 644–651 (1970).

[6]T. Jorgensen, *The Physics of Golf* (Springer-Verlag, New York, 1999), 2nd ed.

[7]R. M. Alexander, "Optimum timing of muscle activation for simple models of throwing," J. Theor. Biol. **150**, 349–372 (1991).

[8]R. Cross, "Physics of overarm throwing," Am. J. Phys. **72**, 305–312 (2004).

[9]C. Putnum, "Sequential motions of body segments in striking and throwing skills: Descriptions and explanations," J. Biomech. **26**, Supp. 1, 125–135 (1993).

[10]S. J. Phillips, E. M. Roberts, and T. C. Huang, "Quantification of intersegmental reactions during rapid swing motion," J. Biomech. **16**, 411–417 (1983).

[11]R. Bartlett, *Sports Biomechanics: Reducing Injury and Improving Performance* (E. and FN Spon, London, 1999).

[12]R. J. Neal and B. D. Wilson, "3D kinematics and kinetics of the golf swing," Int. J. Sport Biomech. **1**, 221–232 (1985).

[13]K. Miura and T. Naruo, "Accelerating and decelerating phases of the wrist motion of the golf swing," in *The Engineering of Sport*, edited by S. J. Haake (Blackwell Science, Cambridge, 1998), pp. 455–463.

[14]E. J. Sprigings and R. J. Neal, "An insight into the importance of wrist torque in driving the golfball: A simulation study," J. Appl. Biomech. **16**, 356–366 (2000).

[15]H. Brody, R. Cross, and C. Lindsey. *The Physics and Technology of Tennis* (Racquet Tech, Solana Beach, CA, 2002).

[16]R. Cross, "Impact of a ball with a bat or racket," Am. J. Phys. **67**, 692–702 (1999).

[17]If a heavy ball strikes a light ball head-on and if the light ball is initially at rest, a large fraction of the initial kinetic energy of the heavy ball is retained by the heavy ball. The energy transfer efficiency can be improved by using a third ball of intermediate mass, in such a way that the heavy ball first strikes the third ball which then strikes the light ball. The energy transferred to the light ball is a maximum when the mass of the third ball is the geometric mean of the masses of the light and heavy balls.

[18]See, for example, E. Kreighbaum and K. M. Barthels, *Biomechanics* (Allyn and Bacon, Needham Heights, MA, 1996).

[19]L. Smith, J. Broker, and A. Nathan, "A study of softball player swing speed," in *Sports Dynamics: Discovery and Application*, edited by A. Subic, P. Travailo, and F. Alam (RMIT U.P., Melbourne, 2003), pp. 12–17.

[20]R. Cross, "Center of percussion of hand-held implements," Am. J. Phys. **72**, 622–630 (2004).