A double pendulum model of tennis strokes

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The physics of swinging a tennis racquet is examined by modeling the forearm and the racquet as a double pendulum. We consider differences between a forehand and a serve, and show how they differ from the swing of a bat and a golf club. It is also shown that the swing speed of a racquet, like that of a bat or a club, depends primarily on its moment of inertia rather than on its mass.

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I. INTRODUCTION

It is well known that the swing of a golf club can be accurately modeled as a double pendulum.[1,2] Animations illustrating the basic physics are available.[3] The swing of a baseball bat can also be modeled as a double pendulum.[4,5] In this paper, the double pendulum model is used to describe the swing of a tennis racquet. We consider only the forehand and serve, and show how they differ from each other and from the swing of a club and a bat.

A conventional double pendulum consists of two segments joined end-to-end at a pivot point, the upper segment also being pivoted about a fixed axis near its upper end. The two segments normally swing in a vertical plane, driven by gravity alone. A tennis stroke can be described in a simplified manner by treating the forearm as one segment of a double pendulum and the racquet as the other segment, pivoted at the wrist. In a tennis stroke, gravity plays only a minor role; the two segments are driven primarily by muscles in the arm and the hand, and can swing in a horizontal or a vertical plane or in any other plane that the player chooses. The elbow does not act as a fixed pivot point because the upper arm rotates about an axis in the shoulder and because the upper arm translates as a result of forward motion of the player during the stroke. The conventional model of a double pendulum is modified in this paper to take these added complications into account in a manner similar to that described previously in relation to the swing of golf clubs[1,2] and baseball bats.[4,5]

Some features of real strokes are not captured by the double pendulum model, such as rotation of the hand and the forearm or any other body segment about its long axis, nor is the three-dimensional nature of many tennis strokes fully described. Nevertheless, the double pendulum model provides useful insights into the forces and torques required to swing a racquet, the manner in which the forearm and the racquet are mutually coupled, and the dependence of swing speed on the physical properties of a racquet.

It is shown that the swing speed of a racquet, like that of a club or a bat, depends on its moment of inertia rather than on its mass. It is not obvious that this dependence should be the case, given the complex and nonlinear interaction between the arm and the implement in the hand. A common assumption is that light clubs, bats, and racquets can be swung faster than heavy versions.[6,7] This assumption is correct only if the lighter implement has a lower moment of inertia than the heavier implement. If the mass difference is large, then the assumption is likely to be correct. However, the mass of commonly available racquets, bats, and clubs varies over a small range, in which case the assumption might be incorrect.

Recent experiments have established that the primary factor affecting swing speed is the moment of inertia of the implement.[8,9] When throwing an object, the force applied to the object increases with its mass, with the result that throw speed does not depend as strongly on mass as might be expected.[10] Similarly, we show in this paper that swing speed depends only weakly on the moment of inertia. If the moment of inertia of a bat or racquet is doubled, then the swing speed decreases by about 20%. The latter result has been incorporated into regulations governing the performance of baseball and American Softball Association bats,[9] but has not previously been explained in terms of a model.

II. EQUATIONS DESCRIBING A DOUBLE PENDULUM

The geometry of the double pendulum model, as it applies to the swing of a racquet, is shown in Fig. 1. The elbow translates in the x direction at velocity $v_{E,x}=dx_E/dt$ and in the y direction at velocity $v_{E,y}=dy_E/dt$. The forearm and racquet rotate in a clockwise direction in the x-y plane at angular velocities $\omega_1=-d\theta/dt$ and $\omega_2=-d\phi/dt$, respectively; the angles θ and ϕ are defined in Fig. 1.

Normally, the elbow moves down and to the left in Fig. 1 if the ball approaches from the left, in which case, both $v_{E,x}$ and $v_{E,y}$ are negative. If the elbow is moving vertically down in Fig. 1(a) and is decelerating, then the effect is equivalent to a vertical force acting upward at the elbow, with the result that a clockwise torque is exerted on the forearm. An additional clockwise torque arises if the elbow is moving to the left and is decelerating. Both effects are significant when swinging a racquet or any other implement. An identical effect occurs at the wrist. If the angular velocity of the forearm decreases, the resulting force on the racquet at the wrist joint acts to increase the angular velocity of the racquet.

The forearm has mass M_1, length L_1, and its center of mass (point G_1) is a distance h_1 from the elbow. The hand is rigidly attached to the racquet handle when swinging a racquet, and both rotate at the same angular velocity. The racquet-hand system has mass M_2, length L_2, and the center of mass (point G_2) is at a distance h_2 from the wrist. The coordinates of G_1 and G_2 are $x_1=x_E+h_1\sin\theta$, $y_1=y_E-h_1\cos\theta$, and $x_2=x_E+L_1\sin\theta+h_2\sin\phi$, $y_2=y_E-L_1\cos\theta-h_2\cos\phi$. Differentiation with respect to time yields the ve-

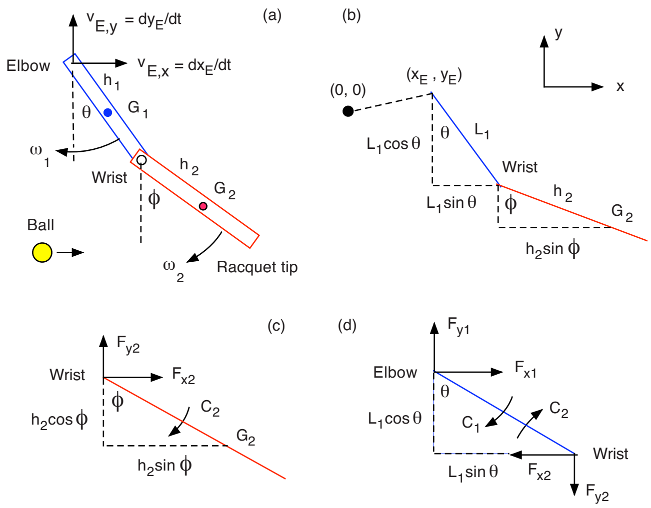

Fig. 1. (a) Geometry of a double pendulum consisting of a forearm and a racquet free to rotate about an axis through the wrist. The elbow is also free to translate or rotate. The subscript E refers to the elbow. G_1 and G_2 denote the center of mass of the forearm and racquet, respectively. The quantity h_1 is the distance between G_1 and the elbow, while h_2 is the distance between G_2 and the wrist. The forearm is inclined at angle θ to the y axis and rotates at angular velocity ω_1. The racquet is inclined at angle ϕ to the y axis and rotates at angular velocity ω_2. (b) The coordinate origin (0,0) and the coordinates of the elbow (x_E, y_E) and wrist; L_1 is the length of the forearm. (c) The force components (F_{x2}, F_{y2}) and the couple, C_2, acting on the racquet at the wrist. (d) The force components (F_{x1}, F_{y1}) acting on the forearm at the elbow and wrist, and the couples C_1 and C_2 applied to the forearm.

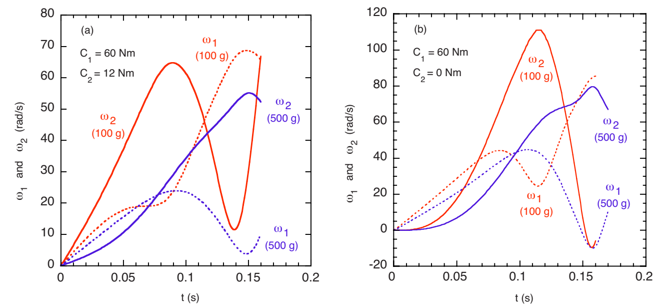

Fig. 2. (a) Results for 100 and 500 g racquets swung with a 1.5 kg forearm and with $C_1=60$ N m and $C_2=12$ N m, both held constant throughout the swing. In (b), $C_2=0$, ω_1 is the angular velocity of the forearm, and ω_2 is the angular velocity of the racquet.

locity components of G_1 and G_2, and second derivatives yield the acceleration and hence the force components acting on the arm and racquet.

The upper arm exerts a force with components $F_{x,1}$ and $F_{y,1}$ on the forearm at the elbow joint, and the forearm exerts a force with components $F_{x,2}$ and $F_{y,2}$ on the racquet at the wrist joint. The racquet and the hand exert a force on the forearm at the wrist joint, as indicated in Fig. 1(d), which is equal and opposite to the force of the forearm on the racquet and hand. In addition to the forces at each joint, muscles in the upper arm exert a couple C_1 on the forearm and muscles in the forearm exert a couple C_2 on the hand and racquet. A couple consists of equal and opposite forces, and thus there is no net force exerted by a couple. If $C_2=0$, the wrist acts as a free hinge and is described as "relaxed," although the hand must still maintain a firm grip on the handle. The hand and racquet exert an equal and opposite couple C_2 on the forearm. The equations of motion for the double pendulum are derived in the Appendix in terms of the torques acting about G_1 and G_2 and are given by Eq. (A10). The effect of gravity can be ignored if the racquet and forearm swing in a horizontal plane. The effect of gravity was included in serve calculations, including those shown in Fig. 2.

III. QUALITATIVE FEATURES OF A DOUBLE PENDULUM

The behavior of a double pendulum is nonlinear and not intuitively obvious because the motion of one segment normally has a strong effect on the motion of the other segment. It is useful to summarize briefly the essential features as they pertain to the swing of a racquet. The strength of the coupling between the two segments of a double pendulum depends on their relative mass.[4] The term "coupling" here refers to the influence of one pendulum segment on the other segment. When swinging a light tennis racquet (or a squash or badminton racquet) of mass less than about 100 g, the coupling between a 1.5 kg forearm and the racquet is weak. A constant couple applied to each segment results in a monotonic increase in the angular velocity of each segment, at least for the duration of a typical stroke. A different result is obtained when the racquet mass is about 250 g or more because the coupling between the forearm and the racquet is then stronger. In that case, a constant couple applied to each segment results in an initial monotonic increase in the angular velocity of the forearm, followed by a monotonic decrease. As the angular velocity of the forearm decreases to a minimum, the angular velocity of the racquet increases to a maximum. The latter process can be described as a transfer of rotational energy or angular momentum from the arm to the racquet. The process can be enhanced, and the racquet speed increased if the couple applied to the forearm is decreased shortly before the angular velocity of the racquet reaches its maximum value. In that case, the angular velocity of the forearm decreases more rapidly and the forearm transfers a greater fraction of its rotational energy to the racquet.

Some of these features are illustrated in Fig. 2(a) for a 1.5 kg forearm with length of 0.3 m swinging either a 100 or a 500 g racquet. Both racquets were assumed to have a uniform mass distribution, were swung with a 500 g hand, with constant couples $C_1=60$ N m and $C_2=12$ N m applied throughout the swing, and with $\beta=\phi-\theta=90°$ at $t=0$. Elbow motion was ignored for the calculations shown in Fig. 2. The couple applied to each racquet was not chosen to ensure that the racquet was correctly aligned on impact with the ball. Rather, the same couple C_2 was used for both racquets to illustrate the behavior of a double pendulum when the only parameter that is altered is the racquet mass. Figure 2(b) illustrates the effect of decreasing C_2 to zero, with the surprising result that the angular velocity of each racquet increases as C_2 decreases. The same effect has been observed in experiments with a mechanical double pendulum.[4]

The parameters chosen in Fig. 2 and in later calculations are typical of those that apply to tennis. Forearm and hand mass data are given in most biomechanics textbooks as a percentage of total body mass and is typically 1.8% and 0.65%, respectively, for adult males.[11] The couple applied to a racquet or the forearm has not been measured as far as we know, although the net torque (which includes the couple) is commonly measured when studying the biomechanics of tennis strokes. The peak elbow torque generated in a serve was found to be 78.3 N m averaged over eight male players competing at the 2000 Olympic Games, and the average elbow torque during the serve was 67.6 N m for the same players.[12] In a similar study by Bahamonde and Knudson,[13] the peak elbow torque in a forehand stroke was found to vary from 27 to 62 N m for 15 different players.

The elbow couple was taken as 60 N m in Fig. 2 for the purpose of illustrating the qualitative features of the double pendulum motion. The resulting angular and linear velocities of the forearm and racquet are typical of those measured in a fast serve by male tennis players.[12,14] Most tennis racquets weigh between 250 and 350 g. The racquet mass assumed for the results of the calculations shown in Fig. 2 was extended well beyond the normal range to exaggerate differences between light and heavy racquets and to highlight that the swing speed of a racquet depends only weakly on its mass or its moment of inertia.

For the 100 g racquet, the angular velocity of the forearm is still increasing when the angular velocity of the racquet is a maximum near $t=0.09$ s [see Fig. 2(a)]. The behavior of the 100 g racquet after 0.1 s is not of practical significance because a player would reverse the direction of C_1 after impacting a ball. The 100 g racquet does not swing five times faster than the 500 g racquet, as might be expected. It swings five times faster at the start of the swing, but the maximum swing speed is only 22% greater because the transfer of energy from the arm to the racquet is less efficient with a light racquet and the 500 g racquet is accelerated over a longer period of time. For the 500 g racquet, the forearm is almost stationary when the angular velocity of the racquet is a maximum. An increase in the coupling between the forearm and the racquet can be achieved if the couple applied to the racquet is reduced, as indicated in Fig. 2(b). As a result, a greater fraction of the energy in the arm is transferred to the racquet and the maximum angular velocity of the racquet is increased.

The results in Fig. 2(b) highlight the point made by Jorgensen[1] that golfers can expect lower swing speeds if the wrists are used deliberately in an attempt to swing the club faster. However, the result depends sensitively on the timing of the wrist couple. If a wrist couple is applied late in the swing, then the swing speed is increased.[2]

When using a heavy racquet or a baseball bat, the coupling between the forearm and the racquet or the bat is so strong that the racquet or the bat tends to rotate too fast to line up correctly at the intended impact point. It may then be necessary to reduce the angular velocity of the racquet so that it arrives in the correct orientation to meet the ball. The latter result can be achieved by increasing the couple applied to the forearm just before impact or by decreasing the couple applied to the racquet just prior to impact with the ball. When swinging a baseball or a softball bat, it is necessary to apply a negative couple to the bat, using both hands, prior to impacting the ball.[5]

IV. A TYPICAL FOREHAND

A forehand in tennis is often struck with topspin by swinging the racquet upward as well as forward. We will ignore vertical motion of the racquet and assume that both the forearm and the racquet swing in a horizontal plane. Forward motion of the racquet usually commences when the forearm is pointing in a direction approximately toward the back fence ($\theta=90°$ in Fig. 1) and the racquet is about 50° further around, with $\phi=140°$. Players do not always begin forward motion of the racquet from that position, but the values given here are common and will be used in the following. It is known that a double pendulum is chaotic and sensitive to the initial conditions, but good players do not swing a racquet or a golf club in a chaotic manner. The reason is not that all players start their swing from exactly the same position, but that chaos becomes important only after a few complete cycles of oscillation of the pendulum. The first half cycle of oscillation of a double pendulum is reproducible.[4]

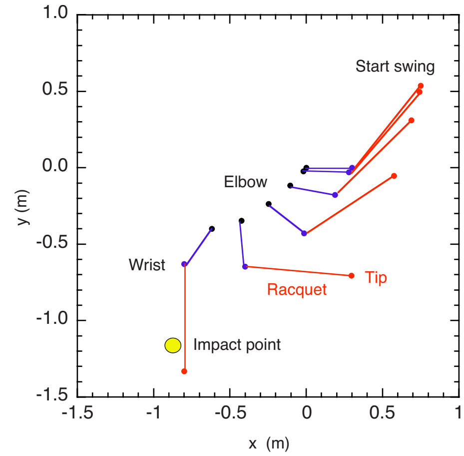

Fig. 3. A bird's-eye view showing the forearm and racquet at 0.05 s intervals for a 300 g racquet swung in a horizontal plane to impact a ball when the racquet is aligned with $\phi=0$. The elbow swings forward due to rotation about the shoulder and forward motion of the player.

In a tennis forehand, the ball is normally struck in front of the body with the forearm extended forward while the racquet is aligned nearly parallel to the net at the point of impact. We will assume that the ball is struck when $\theta=-40°$ and $\phi=0$ so that the outgoing ball heads toward the net. This type of swing differs from that in golf or baseball, where $\beta=\phi-\theta$ is approximately 90° at the start of the swing and remains close to 90° while the wrists remain locked. Soon after the start of a golf or baseball swing, the wrists relax and the striking implement then rotates rapidly to align with the arms at the point of impact so that $\beta=0$.

A realistic forehand can be modeled by assuming that the couples C_1 and C_2 remain constant in time. A realistic golf swing can also be modeled when the couple C_1 remains constant.[1] If the wrist remains locked, then the forearm and the racquet swing at the same rotational speed at the beginning of the swing, and C_2 decreases with time. If C_2 remains constant in time, then the forearm swings faster than the racquet at the beginning of the stroke, while the racquet swings faster than the forearm toward the end of the stroke. Both types of forehand are common and can be viewed in slow motion.[15] The effects of allowing C_2 to vary with time are considered in Sec. V.

Calculations for a medium pace forehand are shown in Figs. 3 and 4 for a racquet with length of 70 cm, mass of 300 g, swing weight of 310 kg cm^2, and with center of mass 35 cm from the butt end of the handle. The mass of the forearm was taken as 1.5 kg, and the mass of the hand was 0.5 kg. The swing weight of a racquet is an industry term referring to its moment of inertia about an axis in the handle located

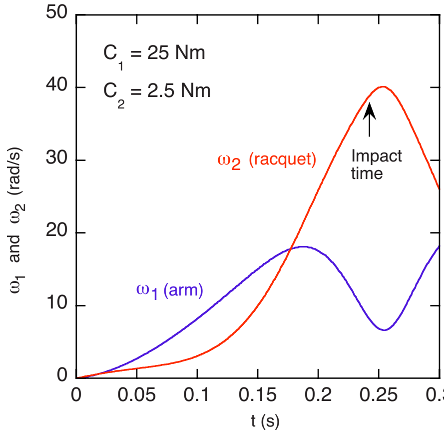

Fig. 4. Angular velocities of the forearm and the racquet for the forehand shown in Fig. 3.

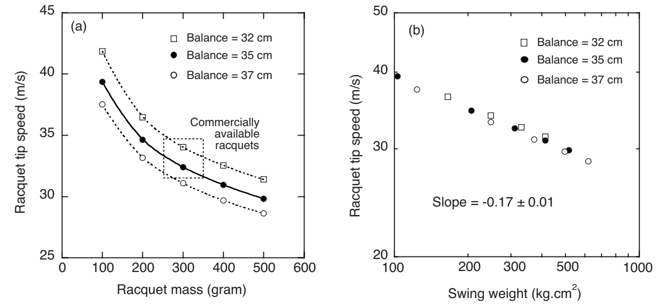

Fig. 5. Velocity of the racquet tip at impact versus (a) racquet mass (linear plot) and (b) swing weight (log-log plot) for 15 different racquets all swung with the same forearm couple C_1 and elbow speed as that shown in Fig. 4. The mass of the racquet was varied from 100 to 500 g in 100 g increments. The swing weight of a racquet is its moment of inertia about an axis 10 cm from the end of the handle.

10 cm from the butt end. The racquet was swung with $C_1 = 25$ N m and $C_2 = 2.5$ N m, resulting in good alignment of the racquet at the nominal impact point.

For the calculations in Figs. 3 and 4, the elbow was assumed to translate in the x direction with a velocity $v_{E,x} = -33t + 69t^2$ and in the y direction with a velocity $v_{E,y} = -42t + 174t^2$. $v_{E,x}$ and $v_{E,y}$ are zero at the start of the swing, $v_{E,x}$ has a maximum value of -4m/s at $t = 0.24$ s (the impact time), and $v_{E,y}$ has a maximum value of -2.5m/s at $t = 0.12$ s and is zero at $t = 0.24$ s. These values are typical of those measured by the author for a medium pace forehand but are not critical to the outcome of the swing. Small changes in $v_{E,x}$ or $v_{E,y}$ or the starting position of the racquet require small variations in C_1 or C_2 to align the racquet, but do not result in large changes in the rotational speed of the racquet.

The angular velocities of the forearm and the racquet are shown in Fig. 4. Despite the fact that C_2 was held constant, rotational energy transferred from the arm to the racquet during the swing, with the result that ω_1 decreased to a minimum and ω_2 increased to a maximum near the end of the swing. The latter effect is well documented in the biomechanics literature but rarely explained in terms of double pendulum mechanics. In an efficient tennis (or golf or baseball) swing, rotational energy is first transferred from the upper arm to the forearm and is subsequently transferred from the forearm to the racquet after a short time delay.[1,2,4–6,12–14]

Calculations similar to those in Figs. 3 and 4 were repeated for a range of different racquets with different values of mass, swing weight, and center of mass location, all swung with the same forearm couple C_1, starting position, and elbow speed as in Figs. 3 and 4. The magnitude of the couple C_2 applied to each racquet was altered so that each racquet swung into correct alignment with the incoming ball, with $\phi = 0$ when $\theta = -40°$. Racquets with a larger moment of inertia require a larger couple for alignment. For example, the 100 g racquet with its center of mass at 35 cm was swung with $C_2 = 1.38$ N m, and the 400 g racquet with its center of mass at 35 cm was swung with $C_2 = 2.54$ N m.

Racquets with a 35 cm center of mass were assumed to have a uniform mass distribution with a moment of inertia $ML^2/12$ about the center of mass. The calculations were repeated for racquets having a center of mass at 37 cm by increasing the swing weight of each racquet by 20% (compared with the uniform mass distribution value) and also for a group of racquets having a center of mass at 32 cm by decreasing the swing weight of each racquet by 20%. The results of the calculations are shown in Fig. 5, giving the linear velocity of the tip of the racquet at impact versus the mass of the racquet [Fig. 5(a)] and the swing weight of the racquet [Fig. 5(b)]. In terms of the outgoing ball speed, the parameter of greatest interest is the linear velocity of the racquet at the impact point on the strings. The velocity is a maximum at the tip of the racquet.

The results in Fig. 5 show that the swing speed of a racquet depends primarily on the swing weight of the racquet rather than on its mass. Figure 5(a) shows a strong correlation between swing speed and racquet mass if the center of mass remains constant and also shows that the center of mass can vary over a relatively wide range in practice. For an arbitrary selection of racquets having different centers of mass, there is only a weak correlation between swing speed and racquet mass, especially when the mass is restricted to commercially available racquets [indicated by the square box in Fig. 5(a)]. Figure 5(b) shows a much stronger correlation with swing weight, given that Fig. 5(b) includes the same 15 racquets as those in Fig. 5(a) and therefore includes racquets with a wide range of centers of mass. The results in Fig. 5(b) indicate that the tip velocity is proportional to $1/(\text{swing weight})^n$, where $n = 0.17 \pm 0.01$, indicating that the speed at which a racquet can be swung depends only weakly on its swing weight.

The latter result was found to be insensitive to the choice of initial parameters. If the initial racquet position is changed from $\phi = 140°$ to $\phi = 130°$, keeping the forearm couple fixed at 25 N m, then a slightly larger racquet couple is required to swing the racquet into alignment, resulting in a slight decrease in the velocity of the racquet tip. Results with $\phi = 130°$ for the same racquet selection, as indicated in Fig. 5, gave the same $n = 0.17$ power law dependence of tip speed on swing weight as that shown in Fig. 5(b).

The results in Fig. 5 were obtained by keeping C_1 fixed because it is easier for a player to vary C_2 to align each racquet when swinging different racquets. A player could also vary C_1, while keeping C_2 fixed, to achieve the same result. However, major changes in C_1 are then required to align both the arm and each racquet due to the high angular velocity acquired by low mass racquets. The results shown in

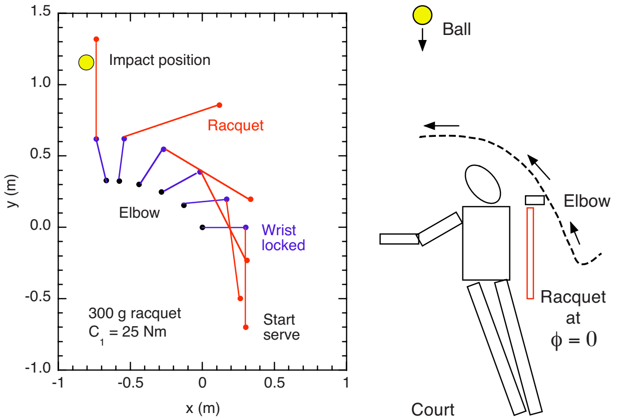

Fig. 6. Calculated positions of the forearm and racquet at 0.02 s intervals for a 300 g racquet swung in a vertical plane to impact a ball when the racquet is aligned with $\phi = 180°$ and $\theta = 195°$. The dashed line at the right shows the motion of the elbow.

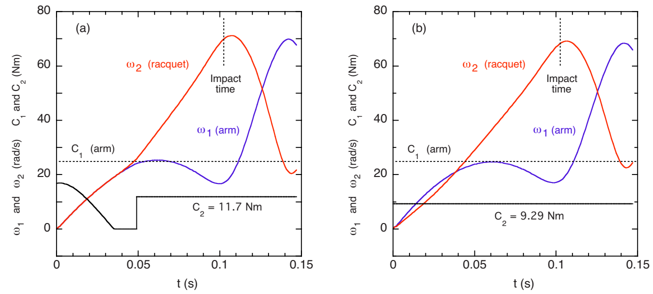

Fig. 7. Angular velocities of the forearm and the racquet, and the couples applied to the forearm and the racquet, for (a) the serve shown in Fig. 6, and (b) a similar serve where C_2 remains constant in time.

Fig. 5 better represent the variation of swing speed under conditions where the effort exerted by the player remains approximately constant for each racquet. The results for a near maximum effort swing are described in the following.

V. A TYPICAL SERVE

A serve in tennis is similar to a golf swing and the swing of a baseball bat in that the server usually starts the swing with the racquet aligned at about 90° to the forearm.[12,14] Impact with the ball occurs when the racquet is approximately in line with the forearm, as it is in golf and baseball. There is also a preliminary stage prior to the commencement of the forward swing, where the server positions the racquet behind his or her back so that the racquet points vertically down to the court. We will assume that the swing starts with $\phi = 0$ (racquet pointing to the court) and the forearm starts in a horizontal position, with $\theta = 90°$. Impact with the ball is assumed to occur when the racquet swings through 180° to $\phi = 180°$ and when the forearm swings through an angle of 105° to $\theta = 195°$ so that the forearm is slightly in front of the server on impact (see in Fig. 6).

The preliminary stage of the swing in a tennis service includes rapid rotation of the upper arm, with the result that the elbow is rising rapidly and moving forward at the instant the racquet is pointing down to the court.[12,14] The effect was measured by filming an elite player at 300 frames/s when serving a ball at 45 m/s (100 mph). The results of the recorded motion of the elbow are shown in Fig. 6. The vertical speed of the elbow decreased from 9 m/s at the nominal start of the forward swing (when $\phi = 0$) to zero at the point of impact with the ball. The horizontal speed of the elbow increased from 5 m/s at the start of the swing to 8 m/s and then decreased to 2.5 m/s on impact with the ball.

Good fits to the elbow data were obtained with $a_x = dv_{E,x}/dt = -125 + 2600t$ and $a_y = dv_{E,y}/dt = -150 + 1250t$. These fits were used in Eqs. (A6) and (A9) for the following calculations, together with the initial conditions $\omega_1 = \omega_2 = 0$. There was negligible rotation of the forearm or the racquet in the x-y plane at the start of the swing because the forearm remained locked at right angles to the upper arm and the racquet remained locked at right angles to the forearm. Rotation of the upper arm caused the forearm and racquet to translate as a rigid body at the same speed as the elbow, but the forearm remained horizontal just prior to the start of the swing. Subsequent deceleration of the elbow in both the x and the y directions resulted in a much smaller couple being needed to rotate the forearm than that shown in Fig. 2. Nevertheless, the torque acting on the racquet is about the same in Figs. 2(a) and 7 because the angular acceleration of the racquet is about the same.

If the wrist remains locked at the beginning of a serve, the couple C_2 applied to the racquet is determined by the condition that $\omega_1 = \omega_2$. As the racquet accelerates, C_2 decreases and the wrist can relax completely when $C_2 = 0$, at least for a short time interval. In a golf swing, C_2 can remain zero until impact occurs with the ball, but in a tennis serve, the centripetal force acting along the forearm is usually too small to swing the racquet into line. Consequently, the server needs to apply a positive couple to the racquet late in the swing to ensure that the racquet impacts the ball in the correct orientation.

Calculations for a typical serve are shown in Figs. 6 and 7 for a 70 cm long, 300 g racquet with a 35 cm center of mass and a swing weight of 310 kg cm^2. The racquet was swung by applying a constant arm couple $C_1 = 25$ N m throughout the swing. In Fig. 6, the wrist remained locked until C_2 decreased to zero. Then, a constant couple $C_2 = 11.7$ N m was applied at $t = 0.05$ s to bring the racquet into alignment. The angular velocities of the forearm and the racquet, as well as C_1 and C_2, are shown in Fig. 7(a).

Figure 7(b) shows a similar serve for the same racquet parameters, the same elbow velocity and the same forearm couple, but a constant racquet couple $C_2 = 9.29$ N m was applied throughout the swing to align the racquet. The second serve resulted in a slightly lower swing speed (67.5 rad/s rather than 69.6 rad/s), but the result indicates that it is not necessary to lock the wrist at the beginning of the serve, nor is it necessary to relax the wrist in the middle of the serve. For both serves represented in Fig. 7, the couple C_2 provided approximately half of the 20 N m torque required to swing the racquet, as described by Eq. (A4).

The effect of serving with lighter or heavier racquets was examined, as for forehand swings, by swinging each racquet using the same arm couple as in Figs. 6 and 7, and the same elbow velocity. A constant couple was applied to each racquet throughout the swing, heavier racquets requiring a larger value of C_2 to align the racquet than lighter racquets. However, 500 g racquets required a lower value of C_2 than 400 g racquets due to the increased coupling between the forearm and the racquet. The results are shown in Fig. 8,

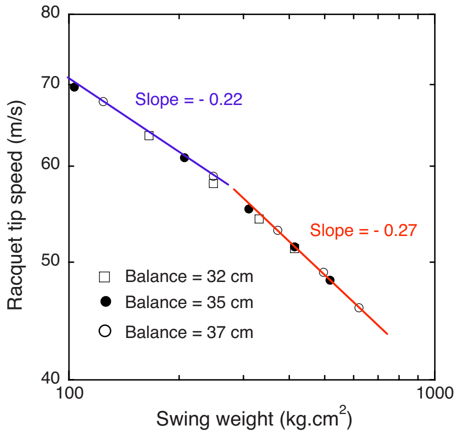

Fig. 8. The speed of a racquet tip at impact versus swing weight for different racquets all swung with a constant forearm couple C_1=25 N m and a constant racquet couple chosen so that the racquet is correctly aligned on impact. The racquet tip speed is proportional to $1/(\text{swing weight})^n$, where n varies slightly with swing weight and is given by the slope of the best fit line segments in this log-log plot.

again indicating that the swing speed of a racquet depends primarily on the moment of inertia of the racquet rather than on its mass. The speed of the tip of the racquet is plotted in Fig. 8 and is a combination of angular speed and translational speed of the racquet. Two best fit lines are shown in Fig. 8, indicating that the slope varies with swing weight. The racquet tip speed is proportional to $1/(\text{swing weight})^n$, where $n=0.22\pm0.01$ for low swing weight racquets and $n=0.27\pm0.01$ for high swing weight racquets. The exponent n is given by the slope of the best fit lines in Fig. 8 because the results are plotted on log-log scales. The exponent n is larger in Fig. 8 than in Fig. 5(b) even though the same set of racquets was compared and the same couple was used to swing the forearm. The difference is presumably due to the larger couple applied to each racquet and the larger elbow speed, both of which acted to increase the torque acting on each racquet.

The general conclusion that swing speed depends primarily on swing weight is one that is specific to the task. A different result is obtained if we assume that the couple applied to each racquet has the same value for all racquets. In that case, the maximum swing speed does not occur at the same arm position for each racquet nor is each racquet correctly aligned on impact with the ball. In that hypothetical situation, the maximum swing speed decreases as the racquet mass increases or as the swing weight increases, at least for racquets having the same center of mass. When comparing racquets with a range of different positions of the center of mass, the correlation between swing speed and swing weight is weak, as is the correlation between swing speed and racquet mass.

VI. DISCUSSION

Clubs and bats are usually swung in a single plane, while racquet motion is often more complex and can involve rotation of many body segments about all three axes. Nevertheless, the essence of a tennis stroke is captured in the double pendulum model, providing useful insights into the basic mechanics. A racquet is light enough to be swung with a single arm, whereas clubs and bats are normally swung using two arms. The forces and torques used to swing a club or a bat are correspondingly larger, with the result that the wrist plays a small role late in the swing and can relax completely when swinging a bat or a club. Because racquets are generally lighter and have a lower moment of inertia, the wrist plays a more significant role in controlling the swing of a racquet, not only at the beginning of the swing but throughout the whole swing. For a fast serve, the wrist actively rotates the racquet by providing a couple amounting to about half of the total torque required to swing the racquet, the other half arising from the force applied at the wrist joint by the forearm.

Golfers and batters in baseball and softball usually lock their wrists at the beginning of the swing to ensure that the club or bat does not get left behind when they swing their arms forward. The same technique can be used when swinging a racquet, but racquets can also be swung effectively without locking the wrist. A more aggressive approach can be used in a tennis forehand, where the arm swings ahead of the racquet at the beginning of the swing and then slows rapidly just before impact with the ball. Likewise when serving with a racquet, it is not necessary to lock the wrist at the beginning of the swing, although players normally apply a couple to the racquet at the beginning of the serve so that the racquet swings forward. It is possible to swing a racquet forward on the end of a rope without applying a couple to the racquet, but that technique is of academic interest only.

In recent years, it has been shown experimentally[8,9] that the swing speed of a sporting implement depends primarily on its moment of inertia I rather than on its mass and that the swing speed for a maximum effort swing is proportional to $1/I^n$, where n is typically about 0.26 for baseball and softball bats and other implements with a swing weight greater than about 300 kg cm^2. It has also been shown experimentally[8] that the exponent n decreases when swinging implements with a swing weight less than 300 kg cm^2. The experimental data on swing speed are consistent with the calculations presented in this paper, at least with those for a near maximum effort serve. A simple explanation of the low value of n is that most of the effort of a batter or a tennis player is expended in swinging the arms and the addition of a light implement in the hand does not alter the swing speed in proportion to the weight or swing weight of the implement.

APPENDIX: EQUATIONS OF MOTION FOR A DOUBLE PENDULUM

The velocity components of the center of mass points G_1 and G_2 in Fig. 1 are, respectively,

$$v_{x1}=dx_1/dt=v_{E,x}-h_1\omega_1\cos\theta, \tag{A1a}$$

$$v_{y1}=v_{E,y}-h_1\omega_1\sin\theta, \tag{A1b}$$

$$v_{x2}=dx_2/dt=v_{E,x}-L_1\omega_1\cos\theta-h_2\omega_2\cos\phi, \tag{A1c}$$

$$v_{y2}=v_{E,y}-L_1\omega_1\sin\theta-h_2\omega_2\sin\phi. \tag{A1d}$$

The net force on the forearm can be calculated in terms of the components $F_{x1}-F_{x2}=M_1dv_{x1}/dt$ and $F_{y1}-F_{y2}-M_1g=M_1dv_{y1}/dt$, giving

$$F_{x1}-F_{x2}=M_1\left(a_x-h_1\cos\theta\frac{d\omega_1}{dt}-h_1\omega_1^2\sin\theta\right), \tag{A2a}$$

$$F_{y1} - F_{y2} = M_1\left(g + a_y - h_1 \sin \theta\frac{d\omega_1}{dt} + h_1\omega_1^2 \cos \theta\right), \tag{A2b}$$

where g is the acceleration due to gravity [Fig. 1(a) is upside-down for a serve]. Similarly, the net force on the racquet-hand system has components

$$F_{x2} = M_2\left(a_x - L_1 \cos \theta\frac{d\omega_1}{dt} - L_1\omega_1^2 \sin \theta - h_2 \cos \phi\frac{d\omega_2}{dt} - h_2\omega_2^2 \sin \phi\right), \tag{A3a}$$

$$F_{y2} = M_2\left(g + a_y - L_1 \sin \theta\frac{d\omega_1}{dt} + L_1\omega_1^2 \cos \theta - h_2 \sin \phi\frac{d\omega_2}{dt} + h_2\omega_2^2 \cos \phi\right). \tag{A3b}$$

The torque about point G_2 can be calculated from the geometry in Fig. 1(c) and is given by

$$C_2 + F_{x2}h_2 \cos \phi + F_{y2}h_2 \sin \phi = I_{\mathrm{cm},2}\frac{d\omega_2}{dt}, \tag{A4}$$

where $I_{\mathrm{cm},2}$ is the moment of inertia of the racquet-hand system about an axis through G_2. The substitution of Eq. (A3) into Eq. (A4) gives

$$I_2\frac{d\omega_2}{dt} = Q - B \cos \beta\frac{d\omega_1}{dt}, \tag{A5}$$

where

$$Q = C_2 + B\omega_1^2 \sin \beta + R[a_x \cos \phi + (g + a_y)\sin \phi], \tag{A6}$$

and $I_2 = I_{\mathrm{cm},2} + M_2h_2^2$ is the moment of inertia of the racquet-hand system about an axis through the wrist, $B = M_2h_2L_1$, $R = M_2h_2$, and $\beta = \phi - \theta$ is the angle between the racquet and a straight line extension of the forearm.

From the geometry of Figs. 1(a) and 1(d), the torque about point G_1 in the forearm is given by

$$C_1 - C_2 + F_{x1}h_1 \cos \theta + F_{y1}h_1 \sin \theta + F_{x2}(L_1 - h_1)\cos \theta + F_{y2}(L_1 - h_1)\sin \theta = I_{\mathrm{cm},1}\frac{d\omega_1}{dt}, \tag{A7}$$

where $I_{\mathrm{cm},1}$ is the moment of inertia of the forearm about an axis through G_1. Substitution of Eqs. (A2) and (A3) into Eq. (A7) gives

$$A\frac{d\omega_1}{dt} = P - B \cos \beta\frac{d\omega_2}{dt}, \tag{A8}$$

where

$$P = C_1 - C_2 - B\omega_2^2 \sin \beta + S[a_x \cos \theta + (g + a_y)\sin \theta], \tag{A9}$$

and $A = I_1 + M_2L_1^2$, $I_1 = I_{\mathrm{cm},1} + M_1h_1^2$ is the moment of inertia of the forearm about an axis through the elbow, and $S = M_1h_1 + M_2L_1$.

Equations (A5) and (A8) can be combined to give

$$\frac{d\omega_1}{dt} = \frac{PI_2 - QB \cos \beta}{AI_2 - B^2 \cos^2 \beta}, \tag{A10a}$$

$$\frac{d\omega_2}{dt} = \frac{QA - PB \cos \beta}{AI_2 - B^2 \cos^2 \beta}. \tag{A10b}$$

Equation (A10) can be solved numerically to find ω_1 and ω_2 as functions of time for any starting positions of the forearm and the racquet and for the given values of the two couples and the acceleration of the elbow. If the wrist is locked at the beginning of the swing, then $\omega_1 = \omega_2$, and Eq. (A10) can be solved to find C_2 in terms of C_1. The equations simplify if the forearm and racquet are locked at right angles because then $\beta = 90°$ so that $PI_2 = QA$.

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[1]T. Jorgensen, *The Physics of Golf*, 2nd ed. (Springer-Verlag, New York, 1999).
[2]N. Betzler, S. Monk, E. Wallace, S. R. Otto, and G. Shan, "From the double pendulum model to full-body simulation: Evolution of golf swing modeling," Sports Technology **1**, 175–188 (2008).
[3]⟨www.tutelman.com/golf/clubs⟩.
[4]R. Cross, "A double pendulum swing experiment: In search of a better bat," Am. J. Phys. **73**, 330–339 (2005).
[5]R. Cross, "Mechanics of swinging a bat," Am. J. Phys. **77**, 36–43 (2009).
[6]R. Adair, *The Physics of Baseball*, 3rd ed. (Harper Collins, New York, 2002).
[7]R. G. Watts and A. T. Bahill, *Keep Your Eye on the Ball: The Science and Folklore of Baseball* (Freeman, New York, 1990).
[8]R. Cross and R. Bower, "Effects of swing-weight on swing speed and racket power," J. Sports Sci. **24**, 23–30 (2006).
[9]R. Cross and A. M. Nathan, "Performance versus moment of inertia of sporting implements," Sports Technology **2**, 7–15 (2009).
[10]R. Cross, "Physics of overarm throwing," Am. J. Phys. **72**, 305–312 (2004).
[11]E. Kreighbaum and K. M. Barthels, *Biomechanics*, 4th ed. (Allyn and Bacon, Boston, 1996).
[12]B. Elliott, G. Fleisig, R. Nicholls, and R. Escamilia, "Technique effects on upper limb loading in the tennis serve," J. Sci. Med. Sport **6**, 76–87 (2003).
[13]R. E. Bahamonde and D. Knudson, "Kinetics of the upper extremity in the open and square stance tennis forehand," J. Sci. Med. Sport **6**, 88–101 (2003).
[14]B. J. Gordon and J. Dapena, "Contributions of joint rotations to racquet speed in the tennis serve," J. Sports Sci. **24**, 31–49 (2006).
[15]See websites such as YouTube and ⟨www.tennisplayer.net⟩.