9850 8923

PHYS378 – GENERAL RELATIVITY AND COSMOLOGY

Welcome to our new astrophysics unit covering some aspects of general relativity and modern ideas on cosmology. This is the first time this unit has been offered so feel free to comment on any part of the unit in order to make it better.

Some important information is as follows:

UNIT DETAILS

Offering	D2
Credit	3cpt
Prerequisites	PHYS202(C), MATH235(C) If you have not satisfied these pre-requisites please see Dr Alan Vaughan.

LECTURERS

Dr Alan Vaughan	E7A 206	9850 8904
Dr Jim Cresser	E7A 208	9850 8906
Dr Mike Wheatland	E7A 306	9850 8923

TEXT BOOK

There will be reference to various texts

LECTURE CONTENT (Up to 39 lectures)

Review of Special Relativity
Gravity and the equivalence principle
Tensors
Spacetime – metrics and curvature
Schwarzschild metric and black holes
Experimental tests of general relativity
Gravitational radiation

Cosmological ideas
Hubble expansion – the FRW metric
Cosmological models
Observational cosmology
Nucleo-synthesis of the light elements
GUT and Inflation
Structure in the Universe
Cosmogony

ASSIGNMENTS

There will be 4 assignments covering the material in the lectures.

ESSAY

You will be asked to write a 1500 word essay relevant to the unit material. Topic choices will be advised over the next few weeks.

ASSESSMENT

Final Exam	70%
Assignments	20%
Essay	10%

TOPICS COVERED IN GR LECTURES, PHYS378 2000

THE EQUIVALENCE PRINCIPLE

- gravitational redshift, deflection of light

CURVATURE OF 2-D SURFACES

- geodesics, parallel transport, geodesic deviation, Gaussian curvature

CURVES IN 3 SPACE

- Frenet formulas

SPECIAL RELATIVITY & SPACE-TIME CURVATURE

- Minkowski metric, space-time diagrams, time-like & space-like, impossibility of signal propagation at $>c$, metrics for curved space-time

TENSOR ANALYSIS

- contravariant & covariant, algebra of tensors, Maxwell equations in the tensor formalism, the covariant derivative & metric connections, Fundamental Theorem of Riemannian geometry, parallel transport, ~~geodesic equation as a description of free fall, generalized covariance~~

THEORY OF GR I

- the geodesic equation as a description of free-fall, generalized covariance, the covariant formulation of Newton's 2nd law, the covariant formulation of Maxwell's equations, geodesics as paths of extremal path length (calculus of variations), metric connections & geodesics on a sphere, classical free-fall

THEORY II

- Einstein's equations, Riemann curvature tensor, (derivation in terms of parallel transport), Riemann curvature tensor on a sphere, Ricci tensor & scalar, geodesic deviation, stress-energy tensor for dust, conservation of mass/energy & momentum, identification of Einstein tensor, Bianchi identities & the divergence of $G_{\mu\nu}$, cosmological constant, alternative forms for the Einstein equations, the Newtonian limit, the Schwarzschild metric

TESTS OF GR IN THE SOLAR SYSTEM

- Advance of the perihelion of Mercury, deflection of light by the Sun, radar echo delays from Venus & Mars

BLACK HOLES

- The Schwarzschild radius & the event horizon, difference between co-ordinate & proper time, Eddington-Finkelstein co-ordinates, rotating black holes (qualitative), formation of black holes, Hawking radiation

GRAVITATIONAL RADIATION

- Linearised field equations & the wave equation, plane wave solutions & the + & × modes, possibility of direct detection.

OUTLINE OF COURSE

~ 24 lectures : 4 on S.R., given by Jim Cresser
Might as well follow Kenyon – at least broadly

CHAPTER	1	INTRO	– includes review of SR
	2	EQUIVALENCE PRINCIPLE	– do
	3	SPACE CURVATURE	} order?
	4	SPACE-TIME CURVATURE	
	5	TENSORS	
	6	EINSTEIN I	
	7	EINSTEIN II	
	8	TESTS OF GR	
	9	BLACK HOLES	
	10	GRAVITATIONAL RAD.	
	11	COSMOLOGY	– omit
	12	QUANTUM GRAVITY	– may be interesting

PHYS378 2000
GENERAL RELATIVITY AND COSMOLOGY

I taught five and a half weeks of this course, covering the basics of General Relativity. Jim Cresser stood in for the first week and a half and gave a review of Special Relativity, and the second half of the course (Cosmology) was taught by Alan Vaughan. I was called upon to give the course at short notice (I had a week and a half to prepare for my first lecture), and this had some effect on my approach. I chose to follow the textbook by Kenyon fairly closely, and with hindsight I would not do this again. For example, the development of GR in Kenyon uses a series of analogies with results from the curvature of two-dimensional surfaces, but the topic of the curvature of surfaces is not itself treated well in the book. There are many other problems with the presentation in Kenyon.

The division of the course into the topics of General Relativity and Cosmology is sensible, and seems to work well.

The major difficulty with this course is that the students do not have the necessary level of mathematical ability to study General Relativity. None of the students walked away proficient at tensor algebra, despite my spending an inordinate amount of time trying to teach it to them. In the exam a couple of parts of questions involved tensor manipulation, or just writing down correct tensor equations, and no student got full marks on these questions. I don't know what can be done to solve this problem. The subject can be presented qualitatively or via analogies (e.g. Kenyon) but a certain level of mathematical proficiency is needed to fully appreciate General Relativity.

Because I was initially pressed for time in preparing lectures, I handed out too few assignments too late in the course. The first assignment was also much too hard for the students because I began with a belief that they could manipulate tensors. To their credit, many of the students came and asked a lot of questions about the assignments.

With the exam, I made the questions very easy because I appreciated by that point that there was too much in the course and that the students were not coping with the mathematics of the subject. The results of the exam were reasonable given my expectations. I gave an additional lecture at the end of the semester summarising the GR part of the course, at the request of the students. I basically summarised what would be in the exam.

I obtained a student assessment on the course from the CPD. To date I have only received the marks and not the student comments, but the marks were very favourable.

In summary it is difficult to teach a course on General Relativity when the students cannot master the necessary mathematics.

— Mike Wheatland

GENERAL RELATIVITY

Introduction :

GR is a physical theory that links the gravitational force to the structure of space-time.

GR has enjoyed something of a renaissance since the 1960s. The theory was completed by Einstein in 1915, but from that time until the 60s it was considered to be a curiosity - the province of mathematicians & not physicists. The problem was that it predicted small effects (e.g. the deflection of starlight grazing the sun's limb by 1.75"), & appeared divorced from the rest of physics.

The renaissance has been driven by astrophysics & the birth of cosmology, which can be considered to be the application of GR to the universe as a whole. (Alan Vaughan will discuss cosmology in the final part of this course.) Important discoveries in astrophysics that have sparked renewed interest in GR include

- CMB detection in 1965 — compact stars, few kms, intense grav. fields
- discovery of neutron stars in 1967
- first black hole candidates - early 1970s
- gravitational lensing - 1980s

- indirect detection of gravitational waves in slowing of period of binary pulsar 1913+16 (Nobel prize to Hulse & Taylor, 1993)

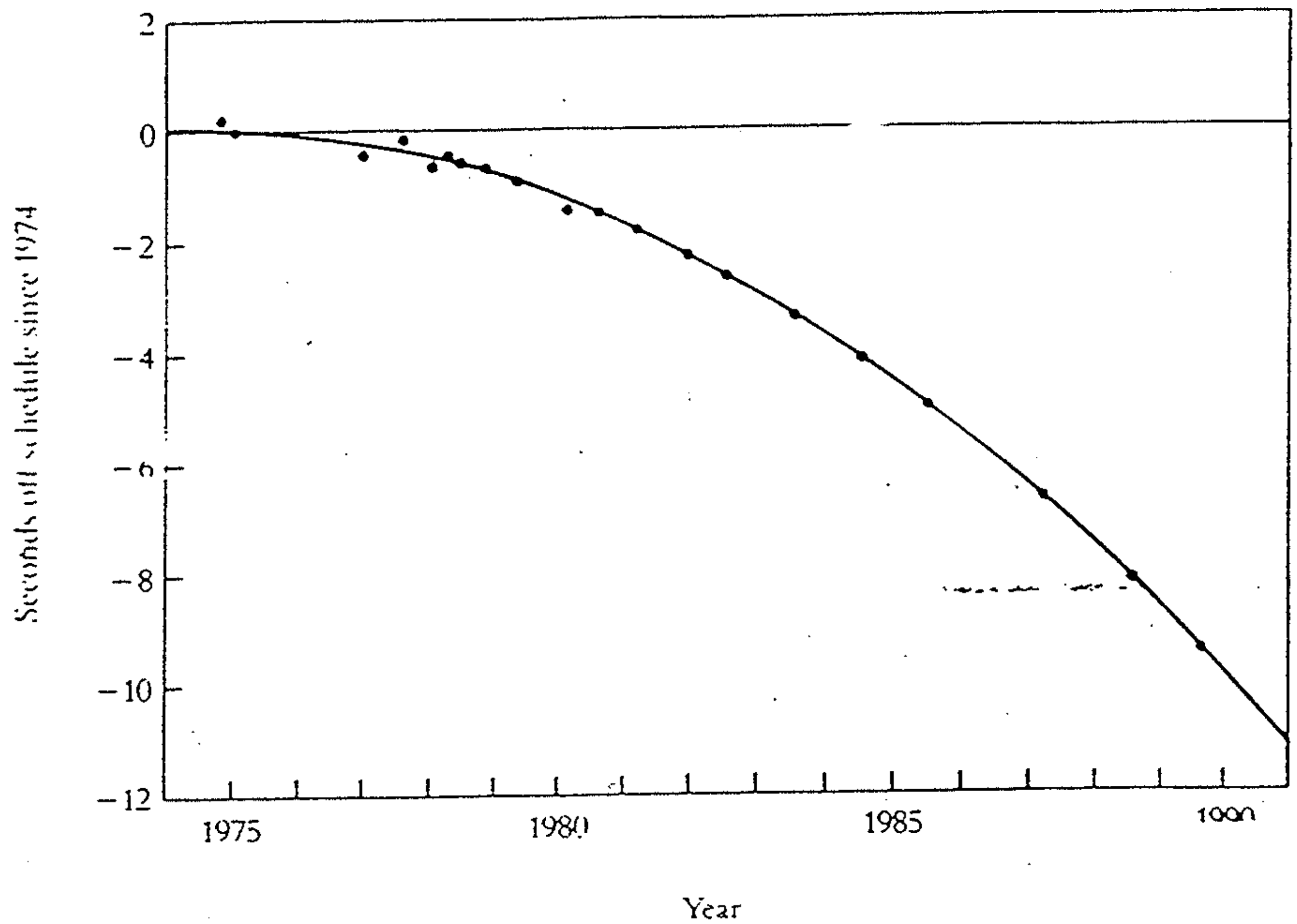
FIGURE

GR is also still an evolving, active discipline. Current topics of research include direct detection of gravitational waves, unification of ~~gra~~ GR & quantum, & even the definition of angular momentum in GR.

GR has a reputation as a difficult subject. The mathematics is unfamiliar, but ~~&~~ it is often made more obscure than it should be. (legacy of its being hijacked by mathematicians?) Here the emphasis is on the physical interpretation of the theory, although there is a certain required amount of mathematical machinery. I will follow the textbook – Kenyon, "General Relativity" – fairly closely, although other recommended books include

D'Inverno – Introducing Einstein's Relativity

Schutz – A first course in GR



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2. GRAVITY & THE EQUIVALENCE PRINCIPLE

- Last week: review of SR

- Newton's law of gravity, $F = \frac{G m_1 m_2}{r^2}$ is inconsistent with SR!

  - no time dependence, so gravitational influence propagates instantly ($> c$)

  - analogy with Coulomb's Law, $F = \frac{k q_1 q_2}{r^2}$. Difficulty there resolved by full time-dependent equations for EM field. Apparently something similar is needed for the gravitational field...
    Einstein developed GR (correct rel. description of gravity) in 10 years following publication of SR (1905–1915)

- Einstein was guided in formulating ~~the~~ GR ~~theory~~ by a series of principles. Most significant is equivalence principle, namely that it is not possible to distinguish between the ~~effects~~ force of gravity & inertial forces assoc. with acceleration.
  e.g. rotating space station in movie 2001 provides gravity

- Equivalence dates back to Galileo, & his experiments to determine if all bodies fall with same acc'n, i.e. g.

  - Analyse: $m_g g = m_i a$

    m_i: inertial mass (measure by applying known forces, or from mom. considerations)

    m_g: gravitational mass (measure from weight)

(2.)

So $a = \frac{m_g}{m_i} g$

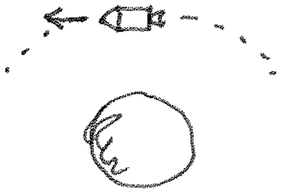
See pp. 12–15 of Kenyon for sensitive experiments that show $\frac{m_g}{m_i}$ const.

& since a is the same for all bodies

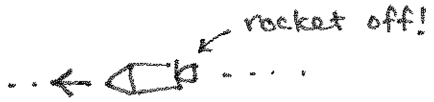
$\Rightarrow \frac{m_g}{m_i}$ is the same, i.e. unity (NEUTRAL)

- EINSTEIN ("weak" EP or WEP): motion of test particle released at a given point in spacetime is independent of its composition

- Next, consider two spacecraft: one is in orbit, other adrift in intergalactic space (where gravity is negligible)



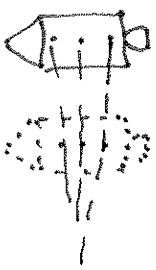
1.



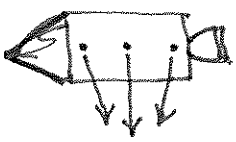
2.

— can an astronaut inside decide between these situations, w/o looking out the window? (In both cases he falls at the same rate as the s/c, i.e. is "weightless")

— If he is careful, he can. The gravitational field of the planet is not uniform, so there is a small component of acc'n towards the centre of the s/c



so particles released at each end move together slowly. (Also radial gradient!)



(so if you're hoovering in the space shuttle...)

3.

just as tides arise from non-uniform field of moon (also sun)

These are tidal effects due to the nonuniformity of the field. Over distances where field variations are small ("locally"), impossible to distinguish situations 1. & 2.

- We have been talking about dynamical expts, but Einstein generalized to include EM as well, arriving at the strong EP (SEP)

  1. Results of all local exp'ts in a frame in free-fall are independent of the motion ~~of~~

  2. ~~Results are the same at all times and all places~~ doesn't add anything

  2. results of all local expts in free fall are consistent with SR

     this because SR works!

- SEP extends first postulate of SR (result of experiment same for all inertial frames); inertial frame is a special case of a free-falling frame, i.e. when $g=0$ (it is case 2 from above). But also MORE RESTRICTIVE: only local exp'ts unchanged. Can always locally "get rid of" gravity by transforming to a frame with $\underset{\sim}{a} = \underset{\sim}{g}$, but cannot cover the entire universe with this frame (cf. SR)

Consequences of SEP:

- Gravitational red-shift

* ARGUMENT IN KENYON IS FLAWED: Here is a different argument. (D'Inverno, "Introducing Einstein's Relativity")

- Any theory consistent with SEP predicts a gravi red shift of light emerging from a gravitational field (so this is not specifically a testable prediction of GR)

- consider FIGURE. chain of buckets of 2-state atoms in Sun's grav. field. Excited atoms at left have higher energy, hence greater mass, according to SR ($E = mc^2$). So the left hand side falls down. If the excited atoms are de-excited at the bottom, resulting photons beamed to the top (all done with mirrors), can re-excite atoms at top, & a perpetual motion machine results. This contradicts conservation of energy, so something is wrong. What? Photons are red-shifted as they climb out of gravitational field, hence move to lower energy.

Analysis: Total energy of photon at radius r:

$$h\nu - GM(h\nu/c^2)/r$$

Require energy to be conserved as photon goes from $r_1 \to r_2$:

$$\nu_2 - \nu_1 = -\frac{GM}{c^2}\left(\frac{\nu_1}{r_1} - \frac{\nu_2}{r_2}\right)$$

For $\nu_1 \approx \nu_2$, $r_1 \approx r_2$

$$\boxed{\frac{\Delta\nu}{\nu} = -\frac{GM}{c^2}\frac{\Delta r}{r^2}}$$

$\Delta\nu = \nu_2 - \nu_1$

$\Delta r = r_2 - r_1$

NOTE $\Delta\nu < 0$ SO REDSHIFT

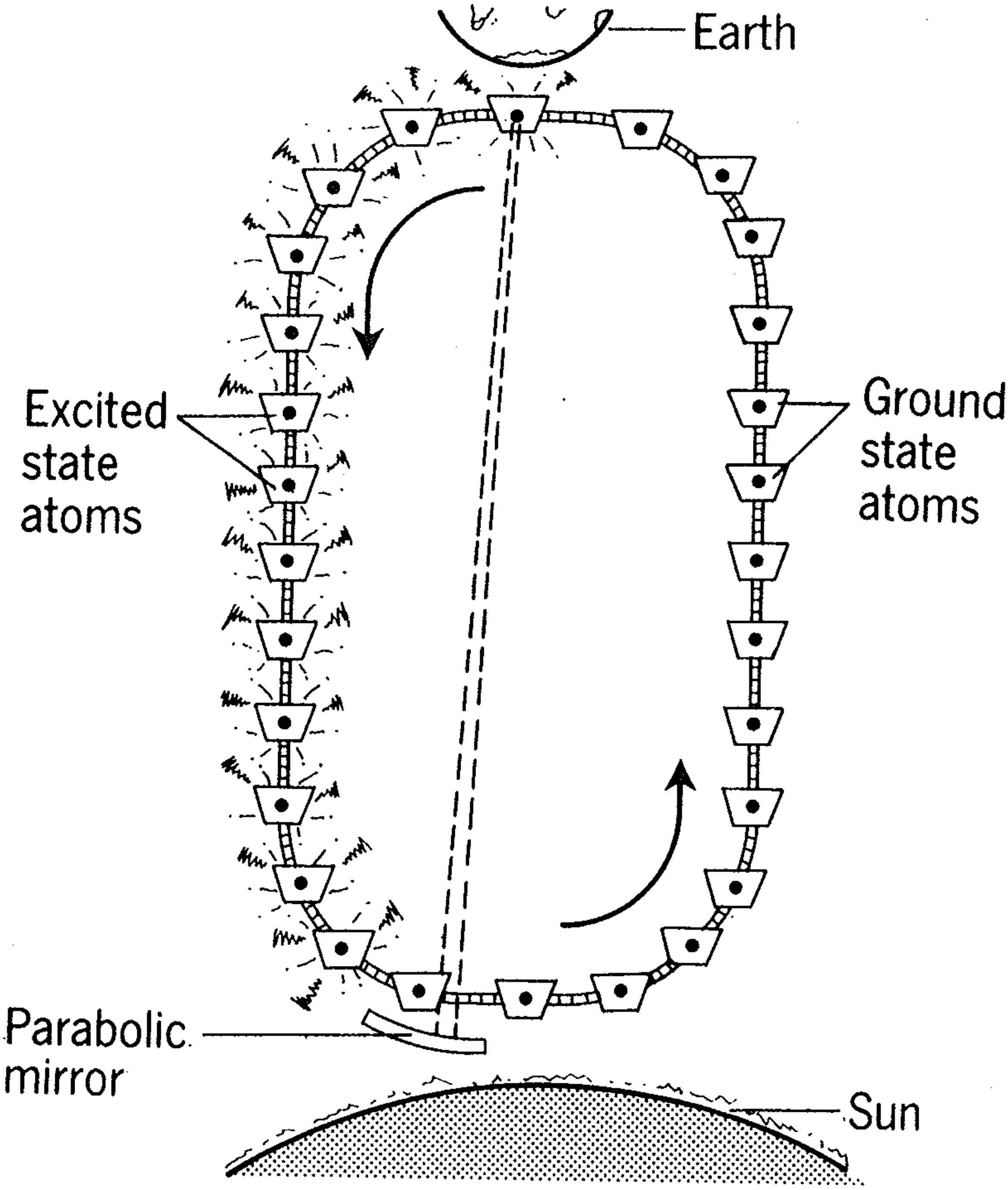
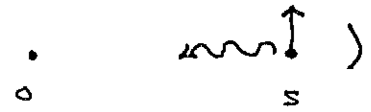


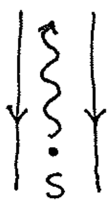
Fig. 15.10 A gravitational *perpetuum mobile*?

- In SR, two contributions to ~~redshift~~ doppler shift
  - relative motion of source, observer
  - time dilation (so there is a transverse doppler shift:



)

- Here, no relative motion, so it suggests that there is time dilation. If we consider a source emitting photons upwards in a gravitational field, then at a certain height r the frequency is ν. The reciprocal of ~~this~~ frequency is a measure of the ~~rate~~ period of the "clock" formed by the source:



$$\frac{1}{t(r)} = \nu$$

Now consider a height $r + \Delta r$:

Kenyon uses $|\Delta\nu|$, so has a minus

$$\frac{1}{t(r+\Delta r)} \approx \nu + \Delta\nu = \nu\left(1 - \frac{GM\Delta r}{r^2c^2}\right)$$

To first order (Binomial theorem):

$$t(r+\Delta r) = t(r)\left(1 + \frac{GM\Delta r}{r^2c^2}\right)$$

i.e. $$\frac{t(r+\Delta r) - t(r)}{\Delta r} = + \frac{GM}{r^2c^2}\, t(r)$$

$$\frac{dt}{dr} = + \frac{GM}{r^2c^2}\, t$$

$$\int_{t(r)}^{t(\infty)} \frac{dt}{t} = + \frac{GM}{c^2}\int_r^\infty \frac{dr}{r^2}$$

$$\left[-\frac{1}{r}\right]_r^\infty = -\left(0 - \frac{1}{r}\right) = \frac{1}{r}$$

$$t(\infty) = t(r)\, e^{\frac{GM}{rc^2}} \approx t(r)\left(1 + \frac{GM}{rc^2}\right)$$

or $$\boxed{t(r) \approx t(\infty)\left(1 - \frac{GM}{rc^2}\right)}$$ Binomial

(Newtonian)

- "Gravitational potential" $\phi = -\frac{GM}{r}$

$$t(\phi) = t(0)\left(1 + \frac{\phi}{c^2}\right)$$

- time interval $t(0)$ measured at a remote point (no grav. field) is CO-ORDINATE TIME
- time interval $t(\phi)$ is LOCAL PROPER TIME (τ often used)

$$\boxed{d\tau^2 = dt^2\left(1 - \frac{2GM}{rc^2}\right)}$$ ← EXACT RESULT IN GR (but deriv. here is approx.)

obtained by squaring both sides, neglecting H.O.T.

- Note that $d\tau < dt$, so remote observer measures time intervals to be dilated

- Precise measurements of gravitational redshift have been made: fractional agreement $\sim 10^{-4}$ for expts involving atomic clocks flown on rockets cf. ones in laboratory

End of L7 '00

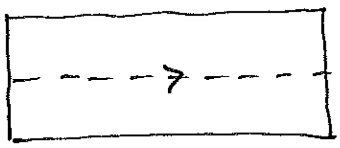
Bending of light in a gravitational field:

- Have already used in an argument above that a photon of energy $E = h\nu$ behaves in a grav. field as if it had an inertial mass $\frac{E}{c^2}$. This suggests the path taken by light might also be bent in a grav. field.

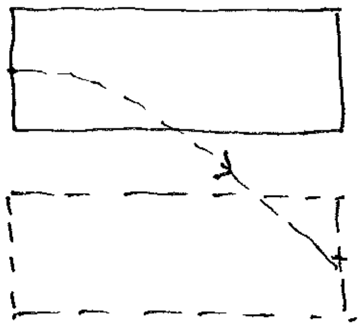
- The following argument (using only SEP) supports this idea

- Consider a space capsule falling radially to earth. An astronaut inside shines a ~~torch~~ laser pointer from one end to the other. His frame is in free fall so result of the exp't is same as usual: light travels in a straight line to other wall (a)
  To an external observer, however, the capsule has fallen a certain distance in the time light takes to reach the far wall, so the beam must follow a curved (parabolic) path (b).

(exaggerated)



a.



b.

- Einstein calculated the deflection of starlight passing close to the limb of the Sun & obtained 1.75". (NB. Argument given above based on SEP gives half this value: we will return to this later). Tested during 1919 solar eclipse: correct. Better tests involve radio sources passing behind Sun; also gravitational lensing.

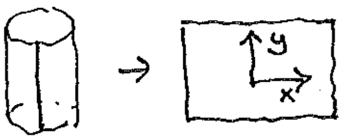
(so this <u>is</u> a test of GR)

$\boxed{\text{FIGURES ?}}$

3. SPACE CURVATURE

GR describes the curvature of space-time due to the presence of matter. Begin to understand curvature by considering 2D surfaces, in particular a sphere & cylinder.

Surfaces of sphere & cylinder both look curved, but there is an important difference: cylinder can be cut (along its length) & laid out flat,



but the sphere cannot. This has been a problem for map makers! The sphere is intrinsically curved, & the cylinder is intrinsically flat.

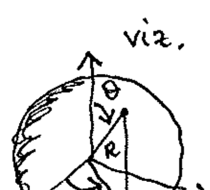
Positions on flat surfaces can be described everywhere by rectangular Cartesian co-ordinates (x,y). The distance between nearby points with co-ordinate separations dx & dy is ds, where

$$ds^2 = dx^2 + dy^2,$$

which is called the metric of the surface.

For a sphere we require generalised co-ordinates to cover the whole surf., viz. (θ, ϕ), & the metric is

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2 \qquad (R \text{ const.})$$



We are interested in Riemann spaces, which are quadratic in their metric. ~~[illegible]~~ These spaces have the nice property that they are LOCALLY flat OR "LOCALLY EUCLIDEAN" (e.g. you can find a tangent plane to a point on a sphere), so locally you

can define Cartesian co-ordinates. We can demonstrate this:

$$ds^2 = g_{11}\,dv^2 + g_{12}\,dv\,dw + g_{22}\,dw^2$$

$$= \left(g_{11}^{\frac{1}{2}}dv + \frac{g_{12}\,dw}{g_{11}^{\frac{1}{2}}}\right)^2 + \left(g_{22} - \frac{g_{12}^2}{g_{11}}\right)dw^2$$

$$= dx^2 \pm dy^2$$ minus case : "pseudo-Euclidean"

where $dx = g_{11}^{\frac{1}{2}}dv + \frac{g_{12}}{g_{11}^{\frac{1}{2}}}\,dw$

& $dy = \left(g_{22} - \frac{g_{12}^2}{g_{11}}\right)^{\frac{1}{2}}$ or $\left(\frac{g_{12}^2}{g_{11}} - g_{22}\right)^{\frac{1}{2}}$

The co-ordinate transformations depend on the g's which vary with position, so the metric takes this form locally.

Note that the space-time of SR (Minkowski spacetime) has the metric:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

which is pseudo-Euclidean

Geodesics are the shortest paths joining points a fixed distance apart on a curved surface. They are also the straightest lines that can be ~~join~~ drawn between the points. An example is great circles on a sphere

Measuring curvature :

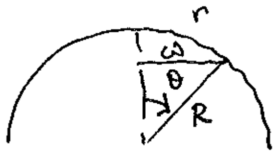
For a 2D surface, fix one end of a string of length r to a point O & make a circuit around O. Measure the length C of that circuit. For a plane, $C = 2\pi r$. If the surface is a dome, $C < 2\pi r$, & for a

Saddle, $C > 2\pi r$.

Define (GAUSSIAN CURVATURE OF 2D SURF.)

$$\boxed{K = \frac{3}{\pi} \lim_{r \to 0} \left(\frac{2\pi r - C}{r^3} \right)}$$

Easy to show that, for the sphere, $K = \frac{1}{R^2}$

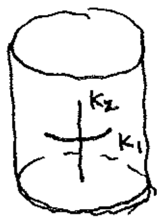


$\omega = R\sin\theta = R\sin\frac{r}{R}$

$C = 2\pi\omega = 2\pi R \sin r/R = 2\pi R\left[\frac{r}{R} - \frac{1}{3!}\left(\frac{r}{R}\right)^3 + \dots\right]$

$\Rightarrow K = 1/R^2$

More generally, the curvature of a surface is different in different directions. ~~At a point~~ (e.g. cylinder). Can define principal curvatures K_1 & K_2 whose product is the Gaussian curvature. K_1 & K_2 are the maximum & minimum curvatures



(For the cylinder, $K_2 = 0$, so $K = 0$, which agrees with our idea that it is not intrinsically curved.) For the sphere, $K_1 = K_2 = \frac{1}{R}$, obviously.

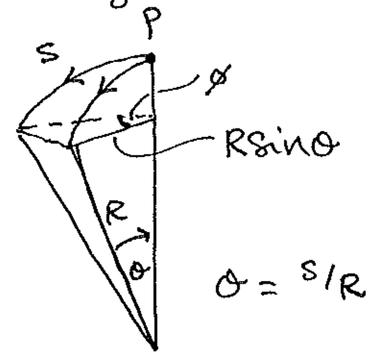
For the saddle, principal directions as shown. Clearly $K < 0$

Other descriptions of curvature:

1. Geodesic deviation

The separation of geodesics with distance gives a measure of curvature.

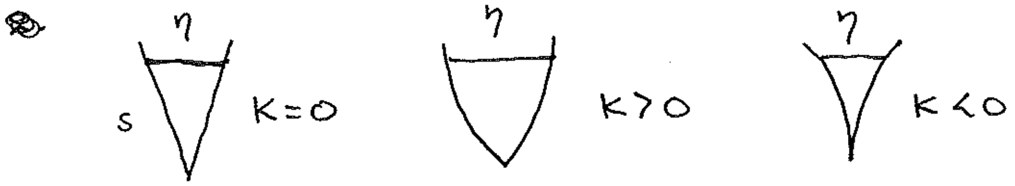
e.g. sphere: $\eta = (R\sin\theta)\phi = R\phi\sin\frac{s}{R}$



diff.: $\frac{d^2\eta}{ds^2} = \frac{-\eta}{R^2}$

end of L2 '00

More generally $\boxed{\dfrac{d^2\eta}{ds^2} = -\eta K}$



$\dfrac{d^2\eta}{ds^2} = 0$ $\qquad \dfrac{d^2\eta}{ds^2} < 0$ $\qquad \dfrac{d^2\eta}{ds^2} > 0$

We will return to geodesics frequently..

2. Parallel transport

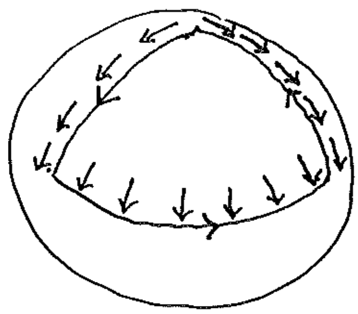
Comparison of local vectors at different places is easy in flat space. E.g. compare cartesian components of $\underset{\sim}{a}$ at A with components of $\underset{\sim}{b}$ at B. However, in a curved space, can't set up a single Cartesian co-ordinate system everywhere. Would like to "carry" $\underset{\sim}{a}$ over to B, keeping length & direction, & compare locally.

PROCEDURE : Move a small distance in the direction of the vector. Repeat...

This traces out a GEODESIC, the straight line equivalent. More generally, can move along a geodesic with the vector at a constant angle to the geodesic. For paths that don't follow a geodesic, split into small steps, each of which coincides with some geodesic...

Call this parallel transport.

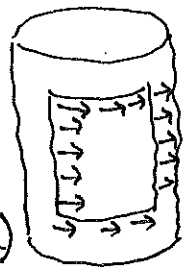
Do this for a sphere:



- Follow a closed path & the vector does not return to the same point with the same orientation.

- Effect of parallel transport in a curved space depends on the path taken

- For a cylinder, it does return to the same!



Turns out rotation, $\phi = \underset{\text{curvature!}}{\underline{\underline{K}}}\,.\,$(area enclosed)

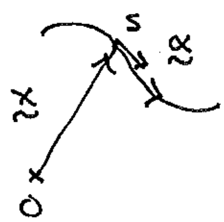
for a surface with constant K. Otherwise $\delta\phi = K\delta A$

For cylinder, $K = 0$.

(OPTIONAL:) <u>CURVES IN 3-SPACE : THE FRENET FORMULAS</u>

Curves in 3-space are conveniently parametrised in terms of their arc length s:

$$\underset{\sim}{x} = \underset{\sim}{\phi}(s)$$



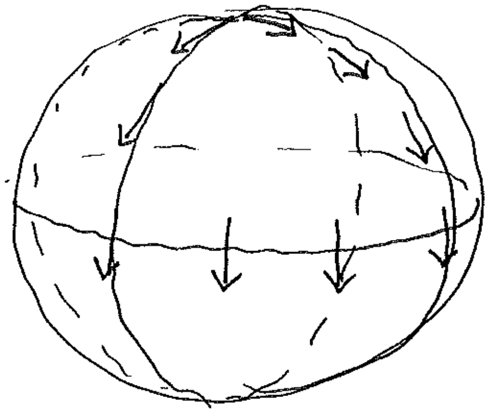
where $ds^2 = d\underset{\sim}{x}\cdot d\underset{\sim}{x}$ – can determine...

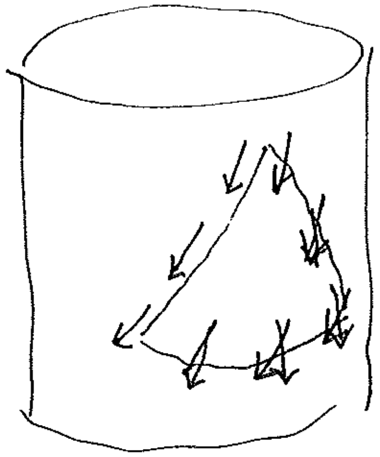
The vector $\underset{\sim}{\alpha}(s) = d\underset{\sim}{\phi}/ds$ is a unit ~~normal~~ tangent vector to the curve.

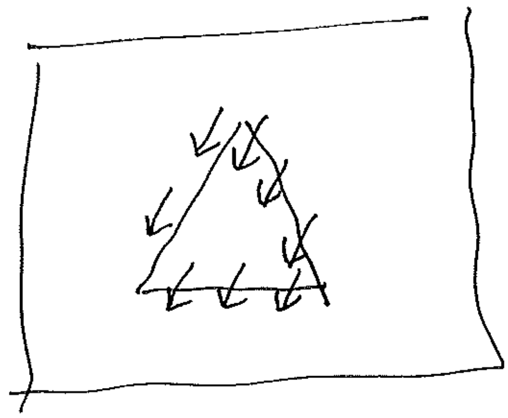
Since $\underset{\sim}{\alpha}\cdot\underset{\sim}{\alpha} = 1$, differentiating: $\underset{\sim}{\alpha}\cdot d\underset{\sim}{\alpha}/ds = 0$, & hence $d\underset{\sim}{\alpha}/ds$ is $\perp$ to the tangent. We can write

$$\frac{d\underset{\sim}{\alpha}}{ds} = K\underset{\sim}{\beta},$$

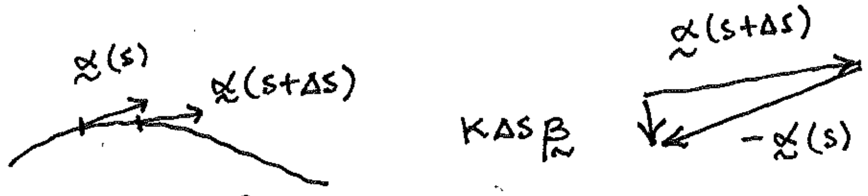
where $\underset{\sim}{\beta}$ is a unit vector chosen so that $K > 0$. The coefficient K is the <u>curvature</u>, Attempts to draw better diagrams:







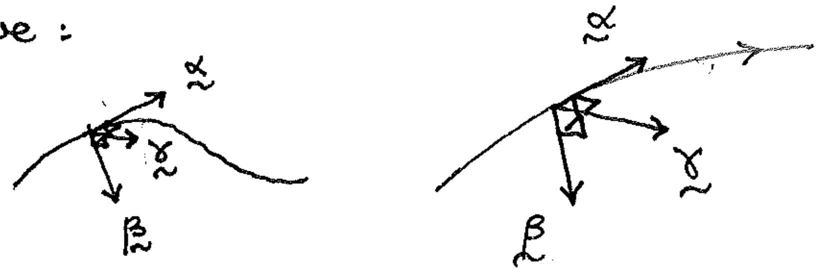
& β is the principal normal to the curve.



Clearly $\kappa = \left| \frac{d^2\phi}{ds^2} \right|$.

"binormal to the curve"

If we define $\gamma = \alpha \times \beta$, then we have a set of basis vectors defined for each point on the curve:



Next consider $\frac{d\beta}{ds}$. Since $\beta \cdot \beta = 1$ we again have that $d\beta/ds$ is $\perp$ to β, so it can be written

$$\frac{d\beta}{ds} = c_1 \alpha + c_2 \gamma$$

in terms of our basis vectors.
From $\alpha \cdot \beta = 0$, differentiating:

$$\alpha \cdot \frac{d\beta}{ds} + \beta \cdot \frac{d\alpha}{ds} = 0$$

$$c_1 + \kappa = 0$$

$$\Rightarrow c_1 = -\kappa.$$

The quantity $(-c_2)$ is written τ, & is called the <u>torsion</u>.
Finally, we consider $d\gamma/ds$.

$$\alpha \cdot \gamma = 0 \Rightarrow \alpha \cdot \frac{d\gamma}{ds} + \gamma \cdot \frac{d\alpha}{ds} = 0 \Rightarrow \alpha \cdot \frac{d\gamma}{ds} = 0$$

$$\underset{\sim}{\beta}\cdot\underset{\sim}{\gamma}=0 \quad\Rightarrow\quad \underset{\sim}{\beta}\cdot\frac{d\underset{\sim}{\gamma}}{ds}+\underset{\sim}{\gamma}\cdot\frac{d\underset{\sim}{\beta}}{ds}=0 \quad\Rightarrow\quad \underset{\sim}{\beta}\cdot\frac{d\underset{\sim}{\gamma}}{ds}=\tau$$

$$\underset{\sim}{\gamma}\cdot\underset{\sim}{\gamma}=1 \quad\Rightarrow\quad \underset{\sim}{\gamma}\cdot\frac{d\underset{\sim}{\gamma}}{ds}=0$$

so $$\frac{d\underset{\sim}{\gamma}}{ds}=\tau\beta$$

& we have derived the <u>Frenet formulas</u>:

$$\frac{d\underset{\sim}{\alpha}}{ds}=\kappa\underset{\sim}{\beta}$$

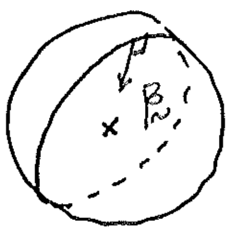
$$\frac{d\underset{\sim}{\beta}}{ds}=-\kappa\underset{\sim}{\alpha}-\tau\underset{\sim}{\gamma}$$

$$\frac{d\underset{\sim}{\gamma}}{ds}=\tau\underset{\sim}{\beta}$$

The first of these gives the formal definition of curvature of a curve.

At this point we note an alternative definition of ~~the~~ a geodesic on a 2-D surface: it is a curve with no component of curvature in the surface, i.e. the ~~binormal~~ principal normal is everywhere $\perp$ to the surface.

e.g. a sphere: clearly the ~~binormal~~ principal normal to a great circle will be in the radial direction, & so is $\perp$ to the surface



4. SPECIAL RELATIVITY REVISITED & SPACE-TIME CURVATURE

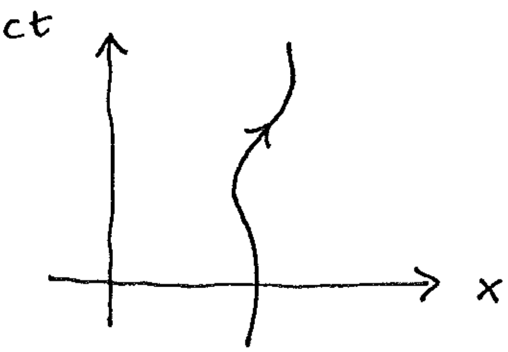
Space-time in SR is flat (one co-ordinate system (ct, x, y, z) can be used everywhere) with the metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

This can also be written

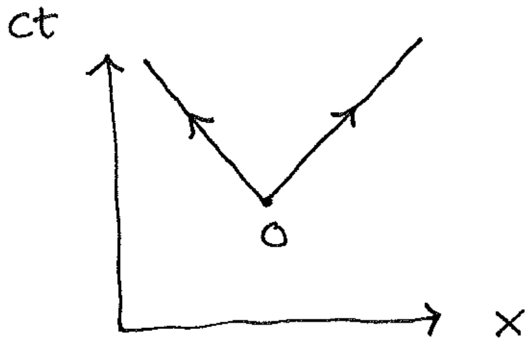
$$ds^2 = c^2 d\tau^2 = c^2 \, dt^2 \left(1 - v^2/c^2\right)$$

where $d\tau$ is the proper time & for a particle following the path $x(t), y(t), z(t)$. The path of the particle can be displayed in a space-time diagram. Suppressing the y & z directions, this is a map of the events (ct, x) corresponding to the position of the particle with time:

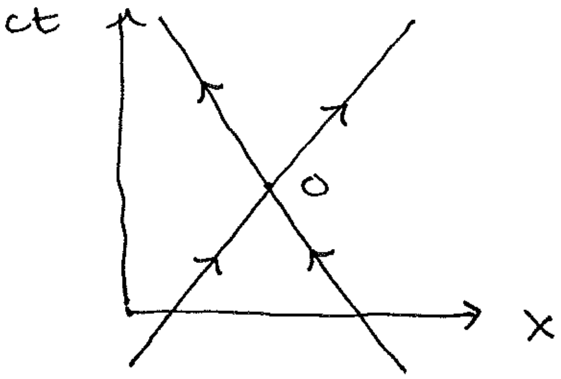


Geometric interpretation, due to Bondi et al.

The path of light in this diagram is described by straight lines with slope ± 1. e.g. if an observer at O sets off a flash of light, the path of the photons is described by the diagram



This is the "forward light cone". There is also a backward light cone, corresponding to light arriving at O.



Next consider the event O & a second event, P, separated from it by ($c\Delta t, \Delta x, \Delta y, \Delta z$). Provided the co-ordinate separation is small, the metric describes the interval between the events:

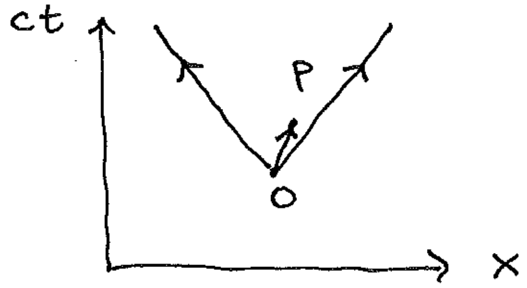
$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$
$$= c^2\Delta t^2 - \Delta r^2$$

& this interval is invariant under Lorentz transformations. The interval Δs^2 can be positive, negative, or zero.

End of L3

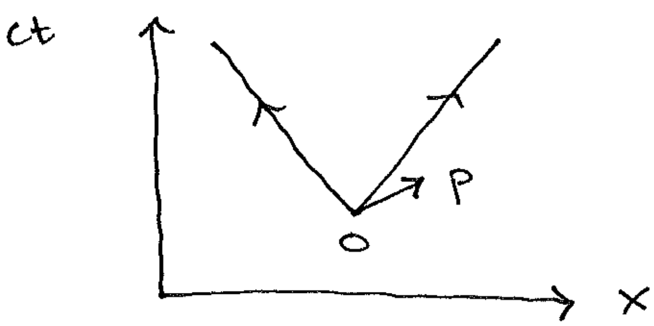
$$\Delta s^2 > 0 \Rightarrow c\Delta t > \Delta r$$

i.e. P can be reached from O by travelling at a speed $< c$. This interval is called <u>timelike</u>, because we can find a Lorentz transformation such that the two events occur at the same location, leaving only a time separation between them. The space-time vector corresponding to a timelike interval lies inside the forward light cone:



$$\Delta s^2 < 0 \quad \Rightarrow \quad c\Delta t < \Delta r$$

i.e. P can only be reached from O by travelling faster than light. This interval is called <u>space-like</u>, & in this case it is possible to Lorentz transform to a frame where the events are simultaneous but are separated by a spatial distance. The space-time vector in this case is outside the light cone:



Because Δs is invariant under L.T., all observers will agree whether an interval is timelike or space-like.

For events separated by a space-like interval, it is possible to Lorentz transform to frames where the events occur in the opposite order, as follows.

Start with the Lorentz transformations corresponding to a primed frame moving with velocity v in the $+x$ direction:

$$t' = \gamma(t - xv/c^2)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

For events close in time the separations of co-ordinates will transform like

$$\Delta t' = \gamma(\Delta t - \Delta x v/c^2)$$
$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

where we assume the separation vector is in the (ct, x) plane.

Next we assume that in the unprimed frame, $\Delta x > 0$ & $\Delta t > 0$. Then the interval

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2$$
$$= (c\Delta t + \Delta x)(c\Delta t - \Delta x),$$

& if $\Delta s^2 < 0$ (i.e. the separation is space-like), then we must have that

$$\Delta x > c\Delta t$$

Then note that the transformed separation in time can be written

$$\Delta t' = \gamma\Delta t\left(1 - \frac{v}{c}\cdot\frac{\Delta x}{\Delta t}\Big/c\right)$$

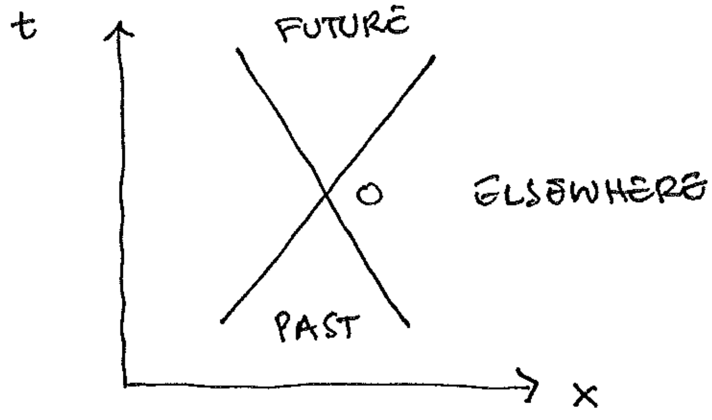
& we will have $\Delta t' < 0$ provided

$$\frac{v}{c} > c\Big/\frac{\Delta x}{\Delta t}$$

The term $c/\frac{\Delta x}{\Delta t}$ is less than unity for a spacelike interval, & hence there is a velocity $v_{<c}$ for which the events occur in the reversed order!

This is the basis for the assertion that influences cannot propagate faster than c. If one event causes another & their space-time separation is spacelike, then there are inertial reference frames in which the ~~event~~ effect precedes ~~the~~ cause. This is paradoxical, & so we conclude that influences / information / signals cannot exceed the speed of light.

Returning to the space-time diagram, for any event O we can define three regions of space: the future, the past, & "elsewhere", depending on whether O can influence the region, ~~can~~ might have been influenced by the region, or cannot influence the region, respectively



Finally, if $\Delta S^2 = 0$ then P is reached from O by moving at c. This interval is sometimes called "NULL", & the vector ~~like~~ lies in the light cone.

Returning to the metric of SR (the Minkowski metric), then we note that it can be rewritten

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

where we relabel our co-ordinates $x^0 = ct$, $x^1 = x$, $x^2 = y$ & $x^3 = z$. More succinctly we can write

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu}\, dx^\mu dx^\nu$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, i.e. $\eta_{\mu\nu}$ is a diagonal matrix with these entries. The summation signs are redundant if we accept the "Einstein summation convention" whereby repeat indices imply summation:

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

<u>Side note</u>: This convention was not introduced until 1916 (by Einstein, as the name suggests). Einstein later made a joke of it to a friend:

"I have made a great discovery in mathematics; I have suppressed the summation sign every time the summation must be made over an index that occurs twice..."

· Now we return to our development of GR. Observations of the deflection of light by gravity imply that space-time is <u>not</u> flat near massive bodies - it is curved. Generalized co-ordinates are then needed to cover space-time (cf. generalized co-ordinates needed with curved 2-D surfaces). If we label these co-ordinates x^μ, then the interval becomes

$$ds^2 = c^2 d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

where $g_{\mu\nu}$ is a set of functions of space & time describing the curvature of space-time. These coefficients define the <u>metric tensor</u>.

The SEP states that we can transform to a freely-falling frame, & then locally SR is valid. This is analogous to the process of finding a local tangent plane to a 2-D surface. Mathematically, for an event y we can find a frame where

$$\left.\begin{array}{l} g_{\mu\nu}(y) = \eta_{\mu\nu} \\ \& \quad \left.\dfrac{\partial g_{\mu\nu}}{\partial x^\rho}\right|_y = 0 . \end{array}\right\} \text{ in FF.}$$

The second condition is consistent with the space being locally flat, like a tangent plane.

Next we introduce tensors, which provide a language for describing physical laws independent of a particular reference frame.

5. TENSOR ANALYSIS

Equations between quantities called tensors are unchanged in co-ordinate transformations between different co-ordinate systems in curved space-time. Hence they are needed to describe physical laws in GR.

Scalars & vectors are tensors. We have seen that in SR, an equality between vectors / scalars that is valid in one inertial frame remains true under Lorentz transformation to another inertial frame. In GR, concerned with general transformations that are no longer linear, e.g. transforming to an accelerated reference frame:

quadratic in t

$$x' = x - \tfrac{1}{2} g t^2$$

We will see later that space-time derivatives, which behave as vectors under Lorentz transformation, need to be redefined to behave as vectors under more general transformations.

General transformations:

Suppose we have two sets of co-ordinates, x^μ & x'^μ, that cover space-time ($\mu = 0,1,2,3$). Then the x'^μ can be expressed in terms of the x^μ,

$$x'^\mu = x'^\mu(x^0, x^1, x^2, x^3),$$

& vice versa. Although these transformations may be complicated, the differentials transform linearly:

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \cdot dx^{\nu} \qquad \text{(MATRIX EQ.)}$$

The quantities dx^{μ} are the prototype for a contravariant tensor of rank 1, or a contravariant vector. Any quantity that transforms under co-ordinate change according to the same rule, i.e.

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \cdot A^{\nu}$$

is similarly a contravariant vector.

In fact dx^{μ} etc. are components of a vector...

Now for a given co-ordinate system x^{μ} we can find basis vectors $\underset{\sim}{e}_{\mu}$ drawn along the local space-time co-ordinate directions. A differential can then be expressed as a vector, $dx^{\mu}\underset{\sim}{e}_{\mu} = dx^{0}\underset{\sim}{e}_{0} + dx^{1}\underset{\sim}{e}_{1} + dx^{2}\underset{\sim}{e}_{2} + dx^{3}\underset{\sim}{e}_{3}$. The length of this vector is an invariant (independent of co-ordinates),

$$ds^2 = (dx^{\mu}\underset{\sim}{e}_{\mu}) \cdot (dx^{\nu}\underset{\sim}{e}_{\nu})$$

separate summations

$$= \underset{\sim}{e}_{\mu} \cdot \underset{\sim}{e}_{\nu}\, dx^{\mu} dx^{\nu}$$

$$= g_{\mu\nu}\, dx^{\mu} dx^{\nu}$$

& we identify $g_{\mu\nu} = \underset{\sim}{e}_{\mu} \cdot \underset{\sim}{e}_{\nu}$.

NB. that $g_{\mu\nu} = g_{\nu\mu}$ "symmetric"

Next consider a scalar function $\phi(x^0, x^1, x^2, x^3)$

$$\phi = \phi\left[x^{\mu}(x'^0, x'^1, x'^2, x'^3)\right]$$

so: $$\frac{\partial \phi}{\partial x'^{\mu}} = \frac{\partial \phi}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}} = \frac{\partial x}{\partial x'^{\mu}} \cdot \frac{\partial \phi}{\partial x^{\nu}}$$

so if we consider the quantities $f_\mu = \frac{\partial \phi}{\partial x^\mu}$ then we have that

$$f'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} f_\nu$$

so the f_μ transform differently to dx^μ. The gradient operator is the prototype for a covariant tensor of rank 1, or a covariant vector, & any set of quantities that transform this way is likewise a covariant vector.

The end of L4 '00

N.B. COVARIANT: indices below

Any vector can be expressed in terms of its contravariant or its contravariant components; they contain the same physical information.

consider the metric again:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

DEFINE $\quad dx_\mu = g_{\mu\nu}\, dx^\nu$

$$\begin{aligned} \text{then } ds^2 &= dx_\mu dx^\mu \\ &= dx'_\alpha dx'^\alpha \\ &= dx'_\alpha \frac{\partial x'^\alpha}{\partial x^\mu} dx^\mu \end{aligned}$$

(invariant under co-ord transf.)

& comparing the first & last lines,

$$dx_\mu = \frac{\partial x'^\alpha}{\partial x^\mu} dx'^\alpha$$

& interchanging primed & unprimed:

$$dx'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu} dx^\alpha$$

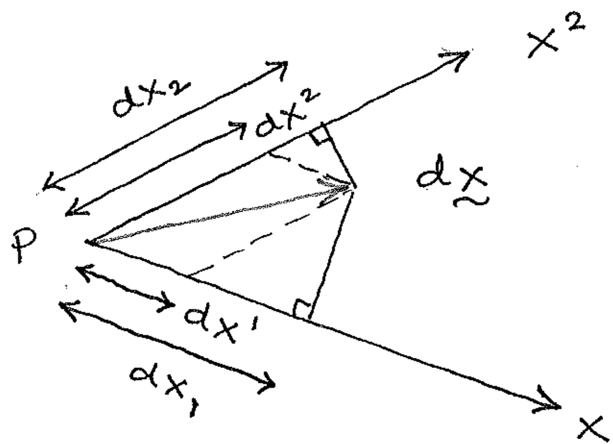
i.e. dx_μ is a covariant vector, or as Kenyon calls it, a covector. The operation

$$dx_\mu = g_{\mu\nu}\, dx^\nu$$

is a way to generate covector components from a contravariant vector (Kenyon calls these simply vectors). In other words, $g_{\mu\nu}$ lowers indices

The physical interpretation of the two sets of components is obtained by considering the inner product of $\underset{\sim}{dx} = dx^\mu \underset{\sim}{e}_\mu$ with $\underset{\sim}{e}_\nu$:

$$\begin{aligned} \underset{\sim}{e}_\nu \cdot \underset{\sim}{dx} &= \underset{\sim}{e}_\mu \cdot \underset{\sim}{e}_\nu\, dx^\mu \\ &= g_{\mu\nu}\, dx^\mu \\ &= dx_\nu \end{aligned}$$



Clearly dx^μ are the projections of $\underset{\sim}{dx}$ drawn parallel to the axes, & dx_μ are the orthogonal projections of $\underset{\sim}{dx}$ along the axes.

In an orthogonal co-ordinate system, there is no distinction between these components. – dealing with cartesian tensors is considerably easier as a result.

We have already been dealing with $g_{\mu\nu}$, which has two indices. This is a COVARIANT tensor of rank 2. How does it transform?

$$ds^2 = g'_{\mu\nu}\, dx'^\mu dx'^\nu$$
$$= g'_{\mu\nu} \underbrace{\left(\frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x'^\nu}{\partial x^\rho} dx^\sigma dx^\rho \right)}_{\text{transformation of } dx^\sigma dx^\rho}$$

but $\quad ds^2 = g_{\sigma\rho}\, dx^\sigma dx^\rho \quad$ (indep. of co-ords)

$$\therefore \quad g'_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x'^\nu}{\partial x^\rho} = g_{\sigma\rho}$$

& interchanging roles of primed, unprimed:

$$g'_{\sigma\rho} = \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial x^\nu}{\partial x'^\rho}\, g_{\mu\nu}$$

& relabelling indices

$$g'_{\mu\nu} = \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu}\, g_{\sigma\rho}$$

So COVARIANT 2nd rank tensors transform in this way. To summarise the transformation rules for 2nd rank tensors:

CONTRAVARIANT: $\quad A'^{\alpha\beta} = \dfrac{\partial x'^\alpha}{\partial x^\mu} \dfrac{\partial x'^\beta}{\partial x^\nu} A^{\mu\nu}$

COVARIANT : $\quad A'_{\alpha\beta} = \dfrac{\partial x^\mu}{\partial x'^\alpha} \dfrac{\partial x^\nu}{\partial x'^\beta} A_{\mu\nu}$

MIXED : $\quad A'^\alpha{}_\beta = \dfrac{\partial x'^\alpha}{\partial x^\mu} \dfrac{\partial x^\nu}{\partial x'^\beta} A^\mu{}_\nu$

note rules on "free" indices; up or down on both sides

"2nd rank, 1st order cov. 1st order cont."

— Generalization to high order tensors obvious.

Next we introduce the "Kronecker delta":

$$\delta^{\alpha}{}_{\beta} = \frac{\partial x^{\alpha}}{\partial x^{\beta}} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

If we define $g^{\mu\nu}$ such that

$$g_{\mu\rho}\, g^{\rho\nu} = \delta^{\nu}{}_{\mu}$$

then

$$\begin{aligned} g^{\mu\nu} dx_{\nu} &= g^{\mu\nu} g_{\nu\rho}\, dx^{\rho} \\ &= \delta^{\mu}{}_{\rho}\, dx^{\rho} \\ &= \delta^{\mu}{}_{0}\, dx^{0} + \delta^{\mu}{}_{1}\, dx^{1} + \delta^{\mu}{}_{2}\, dx^{2} + \delta^{\mu}{}_{3}\, dx^{3} \\ &= dx^{\mu} \end{aligned}$$

So here we have learnt two things:

1. $g^{\mu\nu}$ acts to "raise an index"
2. $\delta^{\alpha}{}_{\beta}$ "replaces" the index

i.e. $\delta^{\mu}{}_{\rho}\, dx^{\rho} = dx^{\mu}$

Is $\delta^{\alpha}{}_{\beta}$ a tensor? To establish this we repeatedly use a trick, $\frac{\partial}{\partial x'^{\beta}} = \frac{\partial x^{\mu}}{\partial x'^{\beta}} \frac{\partial}{\partial x^{\mu}}$

$$\begin{aligned} \delta'^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x'^{\beta}} &= \frac{\partial x^{\mu}}{\partial x'^{\beta}} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \quad \text{(used once)} \\ &= \left(\frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x^{\mu}}{\partial x^{\nu}} \right) \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \quad \text{(twice)} \\ &= \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x^{\mu}}{\partial x^{\nu}} \\ &= \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \delta^{\mu}{}_{\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x'^{\beta}} = \partial x'^{\alpha} / \partial x'^{\beta} \end{aligned}$$

transformation rule for mixed 2nd rank tensor, as expected $\rightarrow$

Finally, we have shown $g_{\mu\nu}$ & $g^{\mu\nu}$ can be used to raise & lower indices, but we have introduced $g^{\mu\nu}$ via the Kronecker delta. Is it possible to obtain one from the other by repeated raising / lowering?

$$g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu} = \left(g^{\alpha\mu} g_{\mu\nu}\right) g^{\beta\nu}$$

$$= \delta^{\alpha}{}_{\nu}\, g^{\beta\nu} = g^{\beta\alpha}$$

~~whence the symmetry of $g^{\mu\nu}$ is used~~

But $g^{\alpha\beta} = g^{\beta\mu} g^{\alpha\nu} g_{\mu\nu\ \nu\mu}$

$= g^{\beta\mu} g^{\alpha\nu} g_{\cancel{\nu\mu}\ \mu\nu}$ ($g_{\mu\nu}$ is symmetric)

$= g^{\beta\nu} g^{\alpha\mu} g_{\mu\nu}$ (interchange dummy indices μ, ν)

$= g^{\beta\alpha}$, from above

$$\therefore \quad g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu} = g^{\alpha\beta}, \quad \text{as expected.}$$

Also note that the definition

$$g_{\alpha\mu}\, g^{\mu\nu} = \delta^{\nu}_{\alpha}$$

implies that $g^{\alpha}{}_{\beta} = g_{\beta\mu}\, g^{\mu\alpha} = \delta^{\alpha}{}_{\beta}$.

Not all collections of numbers $F_{\mu\nu}$ form a tensor. One useful way to establish tensorial character is provided by the quotient theorem, which states that if the product of $F_{\mu\nu}$ with an arbitrary tensor is also a tensor, then $F_{\mu\nu}$ is itself a tensor.

We will establish this for the simple case

end of L5

$$F_{\mu\nu} \underbrace{A^{\nu}}_{\text{arbitrary vector}} = \underbrace{B_{\mu}}_{\text{covector}}$$

In some other co-ordinates:

$$F'_{\mu\nu} A'^{\nu} = B'_{\mu}$$

i.e. $$F'_{\mu\nu} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} A^{\sigma} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} B_{\rho}$$

Multiplying by $\partial x'^{\mu} / \partial x^{\beta}$:

$$\frac{\partial x'^{\mu}}{\partial x^{\beta}} \cdot \frac{\partial x'^{\nu}}{\partial x^{\sigma}} F'_{\mu\nu} A^{\sigma} = \frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial x^{\rho}}{\partial x'^{\mu}} B_{\rho}$$
$$= \delta^{\rho}_{\beta} B_{\rho} = B_{\beta}$$

& subtracting $F_{\beta\sigma} A^{\sigma} = B_{\beta}$ from this:

$$\left(\frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} F'_{\mu\nu} - F_{\beta\sigma} \right) A^{\sigma} = 0$$

Since this holds for arbitrary A^{σ}, we must have

$$F_{\beta\sigma} = \frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} F'_{\mu\nu}$$

& interchanging roles of primed, unprimed co-ords,

$$F'_{\beta\sigma} = \frac{\partial x^{\mu}}{\partial x'^{\beta}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} F_{\mu\nu} \, , \quad \text{as required.}$$

The algebra of tensors :

First, from the transformation rules it is clear that we can construct higher rank tensors by multiplying vectors:

$$A^{\mu\nu} = a^\mu b^\nu$$
$$B^{\mu}_{\ \nu\sigma} = a^\mu b_\nu c_\sigma \quad , \text{ etc.}$$

Next, clearly we can add tensors of the same rank & orders, e.g.

$$X^{\alpha}_{\ \beta\gamma} = Y^{\alpha}_{\ \beta\gamma} + Z^{\alpha}_{\ \beta\gamma}$$

& obtain a tensor. Note that an equation of this kind is only meaningful if the ranks & orders of all terms match. Turn into Q.?

Next, a frequent tensor operation is "contraction", which involves setting two indices equal (a summation is of course implied). This is equivalent to multiplying by the Kronecker delta.
e.g. if we take $X^{\alpha}_{\ \beta\gamma\delta}$ & multiply by $\delta^{\beta}_{\ \alpha}$, we obtain

$$X^{\alpha}_{\ \alpha\gamma\delta} = \delta^{\beta}_{\ \alpha} X^{\alpha}_{\ \beta\gamma\delta}$$

& the quantity $X^{\alpha}_{\ \alpha\gamma\delta}$ is a tensor, as follows:

$$X'^{\alpha}_{\ \alpha\gamma\delta} = \underbrace{\frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\alpha}}}_{\delta^{\nu}_{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\gamma}} \frac{\partial x^{\rho}}{\partial x'^{\delta}} X^{\mu}_{\ \nu\sigma\rho}$$

$$= \delta^{\nu}_{\mu} \frac{\partial x^{\sigma}}{\partial x'^{\gamma}} \frac{\partial x^{\rho}}{\partial x'^{\delta}} X^{\mu}_{\ \nu\sigma\rho}$$

$$= \frac{\partial x^{\sigma}}{\partial x'^{\gamma}} \frac{\partial x^{\rho}}{\partial x'^{\delta}} X^{\mu}_{\ \mu\sigma\rho} \quad ,$$

which is the transf. rule for a 2nd rank covariant tensor.

Now we can understand why tensors are important in mathematical physics. Suppose we have a tensor equation that holds in one set of co-ordinates:

$$X_{\alpha\beta} = Y_{\alpha\beta}.$$

Then multiplying by $\dfrac{\partial x^{\alpha}}{\partial x'^{\mu}} \dfrac{\partial x^{\beta}}{\partial x'^{\nu}}$ (repeat matrix multiplication) leads to

$$X'_{\alpha\beta} = Y'_{\alpha\beta},$$

& hence the same equation holds in any other co-ordinate system. Hence tensor equations are an ideal way to express physical laws, which must be independent of our choice of co-ordinates.

But how do we express laws in tensor form? As an example consider the Maxwell equations (in units with $c=1$)

$$\operatorname{div} \underset{\sim}{E} = \rho \qquad\qquad \operatorname{div} \underset{\sim}{B} = 0$$

$$\operatorname{curl} \underset{\sim}{B} - \frac{\partial \underset{\sim}{E}}{\partial t} = \underset{\sim}{j} \qquad\qquad \operatorname{curl} \underset{\sim}{E} + \frac{\partial \underset{\sim}{B}}{\partial t} = 0$$

"source" equations "internal equations"

Define an antisymmetric tensor $F^{\alpha\beta}$ (I will not show this is a tensor) called the EM field tensor or Maxwell tensor:

Q: why 2nd rank?

$$[F^{\alpha\beta}] = \begin{array}{cc} & \xrightarrow{\beta} \\ \alpha \downarrow & \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \end{array} \quad \text{(antisymmetric)}$$

~~&~~ define the current density or source 4-vector:

$$j^\alpha = (\rho, \underset{\sim}{j})$$

Then (exercise) the Maxwell equations can be written

$$\frac{\partial F^{\alpha\beta}}{\partial x^\beta} = j^\alpha$$

$$\frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} = 0$$

which demonstrates the concise nature of the tensor formalism.

I have not justified the tensorial nature of these quantities or these equations. In fact these equations <u>do not</u> retain their form under arbitrary transformations (although they do under certain linear transformations of co-ordinates, e.g. Lorentz transformations). The reason is that they contain space & time derivatives. ~~&~~ of vectors, which do not transform appropriately.

To see this, consider the quantity $D^\alpha{}_\beta = \partial A^\alpha/\partial x^\beta$, where A^α is a contravariant ~~a~~ tensor. How does this transform?

$$D'^\alpha{}_\beta = \frac{\partial A'^\alpha}{\partial x'^\beta} = \frac{\partial}{\partial x'^\beta}\left(\frac{\partial x'^\alpha}{\partial x^\gamma} A^\gamma\right)$$

$$= \frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial}{\partial x^\delta}\left(\frac{\partial x'^\alpha}{\partial x^\gamma} A^\gamma\right)$$

$$D'^{\alpha}{}_{\beta} = \frac{\partial x^{\delta}}{\partial x'^{\beta}}\frac{\partial x'^{\alpha}}{\partial x^{\gamma}}\frac{\partial A^{\gamma}}{\partial x^{\delta}} + \frac{\partial x^{\delta}}{\partial x'^{\beta}}\frac{\partial^2 x'^{\alpha}}{\partial x^{\delta}\partial x^{\gamma}} A^{\gamma} \qquad (33)$$

$$\text{i.e. } D'^{\alpha}{}_{\beta} = \underbrace{\frac{\partial x'^{\alpha}}{\partial x^{\gamma}}\frac{\partial x^{\delta}}{\partial x'^{\beta}} D^{\gamma}{}_{\delta}}_{\text{correct}} + \underbrace{\frac{\partial x^{\delta}}{\partial x'^{\beta}}\frac{\partial^2 x'^{\alpha}}{\partial x^{\delta}\partial x^{\gamma}} A^{\gamma}}_{\text{extra terms}} = \frac{\partial^2 x'^{\alpha}}{\partial x'^{\beta}\partial x^{\gamma}} A^{\gamma} \qquad (*)$$

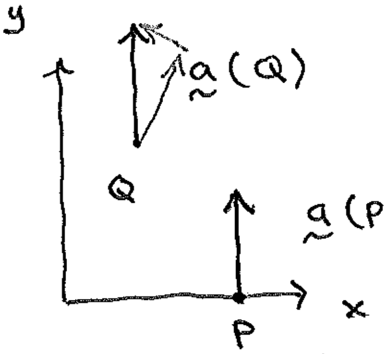
Hence $D^{\alpha}{}_{\beta}$ does not transform correctly, in general.

WHY? ASK Q.

The reason is that the variation of a vector component contains 1. the real variation of the physical object (the vector) & 2. in generalized co-ordinates, the variation of the basis vectors with position.

End of L6

e.g. in polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$



$\underset{\sim}{a}(P)$ has only a θ component

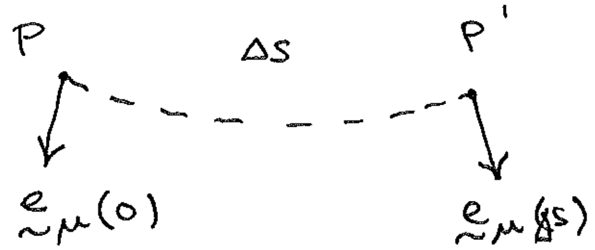
$\underset{\sim}{a}(Q)$ has r & θ components

It turns out that only the variation 1. transforms like a ~~wo~~ tensor: this corresponds to the first term in (*)

A new derivative, called the <u>covariant derivative</u>, is now introduced. This derivative does transform correctly.

COVARIANT DERIVATIVE

Consider how a 4-vector q^μ changes in going from P to P'



Changes in the co-ordinate frame (i.e. the basis vector $\underset{\sim}{e}_\mu$, shown) along the path mean that q^μ changes, even if the underlying physical object (the vector) does not.

Between P & P' we identify

δq^μ : change in q^μ due to variation of co-ordinate direction

Δq^μ : total observed change in q^μ.

The change in the vector due to physical processes is $\Delta q^\mu - \delta q^\mu$. Assuming the path length between P & P' is Δs, we define the covariant derivative:

$$\frac{Dq^\mu}{Ds} = \lim_{\Delta s \to 0} \frac{\Delta q^\mu - \delta q^\mu}{\Delta s}$$

which represents the rate of physical change with path length.

There is a subtlety here, that was anticipated by our discussion of transport of vectors on curved 2-D surfaces. To evaluate δq^μ we need to be able to transport the vector (unchanged) from P to P′ & note how the components vary. How do we do this?

PROCEDURE:

1. Transform to a frame in free-fall at P
2. Locally space-time is flat & SR is valid; so for small Δs can transport $P \to P'$ (Cartesian components same)
3. Transform back to relevant frame at P′

This is generalised "parallel transport".

Suppose q^μ is parallel transported a distance Δx^ρ in the ρ direction. Then if the vector was initially in the σ direction, it will (in general) have final components in all directions, so

$$\delta q^\mu = -\Gamma^\mu{}_{\sigma\rho}\, q^\sigma \Delta x^\rho$$

i.e. there is a linear variation for small Δx^ρ.

The coefficients $\Gamma^\mu{}_{\sigma\rho}$ are called the metric connections, & we expect them to depend on the curvature of space-time, i.e. only on the $g_{\mu\nu}$. We will see soon that this is true. (In other books also see "affine connections," more general coefficients needed when the space is not Riemannian.)

With this form for δq^μ the covariant derivative becomes

$$\frac{Dq^\mu}{Ds} = \frac{dq^\mu}{ds} + \Gamma^\mu{}_{\sigma\rho}\, q^\sigma \frac{dx^\rho}{ds}$$

Multiplying by $\dfrac{\partial s}{\partial x^\nu}$ gives another form:

$$\frac{Dq^\mu}{Dx^\nu} = \frac{\partial q^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\sigma\rho}\, q^\sigma \frac{\partial s}{\partial x^\nu}\frac{dx^\rho}{ds}$$

definition, really

$$= \frac{\partial q^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\sigma\rho}\, q^\sigma \frac{\partial x^\rho}{\partial x^\nu}$$

$$= \frac{\partial q^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\sigma\rho}\, q^\sigma .\, \delta^\rho{}_\nu$$

$$= \frac{\partial q^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\sigma\nu}\, q^\sigma$$

& we can introduce two briefer notations:

$$q^\mu{}_{,\nu} \equiv \frac{\partial q^\mu}{\partial x^\nu} \quad \& \quad q^\mu{}_{;\nu} \equiv \frac{Dq^\mu}{Dx^\nu}$$

& we have $\boxed{q^\mu{}_{;\nu} = q^\mu{}_{,\nu} + \Gamma^\mu{}_{\sigma\nu}\, q^\sigma}$ ⊛

The connections themselves ARE NOT TENSORS, although the covariant derivative is a tensor. The connection part of ⊛ compensates for the non-tensorial behaviour of $q^\mu{}_{,\nu}$ in ⊛, & hence it will also have an extra term under transformation.

The covariant derivative of a scalar is the partial derivative (we have already seen that $\partial\phi/\partial x^\mu$ is a tensor), & the covariant derivative of a covector is

$$q_{\mu;\nu} = q_{\mu,\nu} - \Gamma^\rho{}_{\mu\nu} q_\rho$$

For tensors of second rank (see e.g. Kenyon for justification) each index contributes a term involving the metric connections:

$$A^{\mu\nu}{}_{;\tau} = A^{\mu\nu}{}_{,\tau} + \Gamma^\mu{}_{\rho\tau} A^{\rho\nu} + \Gamma^\nu{}_{\rho\tau} A^{\mu\rho}$$

$$A_{\mu\nu;\tau} = A_{\mu\nu,\tau} - \Gamma^\rho{}_{\mu\tau} A_{\rho\nu} - \Gamma^\rho{}_{\nu\tau} A_{\mu\rho}$$

& the generalisation to mixed tensors & higher rank tensors is obvious.

<u>Calculating $\Gamma^\mu{}_{\sigma\rho}$:</u>

Consider the covariant derivative of the metric tensor,

$$\frac{Dg_{\mu\nu}}{Dx^\sigma} = \frac{\partial g_{\mu\nu}}{\partial x^\sigma} - \Gamma^\tau{}_{\mu\sigma} g_{\tau\nu} - \Gamma^\tau{}_{\nu\sigma} g_{\mu\tau}$$

In free-fall we have locally flat space-time, & the metric connections vanish:

$$\frac{Dg_{\mu\nu}}{Dx^\sigma} = \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \quad \text{(F.F.)}$$

Also, SR is recovered, so as discussed previously

$$g_{\mu\nu} = \eta_{\mu\nu} \quad \& \quad \frac{\partial g_{\mu\nu}}{\partial x^\sigma} = 0$$

Hence for the FF frame $Dg_{\mu\nu}/Dx^\sigma = 0$.

Hence for the frame in free fall $\dfrac{D g_{\mu\nu}}{D x^{\rho}} = 0$.

But this is a tensor equation, so it must hold in all frames. Hence in general

$$\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} = \Gamma^{\tau}{}_{\mu\sigma}\, g_{\tau\nu} + \Gamma^{\tau}{}_{\nu\sigma}\, g_{\mu\tau}$$

$$= \Gamma_{\nu\mu\sigma} + \Gamma_{\mu\nu\sigma}$$

recalling that the $g_{\mu\nu}$ lower indices. In our 'subscript equals partial derivative' notation we have

$$g_{\mu\nu,\sigma} = \Gamma_{\nu\mu\sigma} + \Gamma_{\mu\nu\sigma} \qquad ①$$

In GR all connections can be assumed to be symmetric, i.e.

$$\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta} \Rightarrow \Gamma_{\alpha\beta\gamma} = \Gamma_{\alpha\gamma\beta}$$

Permuting indices on ① we obtain

$$g_{\sigma\mu,\nu} = \Gamma_{\mu\sigma\nu} + \Gamma_{\sigma\mu\nu} \qquad ②$$

& $$g_{\nu\sigma,\mu} = \Gamma_{\sigma\nu\mu} + \Gamma_{\nu\sigma\mu} \qquad ③$$

& using the symmetry property in ② & ③:

$$g_{\sigma\mu,\nu} = \Gamma_{\mu\nu\sigma} + \Gamma_{\sigma\mu\nu} \qquad ④$$

$$g_{\nu\sigma,\mu} = \Gamma_{\sigma\mu\nu} + \Gamma_{\nu\mu\sigma} \qquad ⑤$$

① − ④ + ⑤: $g_{\mu\nu,\sigma} - g_{\sigma\mu,\nu} + g_{\nu\sigma,\mu}$

$$= \Gamma_{\nu\mu\sigma} + \cancel{\Gamma_{\mu\nu\sigma}} - \cancel{\Gamma_{\mu\nu\sigma}} - \cancel{\Gamma_{\sigma\mu\nu}}$$

$+$ [illegible]

end of L7

i.e. $$\boxed{\Gamma_{\nu\mu\sigma} = \tfrac{1}{2}\left(g_{\mu\nu,\sigma} - g_{\sigma\mu,\nu} + g_{\nu\sigma,\mu}\right)}$$

which is the Fundamental Theorem of Riemannian geometry. (our derivation was concerned with the application to GR, but this result is true for all Riemannian spaces. The argument appealing to the SEP is replaced by the argument that Riemannian spaces are locally flat.)

Now we have a complete prescription for the covariant derivative, which describes the physical change in a vector along a path in space-time. If the vector is unchanged it is said to be parallel transported

i.e. $$\frac{Dq^\mu}{Ds} = 0 \Rightarrow \frac{dq^\mu}{ds} = -\Gamma^\mu_{\nu\rho}\, q^\rho \frac{dx^\nu}{ds}$$

i.e. $$\underbrace{\Delta q^\mu}_{\text{observed change}} = -\Gamma^\mu{}_{\nu\rho}\, q^\rho\, \Delta x^\nu$$

i.e. all observed change is due to the change in co-ordinate directions

So far we have used the fact that the covariant derivative of a tensor is a tensor, but we have not proved this. I leave this as an exercise to the student. The procedure involves using the expression for $\Gamma_{\nu\mu\sigma}$ to determine how the metric coefficients transform, & hence how the covariant derivative transforms.

In the previous we have used the path length s in our expressions for the covariant derivative. For a time-like path this can be replaced by $c\tau$, where τ is the proper-time, i.e. the time measured by a clock following the path. When the path is that of light, an alternative parameter can be found to specify path length.

Finally, the following properties of the covariant derivative are useful (the proofs are good exercises in tensor algebra):

1. The covariant derivative obeys the usual product rule for differentiation:

$$(A^{\mu}B^{\nu})_{;\alpha} = A^{\mu}{}_{;\alpha}B^{\nu} + A^{\mu}B^{\nu}{}_{;\alpha}$$

2. The metric tensor & its inverse have zero covariant derivative (the first result was proved in the derivation of the expression for $\Gamma_{\nu\mu\sigma}$):

$$g_{\mu\nu;\alpha} = g^{\mu\nu}{}_{;\alpha} = 0$$

3. The operation of raising/lowering indices commutes with the covariant derivative,

e.g. $$(g^{\mu\nu}A_{\nu})_{;\alpha} = g^{\mu\nu}A_{\nu;\alpha}$$

Principle of
<u>Generalized covariance :</u> (Einstein)

Two parts:

1. Physical laws must be expressible as tensor equations, so that they remain valid under transformation to accelerated frames

2. In the special case of a transformation to a frame in free-fall, the physical laws should reduce to SR.

~~RETURN TO MAXWELL: $F^{\alpha\beta}{}_{;\beta} = j^{\alpha}$ $F_{\beta\gamma;\alpha} + F_{\alpha\beta;\gamma} + F_{\gamma\alpha;\beta} = 0$~~

<u>Covariant formulation of Newton's second law :</u>

The relativistically correct (i.e. invariant under Lorentz transformation) version of Newton's second law is

$$\underset{\sim}{F} = \frac{d\underset{\sim}{p}}{dt} = \frac{d}{dt}(m\underset{\sim}{v})$$

where $m = \gamma m_0$, $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$

This can be put in 4-vector form by defining the 4 vectors $V^{\mu} = \gamma(c, \underset{\sim}{v})$ & $p^{\mu} = (E/c, \underset{\sim}{p})$, where $E = mc^2$ is the relativistic energy & $\underset{\sim}{p} = m\underset{\sim}{v}$ is the relativistic momentum, & by introducing the 4-vector force F^{μ} such that

$$F^{\mu} = \frac{dp^{\mu}}{d\tau}$$

where $d\tau = dt/\gamma$ is the proper time.

Clearly $F^{\mu} = \gamma(\frac{1}{c}dE/dt, \underset{\sim}{F})$

The RHS is not a tensor because of the time derivative. However, it can be made a tensor by replacing the derivative with the covariant derivative:

$$F^\mu = \frac{Dp^\mu}{D\tau}$$

This equation

- is a valid tensor equation
- reduces to the SR form in a frame in free-fall, where the metric connections vanish

$\therefore$ it satisfies the principle of generalized covariance

Have here a general procedure for producing equations valid in accelerating frames: replace derivatives with covariant derivatives.

MAXWELL : $F_{\alpha\beta;\beta} = j^\alpha$, $F_{\beta\gamma;\alpha} + F_{\alpha\beta;\gamma} + F_{\gamma\alpha;\beta} = 0$

Note that the covariant form of Newton's second law can be written

$$\frac{dp^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\tau}p^\rho = F^\mu$$

Geodesics

Now consider the simplest possible motion, i.e. a body in free-fall, that is acted upon by no other forces. Then clearly $F^\mu = 0$

i.e. $$\frac{dp^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\tau}p^\rho = 0$$

or in brief, $Dp^\mu/D\tau = 0$. This means p^μ is being parallel transported. Further, p^μ is in the direction of the tangent to the path of motion, so p^μ is parallel transported along itself. In chapter 3 we saw that this is the prescription for tracing out a <u>geodesic</u>, the straightest possible path. To summarise: a body in free-fall traces out a geodesic in space-time.

Next note that, from the definition of the 4-vector momentum:

$$
\begin{aligned}
p^\mu = (E/c, \underset{\sim}{p}) &= \gamma\,(m_0 c,\ m_0 \underset{\sim}{v}) \\
&= \gamma m_0 \left(\frac{d}{dt} ct,\ \frac{d}{dt} \underset{\sim}{x} \right) \\
&= \gamma m_0 \frac{dx^\mu}{dt} \\
&= m_0 \frac{dx^\mu}{d\tau}
\end{aligned}
$$

Hence the equation of motion becomes

$$\frac{d}{d\tau} \cancel{m_0} \frac{dx^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\tau} \cancel{m_0} \frac{dx^\rho}{d\tau} = 0$$

i.e. $$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

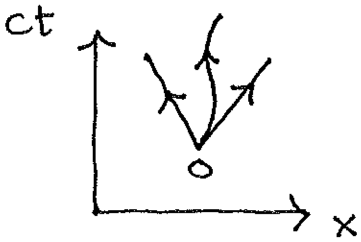
end of L8

which is the <u>geodesic equation</u>. Note that m_0 cancelled out: this is the equivalence principle!

For geodesics followed by light, we introduce a different path parameter, λ:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0.$$

A body in free-fall from a fixed starting event follows different geodesics if it is given different starting velocities. These geodesics lie inside the forward light cone through the starting event; they are time-like with $\int ds^2 > 0$.



When the test body is a photon the path integral is $\int ds^2 = 0$, which defines a null geodesic.

There are also space-like geodesics ($\int ds^2 < 0$), but these correspond to motion with velocity $> c$, & material objects (and light) cannot follow these paths.

In our discussion of the curvature of 2-D surfaces we gave the alternative def'n of a geodesic as a curve with ~~the~~ no component of curvature in the space. This property of geodesics is inherent in the geodesic equation:

$$\frac{Dp^\mu}{D\tau} = 0$$

or

$$\frac{Dp^\mu}{Ds} = 0$$

But $p^\mu = m_0 c \, dx^\mu / ds$, so

$$\frac{D^2 x^\mu}{Ds^2} = 0 .$$

The $D^2 x^\mu / Ds^2$ can be interpreted as the curvature components of the path. So, a geodesic has no curvature in space-time

Geodesics as paths of minimum s : ADDITIONAL?

Require background of the calculus of variations (due to Newton). Consider the problem of maximising / minimising the integral

$$I = \int_A^B L(x^\mu, q^\mu)\, d\tau,$$

where $q^\mu = dx^\mu / d\tau$, & where L can assume different forms. What functional form for L makes I stationary?

We contruct a variation in I :

$$\delta I = \int_A^B \left(\frac{\partial L}{\partial x^\mu} \delta x^\mu + \frac{\partial L}{\partial q^\mu} \delta q^\mu \right) d\tau$$

$$= \int_A^B \left(\frac{\partial L}{\partial x^\mu} \delta x^\mu + \frac{\partial L}{\partial q^\mu} \frac{d}{d\tau} (\delta x^\mu) \right) d\tau$$

$$= \int_A^B \frac{\partial L}{\partial x^\mu} \delta x^\mu \, d\tau + \left[\delta x^\mu \frac{\partial L}{\partial q^\mu} \right]_A^B - \int_A^B \delta x^\mu \frac{d}{d\tau} \frac{\partial L}{\partial q^\mu} d\tau$$

(integrating by parts.)

~~The function L is assumed to be fixed at the endpoints A & B~~

We assume that $\delta x^\mu = 0$ at A & B, & hence the bracketed terms vanish:

$$\delta I = \int_A^B \left(\frac{\partial L}{\partial x^\mu} - \frac{d}{dt} \frac{\partial L}{\partial q^\mu} \right) \delta x^\mu \, d\tau$$

We require $\delta I = 0$. Since this must be true for all δx^μ in the integrand, it follows that

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial L}{\partial q^\mu} = 0,$$

which are the Euler Lagrange equations.

Now we consider the specific problem at hand. We want to extremise

$$s = \int_A^B ds = \int_A^B \left(g_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \right)^{\frac{1}{2}} d\tau$$

$$= \int_A^B \left(g_{\alpha\beta}\, q^\alpha q^\beta \right)^{\frac{1}{2}} d\tau$$

However, the square root signs are messy to deal with. Instead we consider

$$cs = c\int_A^B ds = c^2\int_A^B d\tau$$

$$= \int_A^B \left(\frac{ds}{d\tau}\right)^2 d\tau$$

$$= \int_A^B g_{\alpha\beta}\, q^\alpha q^\beta \, d\tau$$

We have $L = g_{\alpha\beta}\, q^\alpha q^\beta$

so $$\frac{\partial L}{\partial x^\mu} = g_{\alpha\beta,\mu}\, q^\alpha q^\beta$$ (x^α & q^α are independent)

$$\frac{\partial L}{\partial q^\mu} = g_{\alpha\mu}\, q^\alpha + g_{\mu\beta}\, q^\beta$$

$$\frac{d}{d\tau}\left(\frac{dL}{dq^\mu}\right) = \frac{\partial g_{\alpha\mu}}{\partial x^\sigma}\frac{dx^\sigma}{d\tau} q^\alpha + \frac{\partial g_{\mu\beta}}{\partial x^\sigma}\frac{dx^\sigma}{d\tau} q^\beta + g_{\alpha\mu}\frac{dq^\alpha}{d\tau} + g_{\mu\beta}\frac{dq^\beta}{d\tau}$$

or in shorthand,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial q^{\mu}}\right) = g_{\alpha\mu,\sigma}\, q^{\sigma} q^{\alpha} + g_{\mu\beta,\sigma}\, q^{\sigma} q^{\beta} + 2 g_{\mu\alpha}\, dq^{\alpha}/d\tau$$

so the Lagrange equation $\Rightarrow$

$$g_{\alpha\beta,\mu}\, q^{\alpha} q^{\beta} - g_{\alpha\mu,\sigma}\, q^{\sigma} q^{\alpha} - g_{\mu\beta,\sigma}\, q^{\sigma} q^{\beta} - 2 g_{\mu\alpha} \frac{dq^{\alpha}}{d\tau} = 0$$

or $$g_{\alpha\beta,\mu}\, q^{\alpha} q^{\beta} - g_{\alpha\mu,\beta}\, q^{\alpha} q^{\beta} - g_{\mu\beta,\alpha}\, q^{\alpha} q^{\beta} - 2 g_{\mu\alpha} \frac{dq^{\alpha}}{d\tau} = 0$$ relabelling indices

i.e. $\cancel{(g_{\alpha\beta,\mu} - g}$

$$-\left(g_{\alpha\mu,\beta} - g_{\alpha\beta,\mu} + g_{\mu\beta,\alpha}\right) q^{\alpha} q^{\beta} - 2 g_{\mu\alpha} \frac{dq^{\alpha}}{d\tau} = 0$$

Recall metric connection equation:

$$\Gamma_{\nu\mu\rho} = \tfrac{1}{2}\left(g_{\mu\nu,\rho} - g_{\rho\mu,\nu} + g_{\nu\rho,\mu}\right)$$

$\mu \to \alpha$
$\nu \to \mu$
$\rho \to \beta$

$$\Rightarrow \quad -2\Gamma_{\mu\alpha\beta}\, q^{\alpha} q^{\beta} - 2 g_{\mu\alpha} \frac{dq^{\alpha}}{d\tau} = 0$$

i.e. $$-2\Gamma_{\mu\beta\alpha}\, q^{\alpha} q^{\beta} - 2 g_{\mu\alpha} \frac{dq^{\alpha}}{d\tau} = 0$$

end of L9

~~multiply by~~

i.e. $$g_{\mu\nu}\, \Gamma^{\cancel{\mu}\,\nu}{}_{\beta\alpha}\, q^{\alpha} q^{\beta} + g_{\mu\nu} \frac{dq^{\nu}}{d\tau} = 0$$

multiply by $g^{\gamma\mu}$:

$$g^{\gamma\mu} g_{\mu\nu} \left(\Gamma^{\nu}{}_{\beta\alpha} q^{\alpha} q^{\beta} + \frac{dq^{\nu}}{d\tau} \right) = 0$$

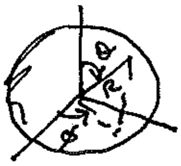
i.e. $$\delta^{\gamma}{}_{\nu} \left(\Gamma^{\nu}{}_{\beta\alpha} q^{\alpha} q^{\beta} + \frac{dq^{\nu}}{d\tau} \right) = 0$$

i.e. $$\Gamma^{\gamma}{}_{\beta\alpha} q^{\alpha} q^{\beta} + \frac{dq^{\gamma}}{d\tau} = 0$$

or $$\frac{d^2 x^{\gamma}}{d\tau} + \Gamma^{\gamma}{}_{\beta\alpha} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} = 0,$$

which is the geodesic equation!

Example : Spherical (2D) surfaces



$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2$$

$$[g_{\mu\nu}] = \mathrm{diag}(R^2, R^2 \sin^2\theta)$$

The contravariant metric tensor is clearly $\mathrm{diag}(R^{-2}, R^{-2}\sin^{-2}\theta)$

i.e.
$$\begin{aligned} g_{\theta\theta} &= R^2 & g^{\theta\theta} &= R^{-2} \\ g_{\phi\phi} &= R^2 \sin^2\theta & g^{\phi\phi} &= R^{-2}\sin^{-2}\theta \\ g_{\theta\phi} = g_{\phi\theta} &= 0 & g^{\theta\phi} = g^{\phi\theta} &= 0 \end{aligned}$$

where we drop the summation convention. Then the metric connections are given by

$$\Gamma^{\mu}{}_{\nu\sigma} = g^{\mu\rho} \Gamma_{\rho\nu\sigma}$$

$$= \frac{1}{2} g^{\mu\rho} \left(g_{\nu\rho,\sigma} - g_{\sigma\nu,\rho} + g_{\rho\sigma,\nu} \right)$$

Writing these out:

$$\Gamma^{\theta}{}_{\theta\theta} = 0$$

$$\Gamma^{\theta}{}_{\theta\phi} = \Gamma^{\theta}{}_{\phi\theta} = 0$$

$$\Gamma^{\theta}{}_{\phi\phi} = -\cos\theta\sin\theta$$

$$\Gamma^{\phi}{}_{\theta\theta} = 0$$

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta$$

$$\Gamma^{\phi}{}_{\phi\phi} = 0$$

only nonzero entries

Recall the geodesic equation:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^{\alpha}{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0$$

This reduces to

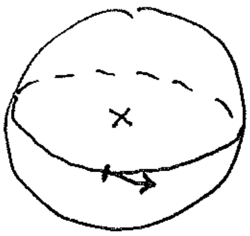
$$\frac{d^2\theta}{ds^2} = \cos\theta\sin\theta\left(\frac{d\phi}{ds}\right)^2$$

$$\frac{d^2\phi}{ds^2} = -2\cot\theta\,\frac{d\theta}{ds}\,\frac{d\phi}{ds}$$

Although these equations look complicated, it is because the initial position, direction & velocity of the geodesic has not been specified. It is easy to see that

$$\phi = \phi_0 + s.\left(\frac{d\phi}{ds}\right)_0, \quad \theta = \frac{\pi}{2}$$

is a solution: this is a great circle lying in the equator. All other solutions represent rotations of this curve.



"Classical" free-fall

How does the geodesic equation for the motion of a test particle in space-time simplify in the classical limit, i.e. the small velocity, weak (& slowly varying) field limit?

The geodesic equation is

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

For small velocities, $dx^0 = c\,dt \gg dx^i$, for $i = 1,2,3$

$$\text{Hence} \quad \frac{dx^0}{ds} \gg \frac{dx^i}{ds}$$

$$\& \quad \frac{dx^0}{ds} \sim 1 \quad \text{(refer to metric)}$$

$$\frac{dx^0}{ds} = \frac{dt}{d\tau} = \gamma \approx 1$$

The dominant terms of the geodesic equ. are then

$$\frac{d^2x^\mu}{d(ct)^2} + \Gamma^\mu{}_{00}\,.1.1 = 0$$

$$\text{i.e.} \quad \frac{d^2x^\mu}{dt^2} + c^2\,\Gamma^\mu{}_{00} = 0$$

Next consider the time ($\mu = 0$) component of this equation. The relevant component of the metric connection is

$$\Gamma^0_{00} = \tfrac{1}{2} g^{00} (g_{00,0} - \cancel{g_{00,0}} + \cancel{g_{00,0}}) + \tfrac{1}{2} g^{01} (g_{01,0} - g_{00,1} + g_{10,0}) + \dots$$

It turns out that for a weak gravitational field, the ~~non~~ off-diagonal terms in the metric tensor are negligible, so

$$\Gamma^0_{00} = \frac{1}{2c} g^{00} \frac{\partial g_{00}}{\partial t}.$$

We also restrict ourselves to the classical limit of fields that very slowly with time, & so this component is negligible. Hence only the spatial ($i = 1,2,3$) components of the geodesic equation need to be considered:

Newtonian limit: interested only in spatial components: $\dfrac{d^2 x^i}{dt^2} + c^2 \Gamma^i_{00} = 0$

Compare this with the equation of motion of a particle undergoing classical free-fall in the j direction (j is fixed):

$$\frac{d^2 x^j}{dt^2} = g$$

& we identify $\Gamma^j_{00} = -g/c^2$. Hence the metric connection corresponds to the components of gravitational acc'n in the classical limit.

Note that the metric connection vanishes when transforming to a frame in free-fall, as

EINSTEIN'S THEORY II

Newton's law of gravity is inconsistent with SR, because it implies that gravitational effects can be transmitted instantaneously to remote locations. (We have seen that SR implies signals must travel at speeds $\leq c$.) So we cannot follow our procedure of replacing normal derivatives by covariant derivatives to obtain a law valid in all frames: we lack a valid law to begin with.

Einstein recognised that there must be a relationship between the distribution of mass/energy & the curvature of space-time, & of course this relationship must be expressible in tensor form. The Einstein equation is

$$G_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4}, \qquad (*)$$

where $G_{\mu\nu}$ is the Einstein tensor, describing the curvature of space-time at a point, $T_{\mu\nu}$ is the stress-energy tensor, & $8\pi G/c^4$ is the constant of proportionality, which includes the gravitational constant G. This equation is the centrepoint of GR. Its application involves the specification of a given stress-energy tensor $T_{\mu\nu}$ & then the solution of the Einstein equation (or equations) for a suitable metric to describe the curvature of space. It should be noted from the outset that $(*)$ is nonlinear. Massive objects produce a gravitational field; this field contains energy, ~~mass~~ which is itself a source of field. This accounts for the difficulty in applying/ solving ⊛, in general. We will now give a more detailed justification of ⊛...

The Riemann curvature tensor

Recall that the curvature of a 2-D surface was described by the Gaussian curvature at a point. For higher-dimensional spaces (e.g. space-time), a more general description of curvature is needed, & this is supplied by ~~the~~ a rank 4 tensor called the Riemann curvature tensor.

We expect that this tensor will depend on the second derivatives, $g_{\mu\nu,\rho\sigma}$ of the metric tensor, for a frame in free-fall. The reason is that the SEP implies that, for a frame in free-fall

$$\left.\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} \\ g_{\mu\nu,\rho} &= 0 \end{aligned}\right\} \text{ at } x \text{ (an event)}$$

~~Hence~~
In other words, the space-time in this frame is locally flat. The departure from flatness must be described by the second derivatives of $g_{\mu\nu}$, i.e. these contain the curvature information.

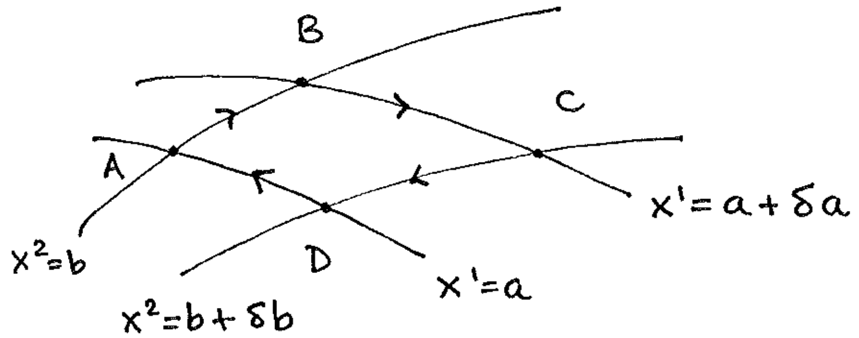
end of L10

$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

LHS OF FIELD EQUATION ($G_{\mu\nu}$)

The curvature tensor can be identified by a procedure that is analogous to the "parallel transport" measure of curvature presented earlier for 2-D surfaces.

Consider a small circuit, defined by changes δa & δb in the co-ordinates x^1 & x^2:



A vector v^μ is parallel transported around the circuit ABCD. In other words,

$$\frac{Dv^\mu}{Dx^\nu} = \frac{\partial v^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\rho} v^\rho = 0$$

everywhere along the route. Since we are only interested in $\nu = 1,2$:

$$\frac{\partial v^\mu}{\partial x^1} = -\Gamma^\mu{}_{\rho 1} v^\rho$$

$$\frac{\partial v^\mu}{\partial x^2} = -\Gamma^\mu{}_{\rho 2} v^\rho$$

where the symmetry of $\Gamma^\mu{}_{\nu\rho}$ is used. Hence when the vector gets to B we have

$$v^\mu(B) = v^\mu(A\text{initial}) + \int_A^B \frac{\partial v^\mu}{\partial x^1} dx^1$$

$$= v^\mu(A\text{initial}) - \int_{x^2=b} \Gamma^\mu{}_{\rho 1} v^\rho dx^1 \qquad (1)$$

where "$x^2=b$" denotes the path AB. Similarly for the path BC:

$$v^\mu(C) = v^\mu(B) - \int_{x^1=a+\delta a} \Gamma^\mu{}_{\rho 2}\, v^\rho\, dx^2 \qquad (2)$$

For CD:

$$v^\mu(D) = v^\mu(C) - \int_{x^2=b+\delta b} \Gamma^\mu{}_{\rho 1}\, v^\rho\, (-dx^1)$$

$$= v^\mu(C) + \int_{x^2=b+\delta b} \Gamma^\mu{}_{\rho 1}\, v^\rho\, dx^1 \qquad (3)$$

where the minus sign is needed because x^1 is decreasing along this path. Similarly for DA:

$$v^\mu(A_{final}) = v^\mu(D) + \int_{x^1=a} \Gamma^\mu{}_{\rho 2}\, v^\rho\, dx^2 \qquad (4)$$

The net change in v^μ is obtained by using (1) – (4):

$$\delta v^\mu = v^\mu(A_{final}) - v^\mu(A_{initial})$$

$$= \int_{x^1=a} \Gamma^\mu{}_{\rho 2}\, v^\rho\, dx^2 + \int_{x^2=b+\delta b} \Gamma^\mu{}_{\rho 1}\, v^\rho\, dx^1$$

$$- \int_{x^1=a+\delta a} \Gamma^\mu{}_{\rho 2}\, v^\rho\, dx^2 - \int_{x^2=b} \Gamma^\mu{}_{\rho 1}\, v^\rho\, dx^1$$

These terms would cancel in pairs if $\Gamma^\mu{}_{\rho\alpha}$ & v^ρ were constants around the loop. However, they are not, & we have

$$\delta v^\mu \approx \left(\Gamma^\mu{}_{\rho 2}\, v^\rho \Big|_{x^1=a}\right)\cdot \delta b$$

$$+ \left(\Gamma^\mu{}_{\rho 1}\, v^\rho \Big|_{x^2=b+\delta b}\right)\cdot \delta a$$

$$- \left(\Gamma^\mu{}_{\rho 2}\, v^\rho \Big|_{x^1=a+\delta a}\right)\cdot \delta b$$

$$- \left(\Gamma^\mu{}_{\rho 1}\, v^\rho \Big|_{x^2=b}\right)\cdot \delta a$$

OR: $$\delta v^\mu \approx \delta a\,\delta b\left[-\frac{\partial}{\partial x^1}\left(\Gamma^\mu{}_{\rho 2} v^\rho\right)+\frac{\partial}{\partial x^2}\left(\Gamma^\mu{}_{\rho 1} v^\rho\right)\right]$$

i.e. $$\delta v^\mu = \delta a\,\delta b\left[-\Gamma^\mu{}_{\rho 2,1} v^\rho - \Gamma^\mu{}_{\rho 2}\frac{\partial v^\rho}{\partial x^1} + \Gamma^\mu{}_{\rho 1,2} v^\rho + \Gamma^\mu{}_{\rho 1}\frac{\partial v^\rho}{\partial x^2}\right]$$

but $\dfrac{\partial v^\rho}{\partial x^\nu} = -\Gamma^\mu{}_{\nu\rho} v^\rho$, so

$$\delta v^\mu = \delta a\,\delta b\left[-\Gamma^\mu{}_{\rho 2,1} v^\rho + \Gamma^\mu{}_{\rho 2}\Gamma^\rho{}_{1\sigma} v^\sigma + \Gamma^\mu{}_{\rho 1,2} v^\rho - \Gamma^\mu{}_{\rho 1}\Gamma^\rho{}_{2\sigma} v^\sigma\right]$$

$$= \delta a\,\delta b\left[\Gamma^\mu{}_{\rho 1,2} - \Gamma^\mu{}_{\rho 2,1} + \Gamma^\mu{}_{\sigma 2}\Gamma^\sigma{}_{1\rho} - \Gamma^\mu{}_{\sigma 1}\Gamma^\sigma{}_{2\rho}\right] v^\rho$$

where the dummy indices have been re-labelled $(\rho \leftrightarrow \sigma)$ in the last two terms. So we have:

$$\delta v^\mu = \delta a\,\delta b . v^\rho\left[\Gamma^\mu{}_{\rho 1,2} - \Gamma^\mu{}_{\rho 2,1} + \Gamma^\mu{}_{\sigma 2}\Gamma^\sigma{}_{\rho 1} - \Gamma^\mu{}_{\sigma 1}\Gamma^\sigma{}_{\rho 2}\right],$$

using the symmetry of $\Gamma^\mu{}_{\alpha\beta}$.

Indices 1 & 2 appear because we have chosen the path to lie along the co-ordinate directions. If instead the path was along directions defined by δa^α & δb^β, we would have

$$\delta v^\mu = \delta a^\alpha \delta b^\beta v^\rho\left[\Gamma^\mu{}_{\rho\alpha,\beta} - \Gamma^\mu{}_{\rho\beta,\alpha} + \Gamma^\mu{}_{\sigma\beta}\Gamma^\sigma{}_{\rho\alpha} - \Gamma^\mu{}_{\sigma\alpha}\Gamma^\sigma{}_{\rho\beta}\right]$$

$$= \delta a^\alpha \delta b^\beta v^\rho .\ R^\mu{}_{\rho\beta\alpha},$$

where we define

$$R^{\mu}{}_{\rho\beta\alpha} = \Gamma^{\mu}{}_{\rho\alpha,\beta} - \Gamma^{\mu}{}_{\rho\beta,\alpha} + \Gamma^{\mu}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\rho\alpha} - \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\rho\beta}.$$

--- HANDOUT >

The quantity δv^{μ} is the difference between local vectors at a point & so is a vector. The δa^{α}, δb^{β} & v^{ρ} are all vectors, & so (by the quotient theorem), $R^{\mu}{}_{\rho\beta\alpha}$ is a tensor: the Riemann curvature tensor. This tensor quantifies space-time curvature, playing a role analogous to the Gaussian curvature of 2-D surfaces. It is worth noting the analogy between our result

$$\delta v^{\mu} = \delta a^{\alpha}\, \delta b^{\beta}\, v^{\rho} R^{\mu}{}_{\rho\beta\alpha}$$

& the earlier result for the change in orientation of a vector parallel transported around a closed curve on a 2-D surface:

$$\varphi = K.(\text{area enclosed}).$$

In the special case of a frame in free-fall the metric connections vanish (but their derivatives do not) & so we have

$$R^{\mu}{}_{\rho\beta\alpha} = \Gamma^{\mu}{}_{\rho\alpha,\beta} - \Gamma^{\mu}{}_{\rho\beta,\alpha} \qquad \text{(free-fall)}$$

The associated tensor $R_{\alpha\beta\gamma\delta} = g_{\alpha\mu} R^{\mu}{}_{\beta\gamma\delta}$ is also of interest. In free-fall this can be evaluated using the fundamental theorem of Riemannian geometry (the equation for $\Gamma_{\nu\mu\sigma}$ in terms of the derivatives of the metric tensor):

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\mu}\left(\Gamma^{\mu}{}_{\beta\delta,\gamma} - \Gamma^{\mu}{}_{\beta\gamma,\delta}\right) \qquad \text{(free-fall)}$$

$$= \Gamma_{\alpha\beta\delta,\gamma} - \Gamma_{\alpha\beta\gamma,\delta}$$

(58.) $\delta\gamma$

$$\Gamma_{\alpha\beta\delta} = \tfrac{1}{2}\left(g_{\beta\alpha,\delta} - g_{\delta\beta,\alpha} + g_{\alpha\delta,\beta}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial x^\gamma}\left(g_{\beta\alpha,\delta} - g_{\delta\beta,\alpha} + g_{\alpha\delta,\beta}\right)$$

$$- \frac{1}{2}\frac{\partial}{\partial x^\delta}\left(g_{\beta\alpha,\gamma} - g_{\gamma\beta,\alpha} + g_{\alpha\gamma,\beta}\right)$$

i.e. $$2R_{\alpha\beta\gamma\delta} = \cancel{g_{\beta\alpha,\delta\gamma}} - g_{\delta\beta,\alpha\gamma} + g_{\alpha\delta,\beta\gamma}$$

$$- \cancel{g_{\beta\alpha,\gamma\delta}} + g_{\gamma\beta,\alpha\delta} - g_{\alpha\gamma,\beta\delta}$$

$$= g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta}$$

(free-fall)

As expected, we see that the Riemann curvature tensor depends on the second derivatives of the metric tensor, in a frame in free-fall.

The Riemann tensor has $4^4 = 256$ components. However, there are a number of symmetries:

$$R_{\beta\alpha\gamma\delta} = R_{\alpha\beta\delta\gamma} = -R_{\alpha\beta\gamma\delta}$$

& $$R_{\gamma\delta\alpha\beta} = R_{\alpha\beta\gamma\delta},$$

& hence there are only 20 independent components in general.

OMIT – Differentiating our expression for the Riemann curvature tensor in a frame in free-fall gives

$$R^\alpha{}_{\beta\gamma\delta,\mu} = \Gamma^\alpha{}_{\beta\delta,\gamma\mu} - \Gamma^\alpha{}_{\beta\gamma,\delta\mu} \qquad \text{(FF)}$$

so: $$R^\alpha{}_{\beta\gamma\delta,\mu} + R^\alpha{}_{\beta\delta\mu,\gamma} + R^\alpha{}_{\beta\mu\gamma,\delta}$$

$$= \cancel{\Gamma^\alpha{}_{\beta\delta,\gamma\mu}} - \cancel{\Gamma^\alpha{}_{\beta\gamma,\delta\mu}} + \cancel{\Gamma^\alpha{}_{\beta\mu,\delta\gamma}} - \cancel{\Gamma^\alpha{}_{\beta\delta,\mu\gamma}}$$

$$+ \cancel{\Gamma^\alpha{}_{\beta\gamma,\mu\delta}} - \cancel{\Gamma^\alpha{}_{\beta\mu,\gamma\delta}}$$

$$= 0 \qquad \text{(FF)}$$

EXAMPLE OF SPHERE:

Recall that for the sphere

$$[g^{\mu\nu}] = \mathrm{diag}(\underset{\substack{\uparrow \\ g^{\theta\theta}}}{a^{-2}}, \underset{\substack{\uparrow \\ g^{\phi\phi}}}{a^{-2}\sin^{-2}\theta})$$

$$\Gamma^{\theta}{}_{\theta\theta} = 0$$

$$\Gamma^{\theta}{}_{\theta\phi} = \Gamma^{\theta}{}_{\phi\theta} = 0$$

$$\Gamma^{\theta}{}_{\phi\phi} = -\cos\theta\sin\theta$$

$$\Gamma^{\phi}{}_{\theta\theta} = 0$$

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta$$

$$\Gamma^{\phi}{}_{\phi\phi} = 0$$

VERIFY:

$$R^{\theta}{}_{\phi\theta\phi} = -R^{\theta}{}_{\phi\phi\theta} = \sin^2\theta$$

$$R^{\phi}{}_{\theta\theta\phi} = -R^{\phi}{}_{\theta\phi\theta} = -1$$

& all other components zero. Lower first index:

$$R_{\theta\phi\theta\phi} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi} = R_{\phi\theta\phi\theta} = a^2\sin^2\theta$$

& all other compounants are zero, which verifies the symmetries.

Ricci tensor:

$$R_{\theta\theta} = 1$$

$$R_{\theta\phi} = R_{\phi\theta} = 0$$

$$R_{\phi\phi} = \sin^2\theta$$

& curvature scalar

$$R = g^{\beta\delta}R_{\beta\delta} = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{2}{a^2}$$ ← twice the Gaussian curvature

EXERCISE: cylinder

This result will be true in all frames if we follow the "," → ";" rule:

$$R^{\alpha}{}_{\beta\gamma\delta;\mu} + R^{\alpha}{}_{\beta\delta\mu;\gamma} + R^{\alpha}{}_{\beta\mu\gamma;\delta} = 0$$

OMIT

This result is known as the "Bianchi identities."
Finally, we can define related tensors, by contraction:

$$R_{\beta\delta} = R^{\alpha}{}_{\beta\alpha\delta} = g^{\alpha\sigma} R_{\sigma\beta\alpha\delta}$$ which is symmetric

(because of the symmetries this is essentially the only contraction...)
is the "Ricci tensor," &

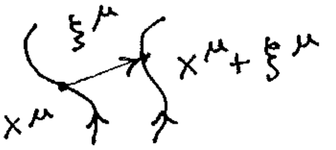
$$R = g^{\beta\delta} R_{\beta\delta} \quad (= R^{\delta}{}_{\delta})$$

is the "Ricci scalar."

EXAMPLE OF SPHERE – P.T.O.

Geodesic deviation, revisited

Recall that the deviation of geodesics on a 2-D surface also gave a measure of the Gaussian curvature of the surface, at a given point. What is the analogous relationship for space-time?



Suppose a vector ξ^μ links nearby geodesics at x^μ & $x^\mu + \xi^\mu$.

Then the geodesics have equations

$$0 = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}{}_{\nu\sigma} \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} \quad ①$$

$$\& \quad 0 = \left.\frac{d^2 x^\mu}{d\tau^2}\right|_{x+\xi} + \Gamma^{\mu}{}_{\nu\sigma}(x+\xi) . \left(\left.\frac{dx^\nu}{d\tau}\right|_{x+\xi}\right)\left(\left.\frac{dx^\sigma}{d\tau}\right|_{x+\xi}\right)$$

The second equation can be expanded in the (small) vector ξ^μ:

$$0 = \frac{d^2x^\mu}{d\tau^2} + \frac{d^2\xi^\mu}{d\tau^2} + \left(\Gamma^\mu{}_{\nu\sigma}(x) + \Gamma^\mu{}_{\nu\sigma,\rho}\Big|_x \xi^\rho + \ldots \right)$$

$$\cdot \left(\frac{dx^\nu}{d\tau} + \frac{d\xi^\nu}{d\tau} \right)\left(\frac{dx^\sigma}{d\tau} + \frac{d\xi^\sigma}{d\tau} \right)$$

i.e. $$0 = \frac{d^2x^\mu}{d\tau^2} + \frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma}(x) \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}$$
$$+ \Gamma^\mu{}_{\nu\sigma,\rho}\Big|_x \xi^\rho \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}$$
$$+ \Gamma^\mu{}_{\nu\sigma}(x) \frac{dx^\nu}{d\tau}\frac{d\xi^\sigma}{d\tau}$$
$$+ \Gamma^\mu{}_{\nu\sigma}(x) \frac{dx^\sigma}{d\tau}\frac{d\xi^\nu}{d\tau} + \mathcal{O}(\xi^2)$$

Subtracting the geodesic equation ①:

$$\frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma,\rho} \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}\xi^\rho + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{d\xi^\sigma}{d\tau}$$
$$+ \Gamma^\mu{}_{\nu\sigma}\frac{dx^\sigma}{d\tau}\frac{d\xi^\nu}{d\tau} = 0$$

In free-fall the coefficients vanish:

$$\frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma,\rho}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}\xi^\rho = 0 \qquad \text{(FF)}$$

②

Consider the covariant derivative of ξ^μ:

$$\frac{D\xi^\mu}{D\tau} = \frac{d\xi^\mu}{d\tau} + \Gamma^\mu{}_{\rho\sigma}\,\xi^\rho \frac{dx^\sigma}{d\tau}$$

& $$\frac{D^2\xi^\mu}{D\tau^2} = \frac{d}{d\tau}\left(\frac{d\xi^\mu}{d\tau} + \Gamma^\mu{}_{\rho\sigma}\,\xi^\rho\frac{dx^\sigma}{d\tau} \right)$$
$$+ \Gamma^\mu{}_{\rho\sigma}\left(\frac{d\xi^\rho}{d\tau} + \Gamma^\rho{}_{\alpha\beta}\,\xi^\alpha\frac{dx^\beta}{d\tau} \right)\frac{dx^\sigma}{d\tau}$$

i.e. $$\frac{D^2 \xi^\mu}{D\tau^2} = \frac{d^2 \xi^\mu}{d\tau^2} + \frac{d\Gamma^\mu{}_{\rho\sigma}}{d\tau} . \xi^\rho \frac{dx^\sigma}{d\tau}$$

+ terms with metric connections as coefficients.

In free-fall the other terms vanish, & writing $\frac{d}{d\tau} = \frac{dx^\nu}{d\tau}\frac{\partial}{\partial x^\nu}$ we have

(FF:) $$\frac{D^2 \xi^\mu}{D\tau^2} = \frac{d^2 \xi^\mu}{d\tau^2} + \Gamma^\mu{}_{\rho\sigma,\nu}\, \xi^\rho \frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau}$$

Combining this with ② gives

(FF:) $$\frac{D^2 \xi^\mu}{D\tau^2} = -\Gamma^\mu{}_{\nu\sigma,\rho}\, \xi^\rho \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} + \Gamma^\mu{}_{\rho\sigma,\nu}\, \xi^\rho \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}$$

$$= \left(\Gamma^\mu{}_{\rho\sigma,\nu} - \Gamma^\mu{}_{\nu\sigma,\rho}\right) \xi^\rho \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}$$

$$= \left(\Gamma^\mu{}_{\sigma\rho,\nu} - \Gamma^\mu{}_{\sigma\nu,\rho}\right) \xi^\rho \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}$$

i.e.

(FF:) $$\frac{D^2 \xi^\mu}{D\tau^2} = R^\mu{}_{\sigma\nu\rho}\, \xi^\rho \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}$$

This equation involves only tensors ($\frac{dx^\mu}{d\tau} = p^\mu/m_0$, which is a tensor) & hence must be true in all frames, not just free-fall. This is the equation of <u>geodetic deviation</u>, which is analogous to the equation found for 2-D surfaces:

$$\frac{d^2\eta}{ds^2} = -K\eta,$$

where K is the Gaussian curvature.

end of L11

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

RHS OF FIELD EQUATION:

THE STRESS-ENERGY TENSOR:

In SR the mass of a particle depends on its speed. The relationship between rest mass m_0, momentum p & energy E is

$$E^2 = p^2c^2 + m_0^2c^4.$$

This suggests that a general law of gravity will also depend on E & p as well as m.

The differential form of Newton's law of gravity can be obtained by analogy with electrostatics:

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \qquad\qquad F = \frac{G m_1 m_2}{r^2}$$

$$\frac{1}{q_p}\underset{\sim}{F} = \underset{\sim}{E} = -\nabla\phi \qquad\qquad \frac{1}{m_p}\underset{\sim}{F} = g = -\nabla\phi$$

$$\nabla\cdot\underset{\sim}{E} = \rho/\varepsilon_0 \qquad\qquad \nabla\cdot\left(\frac{1}{m_p}\underset{\sim}{F}\right) = 4\pi G\rho$$

$$\text{so}\quad \nabla^2\phi = -\rho/\varepsilon_0 \qquad\qquad \text{so}\quad \boxed{\nabla^2\phi = -4\pi G\rho}$$

charge density $\qquad\qquad$ mass density

How can this be generalised in SR? Consider a dust cloud, i.e. a collection of stationary particles. In the rest frame S of the dust the energy density is

$$\rho_0 c^2 = m_0 n_0 c^2$$

where m_0 is the ~~energy~~ rest mass of a single particle & n_0 is the number of particles per unit volume. Viewed in a frame S' moving with velocity v wrt S, each grain is more massive:

$$m_0 \to m' = \gamma m_0 .$$

Also, the volume containing a given number of grains is Lorentz contracted in the direction of motion of S', so that

$$n_0 \to n' = \gamma n_0 .$$

Hence the mass density transforms via:

$$\rho_0 \to \rho' = \gamma^2 \rho_0 .$$

On the basis of this transformation ρc^2 cannot be a scalar (which would be invariant under L.T.), & it can't be the component of a 4-vector (which would undergo a change linear in γ).

However, the behaviour matches the expected transformation of the time-time component of the 2nd rank tensor

$$T^{\mu\nu} = \rho_0 U^\mu U^\nu \qquad (1)$$

where U^μ is the 4 velocity of the dust. Recall that $U^\mu = \gamma(c, u_x, u_y, u_z)$ ← usual vel. comp. with $\gamma^{-2} = (1 + \frac{u_x^2+u_y^2+u_z^2}{c^2})$. In the rest frame of the dust the only non-zero component of $T^{\mu\nu}$ is $T^{00} = \rho_0 c^2$. Under transformation to S'

$$T'^{00} = \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^0}{\partial x^0} T^{00} = \gamma^2 T^{00},$$

as required.

$T^{\mu\nu}$ is the <u>stress-energy tensor</u> (for the special case of dust). Note that $T^{\mu\nu}$ is symmetric. It also satisfies a set of conservation laws:

$$T^{\mu\nu}{}_{,\nu} = 0 \qquad (2)$$

or in other words $T^{\mu\nu}$ is divergenceless. These relations are the SR generalisations of the conservation of mass/energy & momentum, as follows.

Consider the $\mu = 0$ component of (2):

$$\frac{\partial}{\partial x^0}(\rho_0 v^0 v^0) + \frac{\partial}{\partial x^1}(\rho_0 v_0 v^1) + \frac{\partial}{\partial x^2}(\rho_0 v_0 v^2) + \frac{\partial}{\partial x^3}(\rho_0 v_0 v^3) = 0$$

which reduces to

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0,$$

where $\rho = \gamma^2 \rho_0$ is the mass density in the frame in which the dust has velocity (u_x, u_y, u_z). More succinctly

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underset{\sim}{u}) = 0 \qquad (3)$$

which is the form of the usual mass continuity equation of hydrodynamics. Integrating over a volume V & ~~s~~using Stoke's theorem:

$$\frac{\partial}{\partial t}\left(\int_V \rho \, dV\right) = - \int_{S(V)} (\rho \underset{\sim}{u}) \cdot d\underset{\sim}{S}$$

i.e. the rate of change of the mass in the volume is the rate at which mass leaves through the bounding surface.

Similarly the $\mu = 1,2,3$ components of ②, together with ③, lead to

$$\rho \left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} \right) = 0 ,$$

which is recognisable as the LHS of the Navier-Stokes equation, & hence represents momentum conservation.

Finally we note that in curved spacetime, ② is replaced by

$$T^{\mu\nu}{}_{;\nu} = 0,$$

$$\text{or} \quad \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} + \Gamma^{\nu}{}_{\sigma\mu} T^{\mu\sigma} + \Gamma^{\mu}{}_{\sigma\mu} T^{\sigma\nu} = 0 .$$

To summarise the properties of the stress-energy tensor (quite generally; not just for dust):

- it vanishes in the absence of matter
- it is second rank
- it is divergenceless
- it is symmetric

PERFECT FLUID:

So far we have only considered the stress-energy tensor for dust. We write down (but do not justify) the stress-energy tensor for a perfect fluid, which is characterised by a 4-velocity $u^\alpha = dx^\alpha/d\tau$, a proper density $\rho_0 = \rho_0(x)$ & a scalar pressure $p = p(x)$:

$$T^{\mu\nu} = (\rho_0 + p) u^\mu u^\nu - p g^{\mu\nu}$$

end of L12

EINSTEIN'S FIELD EQUATIONS

Einstein recognised the stress-energy tensor as the source of space-time curvature, & suggested the simplest possible relationship between it & the Einstein tensor $G_{\mu\nu}$, which describes space-time curvature:

$$G^{\mu\nu} = kT^{\mu\nu},$$

where k is a scalar constant. Clearly $G^{\mu\nu}$ must be a divergenceless, symmetric second rank tensor, to match the stress energy tensor. It is further reasonable to expect that $G^{\mu\nu}$ is built from contractions of the Riemann tensor, since we know this describes the curvature of spaces of arbitrary dimension. The Ricci tensor, introduced earlier, has the correct rank & is the unique contraction. However, it has a non-zero divergence, but this can be removed by a simple subtraction. In this way Einstein was lead to the choice

$$G^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R,$$

where $R = g^{\beta\delta} R_{\beta\delta} = g_{\sigma\rho} R^{\sigma\rho}$ is the Ricci scalar, & $R_{\mu\nu}$ is the Ricci tensor,

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$$

PROOF THAT $G^{\mu\nu}{}_{;\mu} = 0$: see over

Finally, the constant k is determined by the requirement that, in the limit of weak & slowly varying fields, Einstein's field

PROOF THAT $G^{\mu\nu}{}_{;\mu} = 0$

Recall the expression for the Riemann curvature tensor in FF:

$$R^{\mu}{}_{\rho\beta\alpha} = \Gamma^{\mu}{}_{\rho\alpha,\beta} - \Gamma^{\mu}{}_{\rho\beta,\alpha}$$

so: $R^{\alpha}{}_{\beta\gamma\delta,\mu} + R^{\alpha}{}_{\beta\delta\mu,\gamma} + R^{\alpha}{}_{\beta\mu\gamma,\delta}$

$$= \cancel{\Gamma^{\alpha}{}_{\beta\delta,\gamma\mu}} - \cancel{\Gamma^{\alpha}{}_{\beta\gamma,\delta\mu}} + \cancel{\Gamma^{\alpha}{}_{\beta\mu,\delta\gamma}} - \cancel{\Gamma^{\alpha}{}_{\beta\delta,\mu\gamma}} + \cancel{\Gamma^{\alpha}{}_{\beta\gamma,\mu\delta}} - \cancel{\Gamma^{\alpha}{}_{\beta\mu,\gamma\delta}}$$

$$= 0$$

This result will be true in all frames with the replacement "," → ";":

$$R^{\alpha}{}_{\beta\gamma\delta;\mu} + R^{\alpha}{}_{\beta\delta\mu;\gamma} + R^{\alpha}{}_{\beta\mu\gamma;\delta} = 0$$

which are known as the Bianchi identities.

Next make the replacement $\gamma \to \alpha$ (contraction):

$$R_{\beta\delta;\mu} + R^{\alpha}{}_{\beta\delta\mu;\alpha} + R^{\alpha}{}_{\beta\mu\alpha;\delta} = 0$$

$$R_{\beta\delta;\mu} + R^{\alpha}{}_{\beta\delta\mu;\alpha} - R_{\beta\mu;\delta} = 0$$

$\times g^{\beta\mu}$: $R^{\mu}{}_{\delta;\mu} + R^{\alpha\mu}{}_{\delta\mu;\alpha} - R^{\mu}{}_{\mu;\delta} = 0$

$$R^{\mu}{}_{\delta;\mu} + g^{\alpha\beta} R^{\mu}{}_{\beta\mu\delta;\alpha} - R^{\alpha}{}_{\alpha;\delta} = 0$$

$$R^{\mu}{}_{\delta;\mu} + g^{\alpha\beta} R_{\beta\delta;\alpha} - \delta^{\mu}{}_{\delta} R^{\alpha}{}_{\alpha;\mu}$$

$$R^{\mu}{}_{\delta;\mu} + R^{\alpha}{}_{\delta;\alpha} - \delta^{\mu}{}_{\delta} R_{;\mu}$$

OR : $$R^{\mu}{}_{\delta;\mu} - \frac{1}{2}\delta^{\mu}{}_{\delta}R_{;\mu} = 0$$

$$\left(R^{\mu}{}_{\delta} - \frac{1}{2}\delta^{\mu}{}_{\delta}R\right)_{;\mu} = 0$$

$\times g^{\nu\delta}$: $$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0$$

i.e. $$G^{\mu\nu}{}_{;\mu} = 0$$

equations reduce to Newton's law of gravity, as we will soon see. This leads to

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu},$$

where G is the gravitational constant.

16 equ's in 16 unknowns ($g^{\mu\nu}$, symmetric) but $T^{\mu\nu}{}_{;\nu} = 0 \Rightarrow$ 4 constr. $\Rightarrow$ 10 equ's (constraints equiv. to arb. co-ord choice)

When Einstein's equation was applied to the universe as a whole, it became apparent that it favoured an expanding universe. At the time it was thought that the universe was static, & so Einstein modified his equations by adding a term:

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu},$$

where Λ is a constant called the cosmological constant. This term permits curvature in the absence of matter & radiation ($T^{\mu\nu} = 0$), & by a suitable choice of Λ, a static solution can be arranged. However, by the 1930's the Hubble expansion of the universe became accepted, & Einstein dropped this term. Interestingly, it is now back in vogue, as Alan Vaughn will discuss in the Cosmology section of this course.

We have not derived the field equations. In common with all laws of nature, they cannot be derived. Alternative theories have been proposed (that satisfy the various criteria we have followed), but they are invariably more complicated than Einstein's. More importantly, Einstein's theory has met every observational test to date.

(end of L13)

Before continuing, we note a useful alternative form for the field equations. We start with the covariant version:

$$\circledast \qquad R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} + \Lambda g_{\alpha\beta}$$

& contract with $g^{\alpha\beta}$:

$$g^{\alpha\beta} R_{\alpha\beta} - \tfrac{1}{2} R g^{\beta}_{\ \beta} = \frac{8\pi G}{c^4} T^{\beta}_{\ \beta} + \Lambda g^{\beta}_{\ \beta}$$

from earlier $g^{\beta}_{\ \beta} = \delta^{\beta}_{\ \beta} = \delta^0_{\ 0} + \delta^1_{\ 1} + \delta^2_{\ 2} + \delta^3_{\ 3} = 4$

& $R = g^{\alpha\beta} R_{\alpha\beta}$, so:

$$R - 2R = \frac{8\pi G}{c^4} T^{\beta}_{\ \beta} + 4\Lambda$$

$$\text{OR} \qquad R = -\frac{8\pi G}{c^4} T^{\mu}_{\ \mu} - 4\Lambda .$$

Substituting this back into $\circledast$:

$$R_{\alpha\beta} = -\tfrac{1}{2} g_{\alpha\beta} \left(4\Lambda + \frac{8\pi G}{c^4} T^{\mu}_{\ \mu} \right) + \frac{8\pi G}{c^4} T_{\alpha\beta} + \Lambda g_{\alpha\beta}$$

$$\text{or} \qquad R_{\alpha\beta} = \frac{8\pi G}{c^4} \left(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T^{\mu}_{\ \mu} \right) - \Lambda g_{\alpha\beta},$$

which is the new form. In the absence of matter,

$$R_{\alpha\beta} = -\Lambda g_{\alpha\beta} .$$

end of L13

The Newtonian limit:

For weak & slowly varying fields, the Einstein equations must reduce to Newton's law of gravity. To show this, we begin by assuming the form for the metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, & $|h_{\mu\nu}| \ll 1$. We are assuming that velocities are small ($\ll c$), so that $E^2 = p^2c^2 + m_0^2c^4$ implies that the time component of the 4-momentum (E/c) is much larger than the spatial components. It follows that the dominant term in the stress-energy tensor is T^{00}, which is the energy density. Hence the important part of Einstein's equations is (using the alternative form):

① $$R_{00} = \frac{8\pi G}{c^4}\left(T_{00} - \tfrac{1}{2}T^{0}{}_{0}\, g_{00}\right) - \Lambda g_{00}$$

Since GR $\approx$ SR in this limit, we will use the free-fall version of the Riemann curvature tensor to evaluate the R_{00} term:

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\left(g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta}\right)$$

& $$R_{\beta\delta} = R^{\gamma}{}_{\beta\gamma\delta}$$

$$= g^{\gamma\alpha} R_{\alpha\beta\gamma\delta}$$

$$\approx \tfrac{1}{2}\eta^{\gamma\alpha}\left(h_{\alpha\delta,\beta\gamma} - h_{\beta\delta,\alpha\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta}\right)$$

where we ignore terms of order $|h_{\mu\nu}|^2$.
& use the fact that the derivatives of $g_{\mu\nu}$ are zero.

So: $R_{00} = \frac{1}{2}\eta^{\gamma\alpha}(h_{\alpha 0,0\gamma} - h_{00,\alpha\gamma} + h_{0\gamma,\alpha 0} - h_{\alpha\gamma,00})$.

Next we use the slow moving approximation $\frac{1}{c}\frac{\partial}{\partial t} \ll \frac{\partial}{\partial x^i}$ $(i = 1,2,3)$:

$$R_{00} = -\frac{1}{2}\eta^{ij} h_{00,ij} \qquad (i,j = 1,2,3)$$

$$= \frac{1}{2} h_{00,ii}$$

since $\eta^{ij} = \mathrm{diag}(-1,-1,-1)$.

Next recall our treatment of gravitational red-shift, near the beginning of the course. We arrived at the result

$$d\tau^2 = dt^2(1 + 2\phi/c^2),$$

where $\phi = -GM/r^2$ is the Newtonian potential. This expression can be considered to be the time-component of a metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2\left(1 + \frac{2\phi}{c^2}\right)$$

$$= g_{00}(dx^0)^2$$

where we identify $g_{00} = 1 + 2\phi/c^2$

$$= \eta_{00} + h_{00}$$

Hence we arrive at $h_{00} = 2\phi/c^2$, & our expression for R_{00} becomes

$$R_{00} = \frac{1}{2}\left(\frac{2\phi}{c^2}\right)_{,ii} = \frac{1}{c^2}\nabla^2\phi, \qquad (2)$$

in standard notation.

Next we note that the low velocity expression for the time-time component of the stress-energy tensor is $T^{00} = \rho_0 c^2$. Under our assumptions about the metric $T_{00} = T^{00} = T^0{}_0$ & we can take $g_{00} = 1$ in ①, leading to

$$R_{00} = \frac{8\pi G}{c^4}\left(\rho_0 c^2 - \tfrac{1}{2}\rho_0 c^2\right) - \Lambda$$

$$\text{i.e. } \frac{1}{c^2}\nabla^2\phi = \frac{4\pi G \rho_0}{c^2} - \Lambda \text{ , using ②}$$

$$\text{or } \quad \nabla^2\phi = 4\pi G\rho_0 - c^2\Lambda . \qquad (*)$$

Hence, if $\Lambda = 0$ we obtain Newton's law of gravity (in fact experimental limits on the value of Λ imply that this term is tiny, although it is significant on cosmological scales). This derivation establishes that the constant $\frac{8\pi G}{c^4}$ appearing in the Einstein equations has been correctly chosen.

"Force" corresponding to $(*)$:

$$F = \frac{-GM}{r^2} + \frac{c^2\Lambda r}{3}$$

(A. Vaughan)

Experiments $\Rightarrow$ $|\Lambda| \leq 10^{-52}\, m^{-2}$, so ratio of cosmological term to Newtonian term $\sim 10^{-22}$ for $M = M_\odot$, $r = 1\,AU$. So this term is completely insignificant on solar-system scales, but becomes important over cosmological scales

end of L14

THE SCHWARZSCHILD METRIC :

In 1915 Einstein completed GR. On Jan 16 1916, Einstein read a paper in front of the Prussian academy on behalf of Karl Schwarzschild, who was in the German army at the front at the time. The paper presented an exact solution to the Einstein equations in vacuum for the case of a static, spherically symmetric gravitational field - appropriate, for example, to describe a point mass, or the region external to the Sun. (Incidentally, Schwarzschild died later that year, from an illness contracted at the front.)

For situations involving spherical symmetry it is appropriate to use spherical polar co-ordinates. For example, the Minkowski metric of SR describing flat space-time can be written

$$ds^2 = c^2dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$= c^2dt^2 - dr^2 - r^2 d\Omega^2$$

If $r = R$ (a constant) & $dr = dt = 0$, then we recover the line element for the surface of a sphere, introduced earlier.

The derivation of the Schwarzschild metric proceeds as follows (only an outline will be given - the full derivation is reproduced in most GR textbooks).

A general spherically symmetric, static metric ~~must have the~~ can be written in the form

$$ds^2 = A(r)\,c^2dt^2 - B(r)\,dr^2 - r^2 d\Omega^2$$

i.e. has metric components

$$g_{00} = A(r)$$
$$g_{11} = -B(r)$$
$$g_{22} = -r^2$$
$$g_{33} = -r^2\sin^2\theta,$$

where A & B are arbitrary fns. The metric connections $\Gamma^{\nu}_{\mu\rho}$ can be calculated from this metric, & hence the components of the Riemann tensor & Ricci tensor $R_{\mu\nu}$ can be obtained. These are expressions in A & B & their first & second derivatives. The "alternative" form for the Einstein equations is

$$R_{\mu\nu} = 0,$$

& the resulting DEs can be solved exactly. The result is

$$A = 1 - \frac{2\ell}{r}, \quad B = \frac{1}{1-\frac{2\ell}{r}}$$

where ℓ is a constant with dimension length.

Experiments on gravitational red-shift

confirm the relationship presented early in these lectures:

$$d\tau^2 = dt^2 \left(1 - \frac{2GM}{r^2c^2}\right)$$

This can be interpreted as the time-time component of the metric $ds^2 = c^2 d\tau^2$, & hence by comparison with the functional form of $A(r)$ we have $\ell = GM/c^2$, where M is the mass producing the field.

Hence the Schwarzschild metric is

$$\boxed{ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2 d\Omega^2}$$

There are a number of notable features:

1. The resulting space is asymptotically flat, i.e. the metric approaches the Minkowski metric as $r \to \infty$

2. The term $\frac{2GM}{rc^2}$ determines the severity of the curvature (departure from the Minkowski form).

   - $\frac{2GM}{rc^2} << 1 \Rightarrow$ almost flat

   - $\frac{2GM}{rc^2} \lesssim 1 \Rightarrow$ curvature severe

Space-time in the vicinity of a star whose radius is $r < r_0 = \frac{2GM}{c^2}$, where r_0 is termed the Schwarzschild radius, is so warped that the region interior to r_0 is effectively isolated from the rest of the universe. This is the phenomenon known as a black hole, which will be discussed in greater detail later. For our own Sun the Schwarzschild radius is about

$$r_0 = \frac{2GM}{c^2} = \frac{2.(6.67\times10^{-11}).(2\times10^{30})}{(3\times10^8)^2}\ m$$

$$\approx 3\ km$$

so for the Sun to become a black hole, all of the mass of the Sun would have to be confined within this radius, by some means. Neutrons stars are the result of the explosion of gravitational collapse of stars with mass comparable to the Sun. They have radii ~ 10km, so the Schwarzschild radius is a significant fraction of the radius of a neutron star.

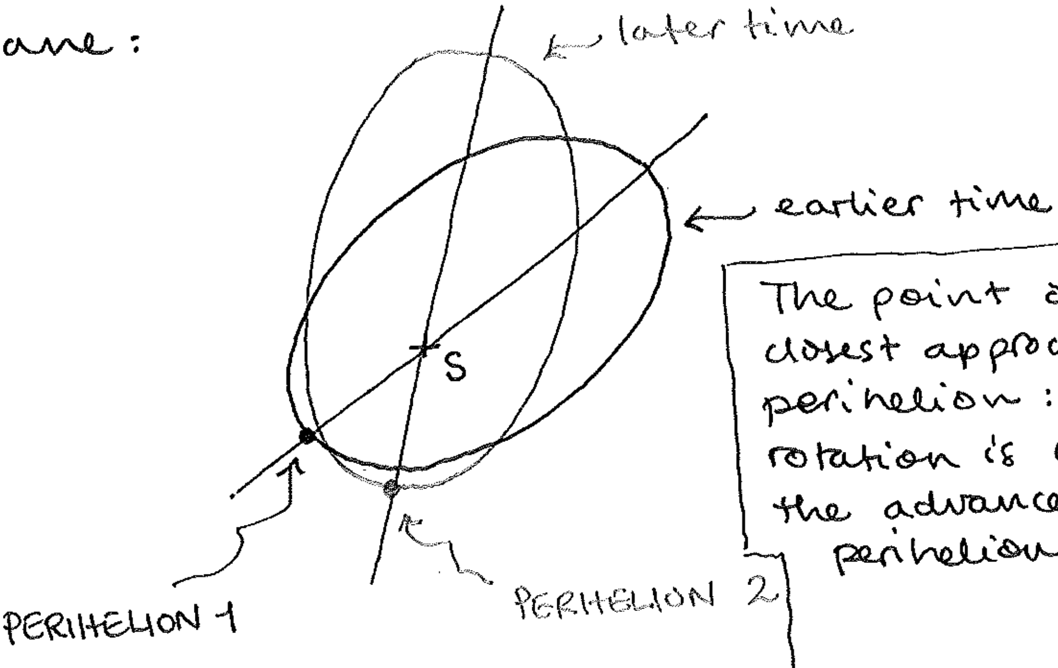
TESTS OF GENERAL RELATIVITY :

We consider measurements of GR effects within the solar system, in particular the "classical tests" :

1. Advance of the perihelion of Mercury
2. Deflection of light by the Sun
3. Radar echo delays from Venus

1. Advance of perihelion of Mercury :

Mercury is the closest planet to the Sun & follows an elliptical orbit with a mean distance of 58 million km from the Sun. Other planets (in particular Jupiter) attract Mercury & perturb its orbit : the net result is that the long axis of the orbital ellipse rotates in the orbital plane :



The point of closest approach is perihelion : so this rotation is called the advance of the perihelion.

The Newtonian prediction is 532"/century, but the measured value departs from this by about 43"/century, a discrepancy that was discovered by Leverrier (Urbain Jean Joseph Le Verrier) (1859). It was proposed that a small, undetected planet (Vulcan) inside the orbit of Mercury was causing the additional precession, but this planet was never discovered (although there is a famous instance of a French astronomer claiming a sighting).

GR provides a very natural explanation for the additional advance of the perihelion. The analysis begins with the Schwarzschild solution. We can ignore the influence of the other planets because this is nearly independent of the GR-induced precession.

The Schwarzschild metric (SM) for motion in the $\theta = \frac{\pi}{2}$ plane is

① $$ds^2 = c^2 d\tau^2 = c^2 Z\, dt^2 - \frac{dr^2}{Z} - r^2 d\phi^2,$$

where $Z = 1 - \frac{2GM}{rc^2} \approx 1 - 5\times10^{-7}$ for the average orbital distance of Mercury.

$\frac{m_0^2}{d\tau^2} \times$ metric : (m_0 is Mercury's mass)

② $$m_0^2 c^2 = m_0^2 c^2 Z\left(\frac{dt}{d\tau}\right)^2 - \frac{m_0^2}{Z}\left(\frac{dr}{d\tau}\right)^2 - m_0^2 r^2\left(\frac{d\phi}{d\tau}\right)^2$$

If $Z=1$ (i.e. in flat space),

$$m_o^2c^2 = m_o^2c^2\gamma^2 - m_o^2\gamma^2 v_r^2 - m_o^2\gamma^2 v_\phi^2$$

i.e. $m_o^2c^4 = E^2 - p^2c^2$,

which is the energy equation of SR. This suggests ② is the analog of the energy equ., for the S.M.

The equations of motion are easily obtained from the variational approach. Recall that the Lagrangian is

$$L = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\mu\nu}q^\mu q^\nu$$

From ① we have explicitly

$$L = c^2 Z (q^t)^2 - \frac{1}{Z}(q^r)^2 - r^2(q^\phi)^2. \qquad ③$$

end of L15 Recall also that the Euler-Lagrange equ's are

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial q^\mu}\right) = 0 \qquad ④$$

The Lagrangian does not depend explicitly on time. Hence

$$\frac{\partial L}{\partial q^t} = \text{const.}$$

i.e. $$2Zc^2\frac{dt}{d\tau} = \text{const.} \qquad (4b)$$

& multiplying by $m_o/2$,

$$Z m_o c^2 \frac{dt}{d\tau} = \text{const.}$$

In flat space the LHS reduces to $\gamma m_o c^2 = E$, the total energy. So we label this constant E:

$$Z m_0 c^2 \frac{dt}{d\tau} = E. \qquad (5)$$

Also $\partial L/\partial\phi = 0$, so

$$\frac{\partial L}{\partial q^\phi} = \text{const.}$$

i.e. $$r^2 \frac{d\phi}{d\tau} = \text{const.} = J, \qquad (6)$$

which is the equivalent of conservation of angular momentum $(r^2\dot\phi = \text{const.})$

Going back to (2) & substituting (5):

$$m_0^2c^2 = \frac{E^2}{c^2 Z} - m_0^2 Z^{-1}\left(\frac{dr}{d\tau}\right)^2 - m_0^2 r^2\left(\frac{d\phi}{d\tau}\right)^2$$

$\times \frac{Z}{m_0}$: $$\left(1 - \frac{2GM}{rc^2}\right) m_0 c^2 = \frac{E^2}{m_0 c^2} - m_0\left(\frac{dr}{d\tau}\right)^2 - Z m_0 r^2\left(\frac{d\phi}{d\tau}\right)^2$$

$\times\frac{1}{2}$, Rearranging:

$$\frac{1}{2}m_0\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}Z m_0 r^2\left(\frac{d\phi}{d\tau}\right)^2 - \frac{GMm_0}{r} = \frac{E^2}{2m_0c^2} - \frac{1}{2}m_0c^2$$

& RHS is a constant, say T:

$$\underbrace{\frac{1}{2}m_0\left(\frac{dr}{d\tau}\right)^2}_{\text{"radial KE" term}} + \underbrace{\frac{1}{2}Z m_0 r^2\left(\frac{d\phi}{d\tau}\right)^2}_{\text{"transverse KE" term}} - \underbrace{\frac{GMm_0}{r}}_{\text{"gravitational P.E." term}} = T \qquad (7)$$

So this is the equivalent of the energy conservation equation for the SM.

Equ's (5), (6) & (7) completely describe the motion of Mercury (or any particle in free-fall in the SM)

Next we solve the equations. From ⑥:

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\cdot\frac{d\phi}{d\tau} = \frac{J}{r^2}\frac{dr}{d\phi}$$

Make the substitution $u = \frac{1}{r}$ (this is Newton's trick, used in the classical orbit calculation):

$$\frac{dr}{d\phi} = \frac{dr}{du}\cdot\frac{du}{d\phi} = -r^2\frac{du}{d\phi}$$

so $$\frac{dr}{d\tau} = -J\frac{du}{d\phi}$$

Substituting this into ⑦ & using ⑥ again:

$$\frac{1}{2}m_0 J^2\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}Zm_0 r^2 . \frac{J^2}{r^4} - GMm_0 u = T$$

$\times\frac{2}{m_0}$:
$$J^2\left(\frac{du}{d\phi}\right)^2 + J^2u^2Z - 2GMu = \frac{2T}{m_0}$$

$$J^2\left(\frac{du}{d\phi}\right)^2 + J^2u^2\left(1-\frac{2GMu}{c^2}\right) - 2GMu = \frac{2T}{m_0}$$

$$J^2\left(\frac{du}{d\phi}\right)^2 + J^2u^2 - \frac{2GM}{c^2}J^2u^3 - 2GMu = \frac{2T}{m_0}$$

$\frac{d}{d\phi}$:
$$2J^2\frac{du}{d\phi}\cdot\frac{d^2u}{d\phi^2} + 2J^2u\frac{du}{d\phi} - \frac{6GM}{c^2}J^2u^2\frac{du}{d\phi} - 2GM\frac{du}{d\phi} = 0$$

$\frac{du}{d\phi}$ cancels: $\left(/2J^2\frac{du}{d\phi}\right)$

$$\boxed{\frac{d^2u}{d\phi^2} + u - \frac{GM}{J^2} = \frac{3GM}{c^2}u^2} \quad ⑧$$

This can be solved exactly in terms of Elliptic functions, but that is overkill for present purposes.

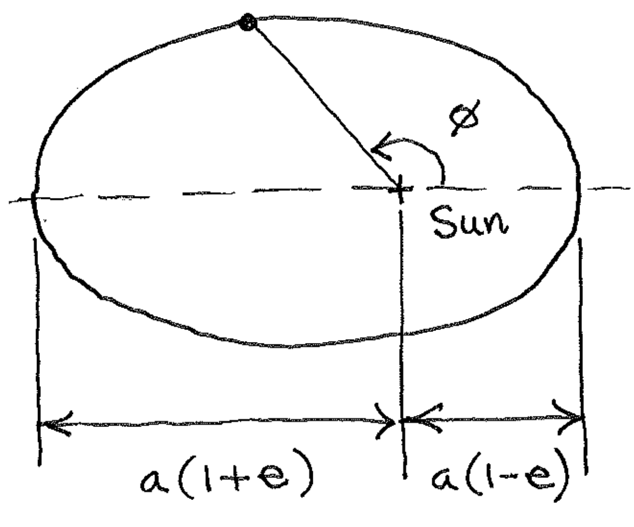
Instead we proceed by identifying how this equation differs from the classical orbit equation (due to Newton). The circled terms above represent the extra ~~GR~~ term ~~effect~~, due to the departure of Z from unity. Hence the classical equation is

$$\frac{d^2u}{d\phi^2} + u - \frac{GM}{J^2} = 0 \qquad (9)$$

This has the solution

$$u_0 = \frac{1 + e\cos\phi}{\ell} \qquad (10)$$

where $\ell = a(1-e^2)$, which represents an ellipse with eccentricity e:



Substituting (10) into (9) we obtain

$$\ell = \frac{J^2}{GM} \qquad (11)$$

The term on the RHS of (8) is small (the orbit is almost a closed ellipse), so we can solve (8) by a perturbation approach.
We identify the dimensionless small parameter

$$\varepsilon = \frac{3GM}{\ell c^2}$$

which will be $\sim 10^{-7}$ for Mercury, as noted before.

Next we assume a solution of the form

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2) \qquad (13)$$

where u_0 is the ellip~~soidal~~tical solution. Substituting this into (8):

$$u_0'' + u_0 - \frac{GM}{J^2} + \varepsilon u_1'' + \varepsilon u_1 = \ell \varepsilon u_0^2 + O(\varepsilon^2)$$

& we ignore terms of order ε^2 (or higher). We know $u_0'' + u_0 - GM/J^2 = 0$, so

$$\begin{aligned} u_1'' + u_1 = \ell u_0^2 &= \ell^{-1}(1 + 2e\cos\phi + e^2\cos^2\phi) \\ &= \ell^{-1}(1 + 2e\cos\phi + \tfrac{1}{2}e^2 \\ &\qquad + \tfrac{1}{2}e^2\cos 2\phi) \\ &= \frac{1+\frac{1}{2}e^2}{\ell} + \frac{2e}{\ell}\cos\phi \\ &\qquad + \frac{e^2}{2\ell}\cos 2\phi \qquad (14) \end{aligned}$$

Next we "guess" the form for the solution:

$$u_1 = A + B\phi\sin\phi + C\cos 2\phi$$

Easy to show (exercise) that

$$u_1'' + u_1 = A + 2B\cos\phi - 3C\cos 2\phi,$$

& comparing this with (14) we have

$$A = \ell^{-1}(1+\tfrac{1}{2}e^2), \quad B = \frac{e}{\ell}, \quad C = -\frac{e}{6\ell}$$

& hence the solution to ⑧, to order ε, is

$$u \approx u_0 + \varepsilon u_1$$
$$= u_0 + \varepsilon\left[\ell^{-1}(1+\tfrac{1}{2}e^2) + \frac{e}{\ell}\phi\sin\phi - \frac{e}{6\ell}\cos 2\phi\right]$$

The most important of the bracketted terms is the $\phi\sin\phi$ term, because this grows steadily in magnitude with each orbit. The other terms are constant & oscillatory, respectively. Hence, keeping the important term, we have

$$u = \frac{1+e\cos\phi}{\ell} + \frac{\varepsilon e}{\ell}\phi\sin\phi$$
$$= \frac{1+e(\cos\phi + \varepsilon\phi\sin\phi)}{\ell} \quad ⑮$$

Now $\cos[(1-\varepsilon)\phi] = \cos\phi\cos(\varepsilon\phi) + \sin\phi\sin(\varepsilon\phi)$

$$= \cos\phi + \varepsilon\phi\sin\phi + O(\varepsilon^2)$$

So we can rewrite our solution

$$u = \frac{1+e\cos[(1-\varepsilon)\phi]}{\ell} \quad ⑮$$

(to order ε). When r is a minimum u is a maximum; so the perihelion points must correspond to

$$(1-\varepsilon)\phi = 2n\pi, \quad n = 0,1,2,\ldots$$

i.e. $\phi \approx 2n\pi(1+\varepsilon) + O(\varepsilon^2)$

$$= 2n\pi + 6n\pi GM/\ell c^2$$

Hence there is an advance of perihelion by $\Delta\phi = 6\pi GM/\ell c^2$ per rotation, & the rate of perihelion advance is

$$\frac{\Delta\phi}{T} = \frac{6\pi GM}{a(1-e^2)Tc^2}, \qquad (16)$$

which is a formula Einstein arrived at in 1916. (The approach here was to start with an exact sol'n & linearise later; Einstein's starting point was a set of linearised equ's.)

Evaluating (16) leads to $\frac{\Delta\phi}{T} \approx 43.03''/\text{century}$.
The best value for the discrep. in the perihelion advance of Mercury is $43.11 \pm 0.45\ ''/\text{century}$, & hence GR accounts for the discrepancy.

end of L16

Mercury has the largest perihelion advance because (see (16)) T is shortest & $1-e^2$ is the smallest among the planets. However, the Earth & Venus also have measurable perihelion advances (after account of perturbations by other bodies), & GR also gets these right.

Finally it should be noted that if the Sun were sufficiently oblate this would also cause Mercury to precess, & in the 1960's Dicke & coworkers claimed evidence for an oblateness that would produce an extra $3''/\text{century}$. The discrepancy in the perihelion advance would then be inconsistent with the GR prediction. Brans & Dicke proposed an alternative, "scalar-tensor" model for gravity. However, other oblateness measurements (before & since) have not substantiated the Dicke et al. value. Also, the Brans-Dicke model has a free parameter ω, & reduces to GR in the limit of large ω. A variety of measurements imply that $\omega > 500$, & hence the Brans-Dicke theory contains nothing new, & is more complicated than GR. For these reasons it is no longer considered a serious contender to GR.

For a fuller discussion see Clifford M. Will, "Was Einstein right?"

2. Deflection of light by the Sun:

The calculation of the orbit of a photon in the SM is similar to the Mercury calculation, with an importance difference. The photon orbit is a null geodesic, so $ds^2 = 0$. Correspondingly we cannot use τ as the parameter in the Lagrangian calculation & it is replaced by λ:

$$L = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$$
$$= g_{\mu\nu} q^\mu q^\nu .$$

The metric (in the equator) is

$$0 = c^2 Z\, dt^2 - \frac{dr^2}{Z} - r^2 d\phi^2 \qquad (7)$$

Or $$0 = c^2 Z \left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{Z}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

The Euler-Lagrange equation involving q^t is

$$2Zc^2 \frac{dt}{d\lambda} = \text{const}, \qquad (18)$$

& the q^ϕ equation is

$$\frac{d\phi}{d\lambda} = J/r^2. \qquad (19)$$

Following the same steps as the Mercury orbit calculation (exercise) leads to the equation of motion

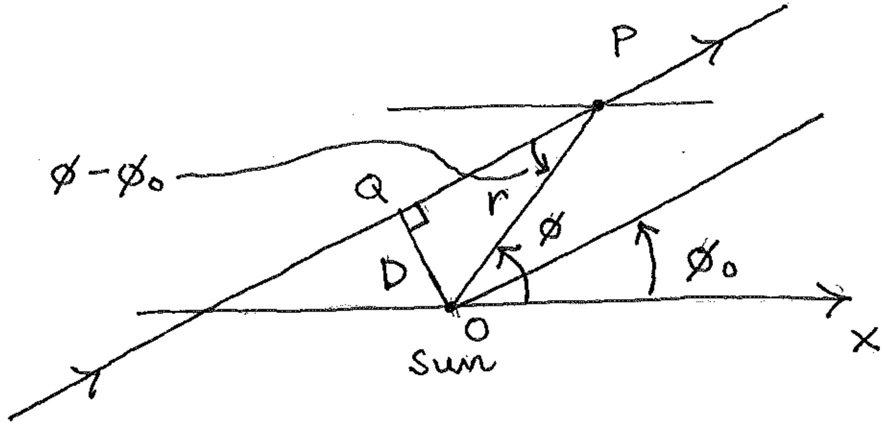
$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2 \qquad (20)$$

Just as with the Mercury calculation, in the ~~flat space~~ classical limit the RHS vanishes, i.e.

$$\frac{d^2u}{d\phi^2} + u = 0,$$

which has the solution

$$u_0 = \frac{1}{D}\sin(\phi - \phi_0). \qquad (21)$$



This is the equation of a straight line, as seen in the diagram: in the triangle OPQ clearly $D = r\sin(\phi-\phi_0) \Rightarrow u = \frac{1}{r} = \frac{1}{D}\sin(\phi-\phi_0)$. D is called the impact parameter.

Once again we seek a solution that is a perturbation of the straight line path, i.e.

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2), \quad (22)$$

where $\varepsilon = 3GM/Dc^2$ is our small parameter. For convenience we assume (w.l.o.g.) $\phi_0 = 0$. Substituting this trial solution into (20) & subtracting the zeroth order solution gives (to order ε)

$$u_1'' + u_1 = D u_0^2 = \frac{\sin^2\phi}{D} \quad (23)$$

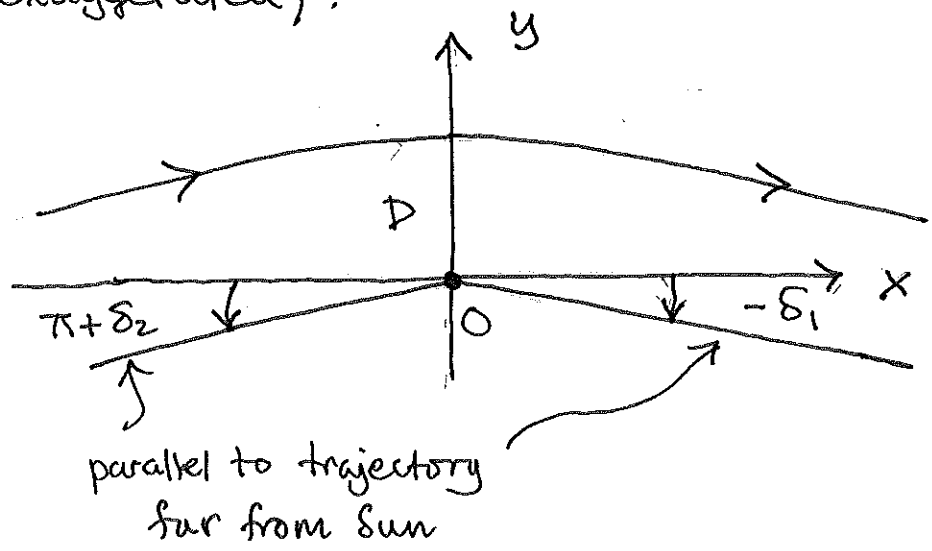
The solution to this ODE (check) is

$$u_1 = \frac{1 + B\cos\phi + \cos^2\phi}{3D},$$

where B is an arbitrary constant. Hence our perturbed solution is

$$u = \frac{\sin\phi}{D} + \varepsilon \cdot \frac{1 + B\cos\phi + \cos^2\phi}{3D}. \quad (24)$$

The new path looks like this (the deflection is exaggerated):



Far from the Sun $u \to 0$ & $\phi \to -\delta_1, \pi+\delta_2$
$(r\to\infty)$
as shown. Using the small angle results

$$\sin\delta_i \approx \delta_i \;, \quad \cos\delta_i \approx 1$$

$$\sin(\pi+\delta_2) = -\sin\delta_2 \approx -\delta_2$$

$$\cos(\pi+\delta_2) = -\cos\delta_2 \approx -1$$

in (24) for $r\to\infty$ we have

$$0 = \frac{-\delta_1}{D} + \varepsilon.\frac{(2+B)}{3D}$$

$$0 = \frac{-\delta_2}{D} + \varepsilon.\frac{(2-B)}{3D} \qquad (25)$$

The total deflection is $\Delta\phi = \delta_1+\delta_2$. Adding (25):

$$0 = -\frac{\Delta\phi}{D} + \frac{4\varepsilon}{3D}$$

OR $$\Delta\phi = \frac{4\varepsilon}{3} = \frac{4}{3}\cdot\frac{3GM}{c^2D} = \frac{4GM}{c^2D} \qquad (26)$$

For a ray grazing the Sun $D = R_\odot = 7\times10^8$ m.
Also $G = 6.67\times10^{-11}$ SI, $M = M_\odot = 2\times10^{30}$ kg, $c = 3\times10^8\,ms^{-1}$, so

$$\Delta\phi = \frac{4.(6.67\times10^{-11}).\,2\times10^{30}}{9\times10^{16}\,.\,7\times10^8}$$

$$\approx 8.5\times10^{-6} \text{ rad}$$

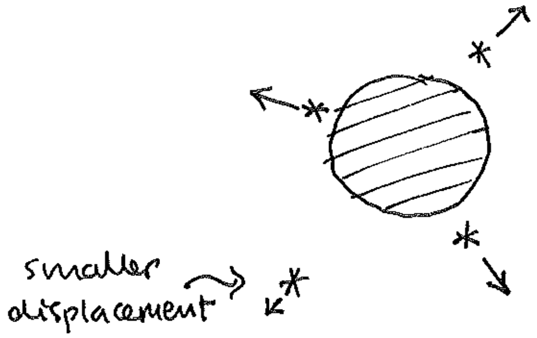
$$= 4.85\times10^{-4} \text{ deg}$$

$$= 4.85\times10^{-4}\times(60)^2 \text{ arc seconds } ('')$$

$$= 1.75''$$

which was Einstein's 1916 prediction.

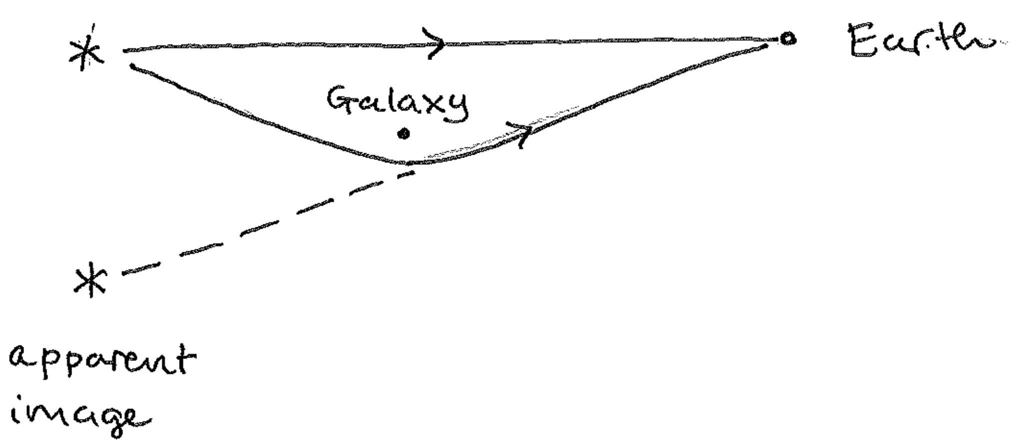
One effect of the deflection is that stars close to the Sun's limb should be radially displaced from their expected locations:



which (in principle) is observable during an eclipse. In 1919 an eclipse observation lead by Eddington "confirmed" Einstein's prediction. However, a repeated versions of this experiment have shown that it is not a particularly definitive test (e.g. the observed deflection of stars also has a significant contribution from atmospheric "seeing"). Hence the results of Eddington & others can only be said to be in qualitative agreement with GR.

The deflection of radio waves from quasars has also been used to test the GR prediction, equ. 26. (The GR prediction is wavelength independent, so radio waves are expected to be deflected the same amount.) This procedure is more accurate, & confirms the GR prediction to a few percent.

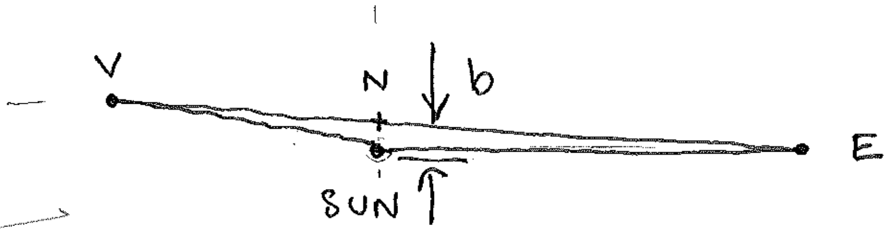
More recently the "gravitational lensing" of quasars by distant galaxies has been identified. The first example was a double quasar discovered in 1979; the second image is due to light bent back into ~~the~~ a line of sight:



In general it is possible to have multiple images, or even, if the intervening galaxy lies along the same line of sight, an "Einstein ring".

3. Radar echo delays

Although the speed of light is locally a constant in any freely-falling reference frame (because SR is valid in this situation), in GR it is not ~~in general~~ generally a constant. Irwin Shapiro realized that this prediction of GR could be tested by measuring the time for radar echoes to return from Venus. GR predicts an excess delay (due to a slowing of light) for measurements made when Venus is ~~in~~ near "superior conjunction", i.e. furthest from the Earth, so that the Sun is almost along the line joining Venus & the Earth:



end of L17

For a null ~~metric~~ path in the SM we have ~~The metric is the SM,~~

$$c^2 Z\, dt^2 - \frac{dr^2}{Z} - r^2 d\phi^2 = 0 \qquad (27)$$

(in the equatorial plane), where $Z = 1 - \frac{2GM}{rc^2}$.
At the point of closest approach $dr = 0$, & we have

$$r\frac{d\phi}{dt} = c Z^{\frac{1}{2}} \qquad (28)$$

The LHS is the velocity of light along the line joining Venus to Earth, at N, as measured by a remote observer. recall t is "co-ordinate time" Clearly $r\frac{d\phi}{dt} < c$, which illustrates the slowing of light.

Returning to the Euler-Lagrange equations for the path followed by light, equ. (18) $\Rightarrow$

$$2Zc\frac{dt}{d\lambda} = \text{const}, \quad \& \qquad \text{(time)}$$

& (19) $\Rightarrow$

$$r^2\frac{d\phi}{d\lambda} = \text{const} \qquad (\phi)$$

Combining these,

$$\frac{d\phi}{dt} = \frac{d\phi}{d\lambda}\cdot\left(\frac{dt}{d\lambda}\right)^{-1} = \frac{Z}{r^2}\cdot \text{const}.$$

which we write as

$$r^2 \frac{d\phi}{dt} = Z.W \qquad (29)$$

where W is a constant of the motion.

Substituting (29) into (27):

$$0 = c^2 Z - \frac{1}{Z}\left(\frac{dr}{dt}\right)^2 - \frac{Z^2W^2}{r^2}$$

$$\text{or} \quad \frac{dr}{dt} = cZ\left(1 - \frac{W^2 Z}{c^2 r^2}\right)^{\frac{1}{2}} \qquad (30)$$

At the point of nearest approach (N) we have $\frac{dr}{dt} = 0$ & $r = b$:

$$\frac{W^2 Z_b}{b^2} = c^2 \quad , \quad Z_b = 1 - \frac{2GM}{bc^2}$$

$$\text{or} \quad W^2 = \frac{b^2c^2}{Z_b} \qquad (3\phi)$$

So we can rewrite (30):

$$\frac{dr}{dt} = cZ\left(1 - \frac{b^2 Z}{r^2 Z_b}\right)^{\frac{1}{2}}. \qquad (31)$$

The ~~total~~ time from nearest approach (N) to the Earth (E) is then

$$t_{NE} = \int_{r_N}^{r_E} \frac{dr}{cZ\left(1 - \frac{b^2 Z}{r^2 Z_b}\right)^{\frac{1}{2}}}$$

& we write $Z = 1 - \varepsilon$ & $Z_b = 1 - \varepsilon_b$, where $\varepsilon = 2GM/rc^2$ & $\varepsilon_b = 2GM/bc^2$ are small parameters.

$$t_{NE} = \int_{r_N}^{r_E} \frac{r\,dr}{c}\,(1+\varepsilon+\ldots)\,\frac{1}{\left(r^2 - b^2\,\frac{z}{z_b}\right)^{\frac{1}{2}}}$$

& $$\frac{z}{z_b} = \frac{1-\varepsilon}{1-\varepsilon_b} = (1-\varepsilon)(1+\varepsilon_b+\ldots)$$
$$= 1 + \varepsilon_b - \varepsilon + \ldots$$

so $$t_{NE} = \int_{r_N}^{r_E} \frac{r\,dr}{c}\,(1+\varepsilon+\ldots)\,\frac{1}{\left[r^2 - b^2(1+\varepsilon_b-\varepsilon+\ldots)\right]^{\frac{1}{2}}}$$

$$= \int_{r_N}^{r_E} \frac{r\,dr}{c}\,(1+\varepsilon+\ldots)\,\frac{1}{(r^2-b^2)^{\frac{1}{2}}}\,\frac{1}{\left[1-\frac{b^2}{r^2-b^2}(\varepsilon_b-\varepsilon+\ldots)\right]^{\frac{1}{2}}}$$

$$= \int_{r_N}^{r_E} \frac{r\,dr}{c(r^2-b^2)^{\frac{1}{2}}}\,(1+\varepsilon+\ldots)\left(1+\frac{b^2}{2(r^2-b^2)}(\varepsilon_b-\varepsilon)+\ldots\right)$$

$$= \int_{r_N}^{r_E} \frac{r\,dr}{c(r^2-b^2)^{\frac{1}{2}}}\left(1+\varepsilon+\frac{b^2}{2(r^2-b^2)}(\varepsilon_b-\varepsilon)+\ldots\right)$$

$$= \int_{r_N}^{r_E} \frac{r\,dr}{c(r^2-b^2)^{\frac{1}{2}}}\left[1+\varepsilon+\frac{b^2}{\cancel{2}(r^2-b^2)}\,\frac{\cancel{2}GM}{c^2}\left(\frac{1}{b}-\frac{1}{r}\right)+\ldots\right]$$

$$= \int_{r_N}^{r_E} \frac{r\,dr}{c(r^2-b^2)^{\frac{1}{2}}}\left[1+\frac{2GM}{rc^2}+\frac{b^{\cancel{2}}}{(r+b)\cancel{(r-b)}}\cdot\frac{GM}{c^2}\,\frac{\cancel{r-b}}{\cancel{b}r}+\ldots\right]$$

$$= \int_{r_N}^{r_E} \frac{r\,dr}{c(r^2-b^2)^{\frac{1}{2}}}\left[1+\frac{2GM}{rc^2}+\frac{GMb}{r(r+b)c^2}+\ldots\right]$$

This is directly integrable, ignoring the HOTs:

$$t_{NE} = \frac{(r_E^2-b^2)^{\frac{1}{2}}}{c} + \frac{2GM}{c^3}\ln\left[\frac{r_E+(r_E^2-b^2)^{\frac{1}{2}}}{b}\right] + \frac{GM}{c^3}\left(\frac{r_E-b}{r_E+b}\right)^{\frac{1}{2}}$$

The 1st term is clearly the flat space-time answer, so the excess time due to the slowing of light is

$$\Delta t_{NE} = \frac{2GM}{c^3} \ln\left[\frac{r_E + (r_E^2 - b^2)^{\frac{1}{2}}}{b}\right] + \frac{GM}{c^3}$$

for $r_E \gg b$. Finally, the total excess time for the journey $E \to V$ & back is

$$\Delta t = 2(\Delta t_{NE} + \Delta t_{VN})$$

$$= \frac{4GM}{c^3} \ln\left[\frac{r_E + (r_E^2 - b^2)^{\frac{1}{2}}}{b} \cdot \frac{r_V + (r_V^2 - b^2)^{\frac{1}{2}}}{b}\right] + \frac{4GM}{c^3}$$

& in the limit $r_E, r_V \gg b$,

$$\Delta t \approx \frac{4GM}{c^3}\left[\ln\left(\frac{4 r_E r_V}{b^2}\right) + 1\right]$$

The round trip to Venus takes ≈ 1300s at superior conjunction, & the delay predicted by this equation is $\approx 200\mu s$. This is a small but measurable effect. ~~Shapiro measured the delay & compared it with the GR prediction using 600 d of observations of the using the a radio telescopes. There are~~ In 1971 Shapiro used 600 d of observations of the radar time delay for reflections of radio waves from Venus to test the GR prediction. The results are shown in the Figure from Kenyon - the solid curve is the GR prediction.

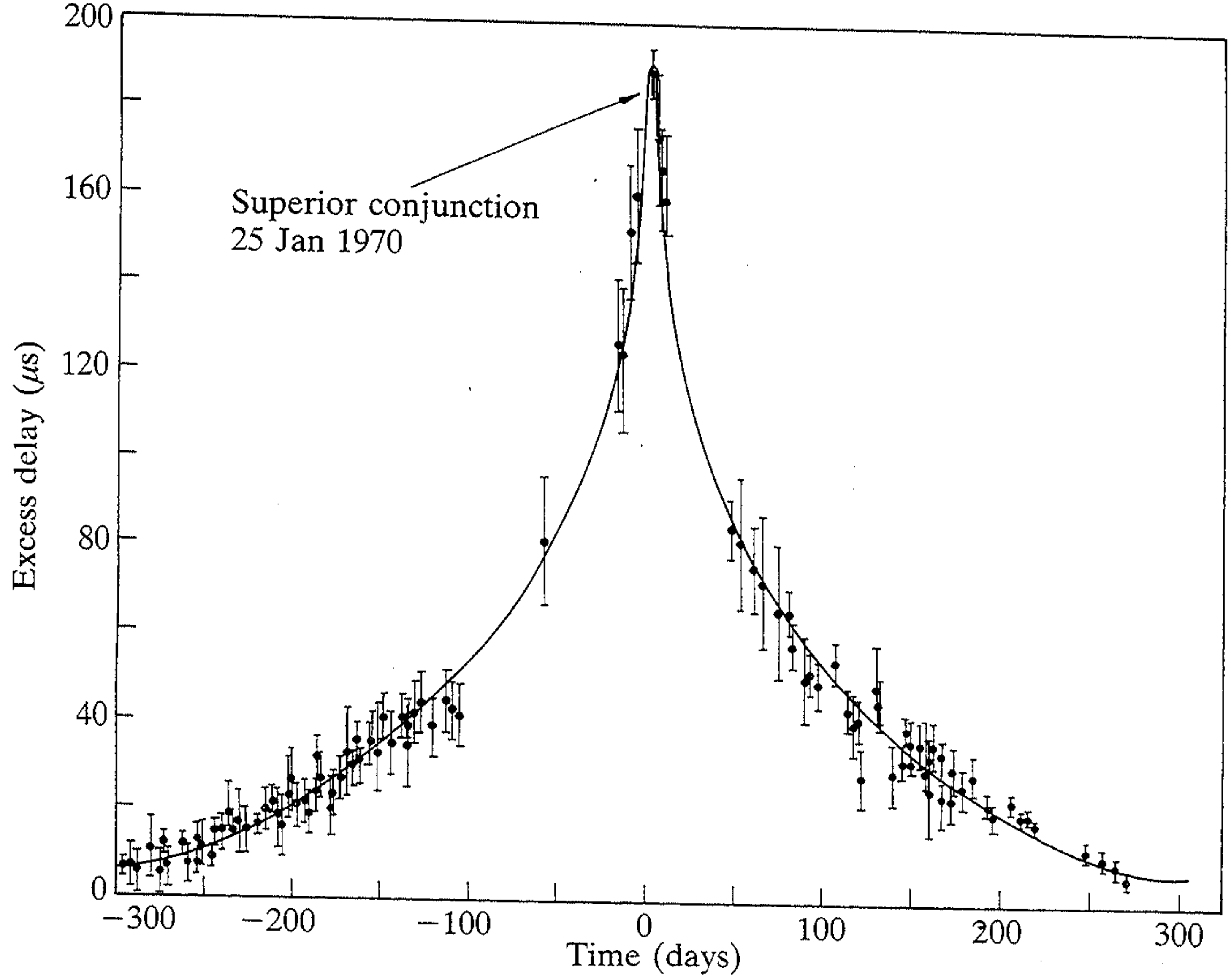


Fig. 8.5 A sample of post-fit residuals for Earth–Venus time-delay measurements: ———, prediction using GR (Shapiro *et al.* 1972). (Courtesy Professor Shapiro and *Physical Review Letters*, published by the American Physical Society.)

(A correction for the effect of the refractive index of the solar corona has been included in the results.) The largest contribution to the uncertainty in timing comes from the topography of Venus. A second experiment conducted in 1979 eliminated this problem by using the Viking Lander probe sitting on Mars to receive & retransmit the signal. The results of this experiment agreed with the GR prediction to ~1 in 10^3, making it one of the most stringent tests of the theory.

end of L18

Black Holes:

Recall the SM,

$$ds^2 = c^2\left(1-\frac{r_0}{r}\right)dt^2 - \frac{dr^2}{1-\frac{r_0}{r}} - r^2 d\Omega^2 .$$

The usual interpretation is that, for a body with $r > r_0$, this metric describes the space-time outside the body. For example, we have applied it repeatedly to the Sun. Inside the Sun a different solution (an "interior solution") with $T_{\mu\nu} \neq 0$ would be needed.

However, if we have a body with $r < r_0$, the "schwarzschild radius" then in principle the same metric applies. GR effects become important for distances $r \approx r_0$ from the body. The gravitational field of the body curves space-time close to the body so much that, for $r < r_0$, not even light can escape.

Interestingly, this result follows even from a naive Newtonian calculation (due to Rev. J. Mitchell, 1784). For a Newtonian gravitational field the escape velocity v is obtained from

$$\underbrace{\tfrac{1}{2}mv^2}_{\text{K.E.}} = \underbrace{\frac{GMm}{r}}_{\text{grav. pot. energy}}$$

$$\text{i.e. } v = \left(\frac{2GM}{r}\right)^{\frac{1}{2}}$$

Hence the escape velocity equals that of light when $r = \frac{2GM}{c^2} = r_0$. Although this calculation gives the correct answer, the details are wrong: in the Newtonian case the particle rises & then falls back. A photon within r_0 never begins to rise, according to GR.

Returning to the metric, for light moving radially in the equator we have $ds^2 = d\Omega^2 = 0$

$$\text{i.e. } c^2 dt^2 \left(1 - \frac{r_0}{r}\right) = \frac{dr^2}{1 - r_0/r}$$

$$\text{i.e. } \quad c\,dt = \frac{dr}{1 - r_0/r}$$

so for a photon released at $r = r_0 + \varepsilon$ the photon takes a longer & longer time to reach the external observer as $\varepsilon \to 0$. Photons released from within the Schwarzschild radius never escape: the surface $r = r_0$ is called the event horizon. A star that shrinks within its event horizon becomes invisible. It ~~turns out~~ is believed & that an object ~~that~~ which collapses to a radius $r < r_0$ is unable to come to equilibrium & continues to collapse, forming a <u>black hole</u>, which ~~is considered here to be~~ in GR is a point mass, described everywhere by the SM.

The form of ① suggests that something strange happens at $r = r_0$. In fact nothing does, & the ~~apparent~~ 'singularities' in the metric is a result of trying to use a co-ordinate system appropriate for flat space-time to describe a highly curved space-time.

To see this, consider what happens to a space probe falling into a black hole. We have already derived the orbital equation for the SM (for $\theta = \frac{\pi}{2}$)

Eq ⑦ : $$\frac{1}{2} m_0 \left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2} m_0 r^2 \left(\frac{d\phi}{d\tau}\right)^2 Z - \frac{GMm_0}{r} = T$$

choosing $\phi = 0$:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} = \frac{T}{m_0}$$

The quantity T is a constant. If we start the body from $r = \infty$ with zero velocity, then $T = 0$, &

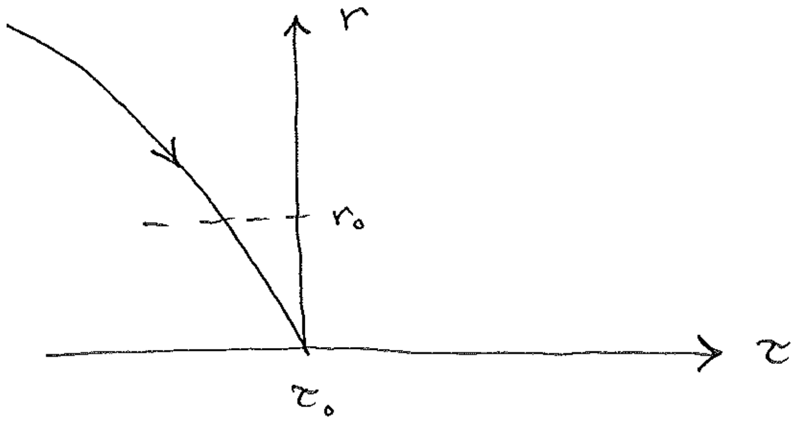
$$\frac{dr}{d\tau} = \pm\left(\frac{2GM}{r}\right)^{\frac{1}{2}}$$

For inward motion the minus sign is appropriate, & integrating:

$$\frac{2}{3} r^{3/2} + C = -(c^2 r_0)^{\frac{1}{2}} \tau$$

Taking $\tau = \tau_0$ at $r = 0$

$$\tau = \tau_0 - \frac{2r_0}{3c}\left(\frac{r}{r_0}\right)^{3/2}$$



Hence the probe falls into the BH in a finite proper time, & nothing remarkable happens at $r=r_0$. (Of course, sooner or later the craft & any ~~its~~ occupants would be torn apart by the tremendous tidal forces, ~~but this may happen before or after passing~~ ~~r_0~~ & the occupants are unlikely to be comfortable close to or within r_0.)

Next consider the journey as observed by a remote observer. Recall from our discussion of Schwarzschild orbits the equation

(4b) ? or (5)

$$2Zc\frac{dt}{d\tau} = \text{const}, \quad Z = 1-\frac{r_0}{r}$$

i.e. $$\left(1-\frac{r_0}{r}\right)\frac{dt}{d\tau} = K.$$

For a probe initially at rest far from the BH, $r\to\infty$ & $dt = d\tau \Rightarrow K=1$, so

$$\left(1-\frac{r_0}{r}\right)dt = d\tau$$

Using our previous result $dr/d\tau = -\left(\frac{2GM}{r}\right)^{\frac{1}{2}}$

$$c\left(1-\frac{r_0}{r}\right)dt = -dr\left(\frac{r}{r_0}\right)^{\frac{1}{2}}$$

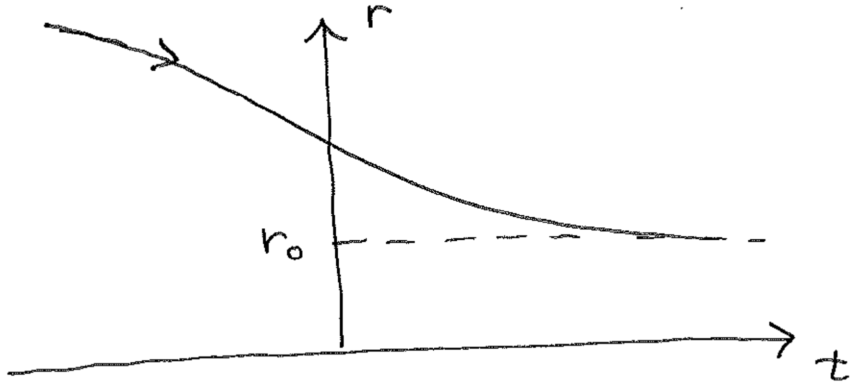
i.e. $$c\,dt = \frac{-\left(\frac{r}{r_0}\right)^{\frac{1}{2}}}{1-r_0/r}\,dr$$

$$= \frac{-1}{r_0^{\frac{1}{2}}}\,\frac{r^{3/2}}{r-r_0}\,dr$$

which can be integrated:

$$t = B + \frac{r_0}{c}\left[-\frac{2}{3}\left(\frac{r}{r_0}\right)^{3/2} - 2\left(\frac{r}{r_0}\right)^{\frac{1}{2}} + \ln\left|\frac{(r/r_0)^{\frac{1}{2}}+1}{(r/r_0)^{\frac{1}{2}}-1}\right|\right]$$

From this equation it is clear that $t \to \infty$ as $r \to r_0$



Hence to a remote observer the probe never crosses the event horizon! (Hovers there!) In accordance with our treatment of red shift early in the course, the light from the probe also becomes more & more heavily red-shifted as $r \to r_0$. Hence the probe becomes dimmer & eventually vanishes from view.

end of L19

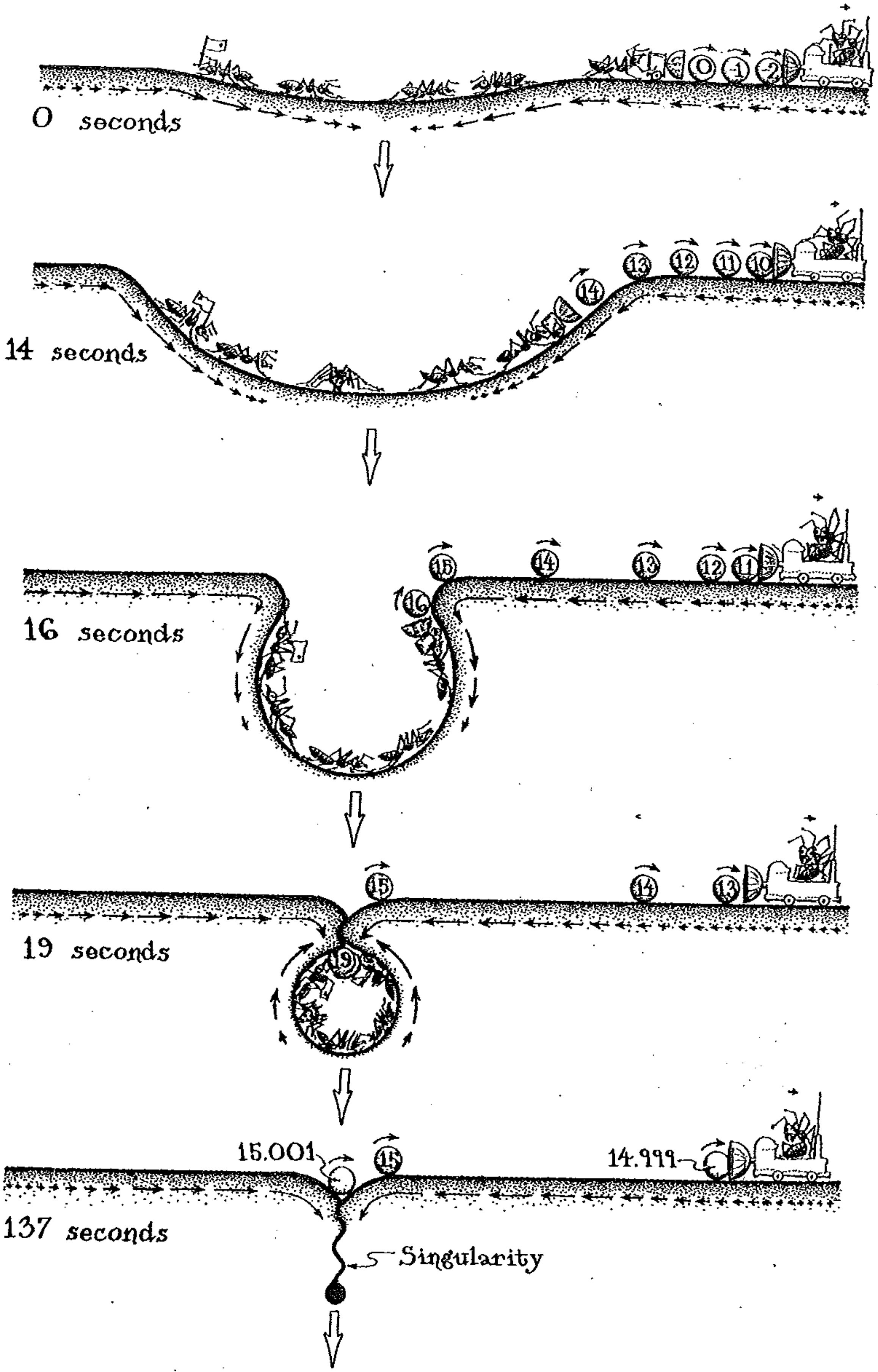
A fanciful illustration of these effects is provided by Kip Thorne's ant diagram. The remote observer never sees the probe fall into the BH because photons cannot reach him once the craft crosses the event horizon.

The difference in behaviour of t & τ arises because of the "co-ordinate singularity" in the metric. We have:

$$g_{00} = 1 - \frac{r_0}{r} \qquad g_{11} = -\left(1 - \frac{r_0}{r}\right)^{-1}$$

The signs are as follows:

	$r > r_0$	$r < r_0$
g_{00}	+	−
g_{11}	−	+



6.6 Collapsing rubber membrane populated by ants provides a fanciful analogue of the gravitational implosion of a star to form a black hole. [Adapted from Thorne (1967).]

FROM KIP THORNE, "BLACK HOLES & TIME WARPS: EINSTEIN'S OUTRAGEOUS LEGACY"

Consider a small change in t at constant r: $ds^2 = c^2 d\tau^2 = g_{00}\, dt^2$

$$\text{so:} \quad ds^2 \begin{cases} < 0 & \text{for } r < r_0 \\ > 0 & \text{for } r > r_0 \end{cases}$$

& the separation in co-ordinate time becomes space-like inside the Schwarzschild radius. The curvature of space-time is so severe that if we insist on using co-ordinates appropriate for flat space-time we find that the roles of time & space in the metric interchange.

This co-ordinate singularity can be removed by a suitable co-ordinate transformation. The Eddington-Finkelstein co-ordinates are suitable for the vicinity of a black hole:

$$\tilde{t} = t + \frac{r_0}{c} \ln\left|\frac{r}{r_0} - 1\right|$$

$$\text{i.e.} \quad d\tilde{t} = dt + \frac{dr}{c\left(\frac{r}{r_0} - 1\right)}$$

Substituting this into the SM leads to (exercise):

$$ds^2 = \left(1 - \frac{r_0}{r}\right) c^2 d\tilde{t}^2 - 2c\,dr\,d\tilde{t}\,\frac{r_0}{r} - \left(1 + \frac{r_0}{r}\right) dr^2 - r^2 d\Omega^2$$

(which no longer has a singularity), i.e. g_{11} is well behaved.

The path of a light ray in the equator is

$$0 = \left(1-\frac{r_0}{r}\right)c^2 d\tilde{t}^2 - 2c\frac{r_0}{r}\,dr\,d\tilde{t} - \left(1+\frac{r_0}{r}\right)dr^2$$

& setting $w = \dfrac{dr}{d\tilde{t}}$

$$0 = \left(1-\frac{r_0}{r}\right)c^2 - 2c\frac{r_0}{r}w - \left(1+\frac{r_0}{r}\right)w^2$$

which is a quadratic in w with solution (exercise):

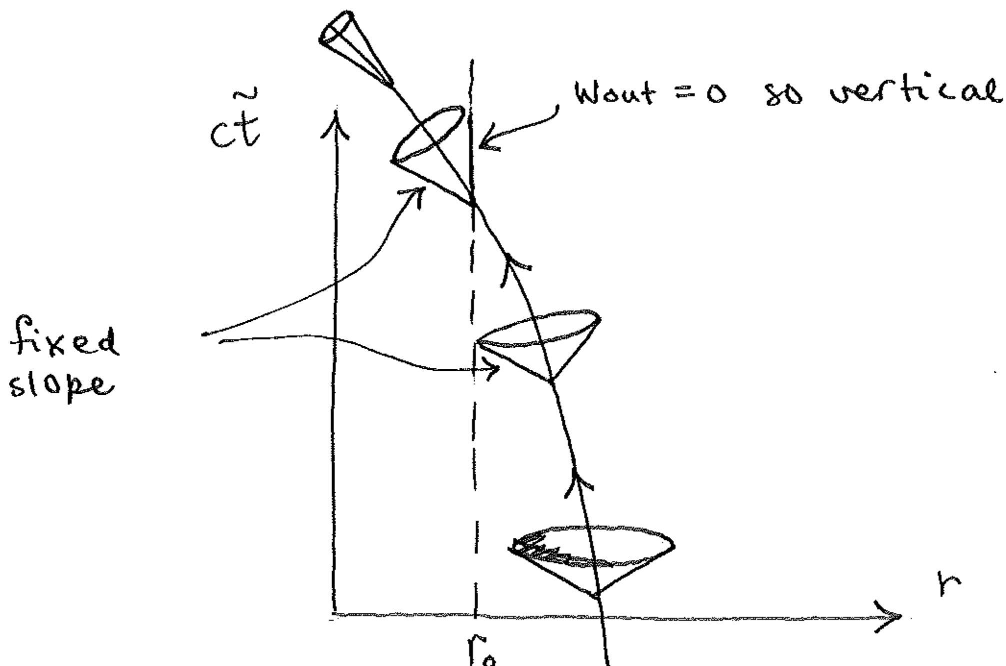
$$w = -c, \quad c \cdot \frac{1 - r_0/r}{1 + r_0/r}.$$

These solutions describe ingoing & outgoing light rays respectively. We see that these co-ordinates make the co-ordinate velocity of the ingoing light a constant*. In the absence of the BH ($r_0 = 0$)

$$w = \pm c, \text{ as expected.}$$

*in fact this motivated the co-ordinate choice

The space-time diagram of the infalling probe, in these co-ordinates is:



close to the hole, the "future lies inward", i.e. the light cones face in. However powerful the rocket on the probe the trajectory will lie inside the light cone, & the fate of the occupants is sealed.

Conditions at the centre of a BH ($r=0$) are a matter of speculation. Obviously we cannot observe this region. GR predicts ∞ curvature, Riemann curvature tensor diverges but it is not -exercise! known whether such a physical singularity can exist. Quantum mechanics must become important on small scales close to $r=0$, but to date there has been no completely successful synthesis of GR & QM.

Rotating black-holes:

No information can be ascertained from within the event horizon of a BH. So what can we "know" about these objects? For the BHs considered so far, clearly the mass, M, can be determined (from particle/light trajectories or orbits). In fact for non-rotating, charge-neutral BHs this is the only property of the hole, according to Steven Hawking & others.

In general there are 3 properties: mass M, total charge Q & angular momentum J. For a rotating (but charge-neutral) BH the Kerr metric applies. It is another exact solution of the Einstein equations. We will not discuss the Kerr metric in detail, but will mention a few features.

There are two event horizons, (surfaces from which light can't emerge) which are both smaller than in the non-rotating case. They are obtained as solutions to

$$r^2 - r r_0 + a^2 = 0,$$

& r_0 is the Schw. rad.

where $a = J/Mc$ describes the rotation. ~~The~~ ~~solutions~~ to this quadratic are

$$r_\pm = \frac{r_0}{2} \pm \left[\left(\frac{r_0}{2}\right)^2 - a^2\right]^{\frac{1}{2}}.$$

For $a > r_0/2$ there are no event horizons, &, in principle, it is possible to observe the singularity in space-time (for a rotating black hole there is a disk singularity). It is controversial as to whether this ~~is possible~~ can actually happen: Penrose has suggested the "cosmic censorship" hypothesis, whereby singularities are always hidden behind event horizons.

There is also an important spheroidal surface called the ergosphere, defined by

$$r_{erg}(\theta) = \frac{r_0}{2} + \left[\left(\frac{r_0}{2}\right)^2 - a^2\cos^2\theta\right]^{\frac{1}{2}}$$

At this radius a particle orbiting the BH in a direction contrary to the direction of rotation of the hole would need to move at c to remain in equilibrium. Hence within $r_{erg}(\theta)$ it is not possible to continuously orbit in a contrary direction. This surface touches r_+ at the poles, where the effect of rotation vanishes.

Formation of BHs

Massive stars ($\geqslant 20 M_\odot$) end their lives when they exhaust their thermonuclear fuel. At this point they have iron cores, because Fe nuclei have the largest binding energy per nucleon (so neither fission nor fusion can release further energy). The iron core may have a density $\sim 10^{11}$ kg m^{-3} & a temperature $\sim 10^9$ K. The only sufficient support against gravitational collapse is electron degeneracy pressure (two e⁻s cannot occupy the same quantum state, according to the Pauli exclusion principle). However, Chandrasekhar (1931) showed that, if the core mass exceeds

$\approx 1.4 M_\odot$, even this pressure is insufficient to resist gravitational collapse. Stellar cores of mass $< 1.4 M_\odot$ are believed to be stable, & to form neutron stars (following an explosive phase in the stellar evolution, a supernova). Stellar cores of greater mass are believed to collapse to form BHs.

Comment

This is the modern view. Interestingly, the possibility of BHs was considered & rejected by Einstein & Eddington in the 1920s & 1930s. Einstein even presented a "proof" of the impossibility of forming BHs in 1939. His argument rested on the point that a gas of particles in equilibrium with a radius $r < \frac{3}{2} r_0$ would have to be moving faster than c. Hence gave an argument that ~~he reasoned~~ the gas would never get to this radius. The flaw is that equilibrium is assumed: the modern view is of gravitational collapse, which is not an equilibrium.

Astrophysical BH candidates:

Isolated BH's offer limited prospects for observation, but BHs in binary systems offer excellent prospects. (For $r > 3r_0$ there are stable orbits around BHs.) There are several promising astrophysical BH candidates – one of the suggested essay topics pursues this point.

HAWKING RADIATION

Stephen?

In 1974 Steven Hawking arrived at the surprising result that, due to a quantum effect, BHs should radiate. The radiation originates from outside the event horizon.

In the quantum picture the vacuum is in a state of constant activity due to the continual creation & annihilation of particle/anti-particle pairs. For example, a pair of photons (the photon is its own anti-particle) can be created close to a BH, with four-momenta $(\Delta E, -\Delta \underset{\sim}{p})$ & $(-\Delta E, \Delta \underset{\sim}{p})$, where $\Delta E = (\Delta p)c$. The negative energy is physically unacceptable, but it can exist for a time Δt where

$$\Delta t \sim \frac{\hbar}{\Delta E},$$

according to the Uncertainty Principle.

For some directions of emission the negative energy photon will cross the event horizon & be lost. The positive energy partner may escape from the BH, in which case it is an example of "Hawking radiation". The temperature of the radiation may be estimated as follows.

The position of a photon emitted near the event horizon can be considered to be uncertain to $\sim r_o$, so

$$\Delta p \sim \frac{\hbar}{r_o},$$

according to the Uncertainty Principle. But we also have

$$\Delta p \approx \frac{k_B T}{c}$$

where T is the photon temperature. Hence

$$\frac{\hbar}{r_o} \approx \frac{k_B T}{c} \Rightarrow \boxed{T \approx \frac{\hbar c^3}{2 k_B G M}}$$

Obviously this is a crude derivation. The exact expression obtained by Hawking was

$$T = \frac{\hbar c^3}{8\pi k_B G M}$$

which differs from the previous only by a factor of 4π! which in astrophysical situations is a great success. Putting in the numbers gives

$$T = \frac{6 \times 10^{-8}}{(M/M_\odot)} \text{ K}$$

where $M_\odot$ is a solar mass. So this is a very low temperature, & H.R. is almost a non-event.

The rate of loss of energy by Hawking radiation is

$$\frac{d(Mc^2)}{dt} = \underbrace{\sigma T^4}_{\text{Stefan-Boltzmann law}} . \underbrace{A}_{\text{surface area}}$$

We have $A \sim r_o^2 \sim M^2$ & $T \sim M^{-1}$. Hence

$$\frac{dM}{dt} \sim M^{-2}$$

& the hole "evaporates" more rapidly with decreasing mass. Integrating this gives

$$\int M^2 dM \sim \int dt$$

which leads to the scaling for the lifetime of the hole

$$\tau \sim M^3$$

i.e. the bigger the black hole, the longer it lives (because it is cooler, & so radiates less). A more careful derivation gives

$$T \approx 10^{10} \text{ years} \cdot \left(\frac{M}{10^{12}\,\text{kg}} \right)^3 .$$

Since a solar mass is 2×10^{30} kg, it follows that stellar mass-size black holes are essentially unaffected by this radiation (their lifetime is >> the accepted age of the universe, ≈ 15 Gy). However, small black holes may have formed early in the history of the Universe & subsequently have evaporated. But this is wild speculation!

GRAVITATIONAL RADIATION:

The initial motivation for seeking a relativistic theory of gravity was that, according to Newton's law, gravitational influences propagate instantaneously. How do they propagate in GR?

We can gain some insight by linearizing the field equations. As before, we write

NB h is symmetric

$$g_{\beta\delta} = \eta_{\beta\delta} + h_{\beta\delta},$$

where $\eta_{\beta\delta}$ is the Minkowski metric, & $h_{\beta\delta}$ is an assumed small correction. First we calculate the metric connections:

$$2\Gamma^{\alpha}{}_{\beta\delta} = 2g^{\alpha\nu}\Gamma_{\nu\beta\delta} = g^{\alpha\nu}(g_{\beta\nu,\delta} - g_{\delta\beta,\nu} + g_{\nu\delta,\beta})$$

$$\text{so} \quad 2\Gamma^{\alpha\,(1)}_{\beta\delta} = g^{\alpha\nu}(h_{\beta\nu,\delta} - h_{\delta\beta,\nu} + h_{\nu\delta,\beta})$$

to first order in h (denoted "(1)"), where we have used the fact that the derivatives of $\eta_{\beta\delta}$ are zero.

Next, recall the Riemann curvature tensor:

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\mu}{}_{\sigma\gamma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\mu}{}_{\sigma\delta}\Gamma^{\sigma}{}_{\beta\gamma}.$$

The terms with products in Γ will be second order in h. Hence

$$R^{\alpha\,(1)}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta}$$

in our notation.

The Ricci tensor is, to 1^{st} order

$$R_{\beta\delta}^{(1)} = R^{\alpha\ (1)}_{\ \beta\alpha\delta}$$

$$= \Gamma^{\alpha}_{\beta\delta,\alpha} - \Gamma^{\alpha}_{\beta\alpha,\delta}$$

$$= \tfrac{1}{2} g^{\alpha\nu}\left(\cancel{h_{\beta\nu,\delta\alpha}} - h_{\delta\beta,\nu\alpha} + h_{\nu\delta,\beta\alpha}\right)$$

$$- \tfrac{1}{2} g^{\alpha\nu}\left(\cancel{h_{\beta\nu,\alpha\delta}} - h_{\alpha\beta,\nu\delta} + h_{\nu\alpha,\beta\delta}\right)$$

i.e. $$R_{\beta\delta}^{(1)} = \tfrac{1}{2} g^{\alpha\nu}\left(h_{\nu\delta,\beta\alpha} - h_{\delta\beta,\nu\alpha} + h_{\alpha\beta,\nu\delta} - h_{\alpha\nu,\beta\delta}\right)$$

So the Einstein equations, $R_{\beta\delta}^{(1)} = 0$ become

$$⊛ \quad \boxed{0 = h^{\alpha}{}_{\delta,\beta\alpha} - h_{\delta\beta,}{}^{\alpha}{}_{\alpha} + h_{\alpha\beta,}{}^{\alpha}{}_{\delta} - h^{\alpha}{}_{\alpha,\beta\delta}}$$

We can construct a solution to this equation as follows. If we require that

$$h^{\alpha}{}_{\alpha} = 0$$

(i.e. the sum of the diagonal terms, or the "trace" vanishes), & we also require that

$$h^{\alpha}{}_{\delta,\alpha} = 0$$

i.e. $h_{\alpha\beta}$ is divergenceless, then we also have that

$$h_{\alpha\beta,}{}^{\alpha} = 0$$

& then the 1^{st}, 3^{rd} & 4^{th} terms in ⊛ vanish, leaving the simplified equation

$$h_{\delta\beta}{}^{\alpha}{}_{\alpha} = 0$$

i.e. $$\eta^{\alpha\gamma} h_{\delta\beta,\gamma\alpha} = 0 \quad \text{(to order } h\text{)}$$

$$\text{i.e. } \eta^{\alpha\gamma} \frac{\partial^2 h_{\delta\beta}}{\partial x^\gamma \partial x^\alpha} = 0$$

$$\text{i.e. } -\frac{\partial^2 h_{\delta\beta}}{\partial (ct)^2} + \frac{\partial^2 h_{\delta\beta}}{\partial x^2} + \frac{\partial^2 h_{\delta\beta}}{\partial y^2} + \frac{\partial^2 h_{\delta\beta}}{\partial z^2} = 0$$

or $$\left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2\right) h_{\delta\beta} = 0,$$

which we recognise as the wave equation with wave speed c. So we have shown that the wave equation is a possible result of the linearised field equations, although it requires certain conditions to be imposed on $h_{\beta\delta}$. These conditions are "gauge conditions", analogous to the gauge choices of EM theory. They arise because of the arbitrariness of the choice of co-ordinates.

There is also sufficient freedom to impose another constraint: $h_{\alpha\alpha} = 0$

In any case the appearance of the wave equation, with wave speed c is an indication that in GR, gravitational influences propagate at c.

We can investigate the nature of (linear) gravity waves by adopting a plane wave as a trial solution:

$$h_{\beta\delta} = A_{\beta\delta} \exp(i k n_\alpha x^\alpha)$$

Substituting this into the wave equation:

$$\eta^{\alpha\gamma} \frac{\partial^2 h_{\beta\delta}}{\partial x^\gamma \partial x^\alpha} = 0$$

$$\Rightarrow \quad \eta^{\alpha\gamma}(ikn_\alpha)(ikn_\gamma)\, h_{\beta\delta} = 0$$

which will be satisfied if

$$\underline{\eta^{\alpha\gamma} n_\alpha n_\gamma = 0}$$

$$\left(\text{i.e.} \quad n^\gamma n_\gamma = 0 \qquad \text{omit} \right)$$

This will be satisfied if $n^\alpha = (1, \underset{\sim}{n})$, where $\underset{\sim}{n}$ is a unit vector. Next we assume the wave is propagating in the z-direction, & so $\underset{\sim}{n} = \underset{\sim}{\hat{z}}$.

One of our gauge conditions was

$$h_{\delta\beta,}{}^{\delta} = 0$$

$$\text{i.e.} \quad \eta^{\delta\alpha} \frac{\partial h_{\delta\beta}}{\partial x^\alpha} = 0$$

$$\text{i.e.} \quad \eta^{\delta\alpha}(ikn_\alpha)\, h_{\delta\beta} = 0$$

which will be satisfied if

$$n^\delta A_{\beta\delta} = 0,$$

or since the wave is in the z direction,

$$A_{\beta 0} + A_{\beta 3} = 0 \qquad (\text{for all } \beta)$$

Our extra gauge conditions was

$$h_{\alpha 0} = 0$$

$$\text{i.e.} \quad A_{\beta 0} = 0 \qquad (\text{for all } \beta)$$

so combining this with the previous,

$$A_{\beta 3} = 0$$

Also, $h_{\alpha\beta}$ is symmetric, so

$$A_{0\beta} = A_{\beta 0} = 0$$

& $$A_{3\beta} = A_{\beta 3} = 0.$$

So we have

$$[A_{\beta\delta}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

but the symmetry of $h_{\beta\delta}$ & the traceless gauge condition $\Rightarrow$ $b = c$ & $a = -d$, so

$$[A_{\beta\delta}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A general solution can be written in terms of two "orthogonal" states:

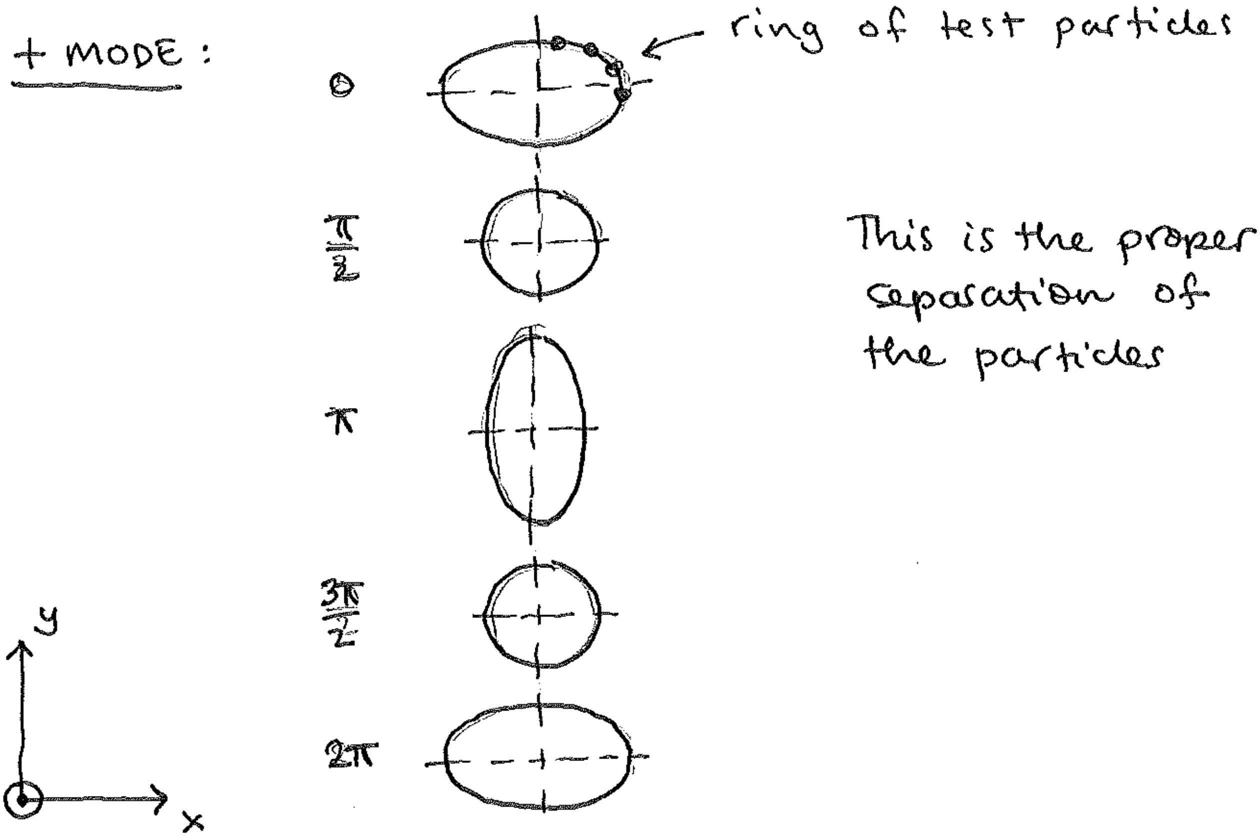
$$h_{\beta\delta} = h_+ (e_+)_{\beta\delta} \cos(\omega t - kz)$$

$$h_{\beta\delta} = h_\times (e_\times)_{\beta\delta} \cos(\omega t - kz + \varphi)$$

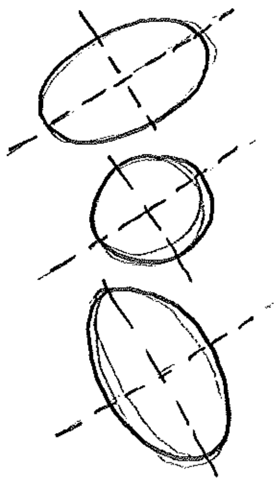
where φ is an arbitrary phase difference, &

$$[(e_+)_{\beta\delta}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [(e_\times)_{\beta\delta}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By considering the metric corresponding to $h_{\beta\delta}$ it is straightforward to obtain the effects of the passage of plane gravitational waves on test particles. For the $+$ mode ~~the particles move in the x & y directions~~:



For the $\times$ mode the motion ~~has axes~~ is



etc.

This motion is quadrupolar (it involves two ⊥ symmetry axes, in both + & × cases). This should be contrasted with EM waves, which are predominantly dipolar.

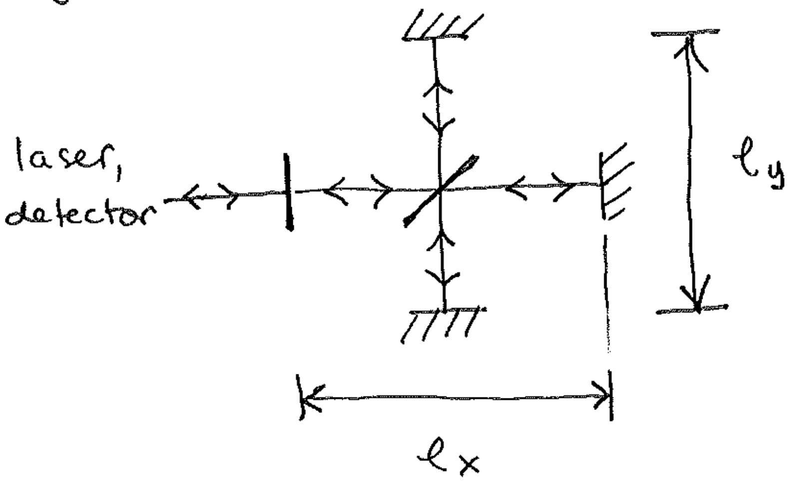
~~It is also~~

In general there are mixtures of the two polarization states (modes). In particular, adding mixtures with ~~equal amplitudes~~ ~~but~~ a phase difference $\Phi = \pm\pi/2$ gives circularly polarized radiation (the ellipse of test particles rotates).

<u>Detecting gravitational waves</u>:

One method of direct detection involves measuring relative displacements & their changes with time

e.g. with a Michelson interferometer



Measurement of interference effects ~~the fringes~~ can reveal changes in l_x & l_y.

There are also many other schemes, & there is an essay topic on this.

The problem is that gravitational waves are weak. The magnitude of the h terms defines the fractional change in proper separations. The sources of gravitational waves that some experimentalists hope to detect are supernovae ~~at the centre~~ in of our galaxy. The asymmetric collapse of a stellar core at the centre of the galaxy is estimated to produce a "strain" at the Earth

$$|h| \sim 10^{-18} \quad \text{(somewhat uncertain).}$$

Such events occur ~ once every 30 years in our galaxy. In principle modern detectors could detect the radiation from such an event.

However, to date there has been no accepted direct detection of gravitational radiation.

There has been an indirect detection, via the observed slowing of the binary pulsar 1913+16, which is believed to be caused by a loss of energy due to the gravitational radiation produced by the two orbiting compact objects. The observed rate of slowing agrees with the GR prediction (which is based on independently determined orbital parameters) to a ~~few~~ 1 part in 500 (& within the uncertainties), which leads most people to believe gravity waves exist.

PHYS378 General Relativity and Cosmology 2000
Assignment 2 due Friday September 8

1. The following steps establish that the covariant derivative transforms tensorially.

   (a) Start with the Fundamental Theorem of Riemannian geometry in co-ordinates x'^{μ}:

$$\Gamma'_{\nu\mu\sigma} = \frac{1}{2}\left(\frac{\partial g'_{\mu\nu}}{\partial x'^{\sigma}} - \frac{\partial g'_{\sigma\mu}}{\partial x'^{\nu}} + \frac{\partial g'_{\nu\sigma}}{\partial x'^{\mu}}\right)$$

   Replace the primed metric tensors on the RHS by unprimed ones, using the transformation rules $[g'_{\mu\nu} = (\partial x^{\alpha}/\partial x'^{\mu})(\partial x^{\beta}/\partial x'^{\nu})g_{\alpha\beta}$, etc.]. Expand the derivatives and use the symmetry of the metric tensor and relabelling of indices to arrive at

$$\Gamma'_{\nu\mu\sigma} = \frac{\partial^2 x^{\alpha}}{\partial x'^{\sigma}\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}g_{\alpha\beta} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\rho}}{\partial x'^{\sigma}}\Gamma_{\beta\alpha\rho}. \qquad (1)$$ 2

   (b) Multiply (1) by $g'^{\nu\tau}\partial x^{\epsilon}/\partial x'^{\tau}$ and simplify terms to obtain

$$\frac{\partial^2 x^{\epsilon}}{\partial x'^{\sigma}\partial x'^{\mu}} = \frac{\partial x^{\epsilon}}{\partial x'^{\tau}}\Gamma'^{\tau}{}_{\mu\sigma} - \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\rho}}{\partial x'^{\sigma}}\Gamma^{\epsilon}{}_{\alpha\rho}. \qquad (2)$$ 2

   (c) Next, recall the transformation rule for a derivative:

6

$$\begin{aligned}\frac{\partial A'_{\mu}}{\partial x'^{\beta}} &= \frac{\partial}{\partial x'^{\beta}}\left(\frac{\partial x^{\nu}}{\partial x'^{\mu}}A_{\nu}\right)\\ &= \frac{\partial^2 x^{\nu}}{\partial x'^{\beta}\partial x'^{\mu}}A_{\nu} + \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial x^{\gamma}}{\partial x'^{\beta}}\frac{\partial A_{\nu}}{\partial x^{\gamma}}. \qquad (3)\end{aligned}$$ 2

   Use (2) to replace the second partial derivative in (3). Rearrange terms and use the definition of the covariant derivative to arrive at

$$A'_{\mu;\beta} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial x^{\gamma}}{\partial x'^{\beta}}A_{\nu;\gamma},$$

   i.e. the covariant derivative of a covariant vector transforms like a second-rank covariant tensor.

2. Starting from the definition of the Einstein tensor $G_{\mu\nu}$:

3

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$ 1, 2

   show that $G_{\mu\nu} = 0$ if and only if $R_{\mu\nu} = 0$.

3. Consider the 3-D space-time with metric

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = dt^2 - dz^2 - [a(t)]^{2}\, d\phi^2,$$

   where $a(t)$ is an increasing function of time. The spatial part of this metric looks like a cylinder that expands with time.

4

   (a) Find the non-zero components of the metric connections $\Gamma^{\alpha}{}_{\beta\gamma}$. 2

   (b) Find the non-zero components of the Riemann curvature tensor $R^{\alpha}{}_{\beta\gamma\delta}$. Show that the space is flat (i.e. the Riemann curvature tensor vanishes) if and only if $\dot{a}(t) = \text{const}$. 2

PHYS 378 GENERAL RELATIVITY & COSMOLOGY 2000

ASSIGNMENT 2 SOLUTIONS

1. (a). We have

$$\Gamma'_{\nu\mu\sigma} = \frac{1}{2}\left(\frac{\partial g'_{\mu\nu}}{\partial x'^{\sigma}} - \frac{\partial g'_{\sigma\mu}}{\partial x'^{\nu}} + \frac{\partial g'_{\nu\sigma}}{\partial x'^{\mu}} \right)$$

$$= \frac{1}{2}\Big[\frac{\partial}{\partial x'^{\sigma}}\left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \right) - \frac{\partial}{\partial x'^{\nu}}\left(\frac{\partial x^{\alpha}}{\partial x'^{\sigma}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} g_{\alpha\beta} \right) + \frac{\partial}{\partial x'^{\mu}}\left(\frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial x^{\beta}}{\partial x'^{\sigma}} g_{\alpha\beta} \right) \Big]$$

$$= \frac{1}{2}\Big[\frac{\partial^2 x^{\alpha}}{\partial x'^{\sigma}\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \;① + \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial^2 x^{\beta}}{\partial x'^{\sigma}\partial x'^{\nu}} g_{\alpha\beta} \;②$$

$$+ \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial g_{\alpha\beta}}{\partial x'^{\sigma}} \;③ - \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\sigma}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} g_{\alpha\beta} \;④$$

$$- \frac{\partial x^{\alpha}}{\partial x'^{\sigma}} \frac{\partial^2 x^{\beta}}{\partial x'^{\nu}\partial x'^{\mu}} g_{\alpha\beta} \;⑤ - \frac{\partial x^{\alpha}}{\partial x'^{\sigma}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial g_{\alpha\beta}}{\partial x'^{\nu}} \;⑥$$

$$+ \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}} \frac{\partial x^{\beta}}{\partial x'^{\sigma}} g_{\alpha\beta} \;⑦ + \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial^2 x^{\beta}}{\partial x'^{\mu}\partial x'^{\sigma}} g_{\alpha\beta} \;⑧$$

$$+ \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial x^{\beta}}{\partial x'^{\sigma}} \frac{\partial g_{\alpha\beta}}{\partial x'^{\mu}} \;⑨ \Big]$$

Term ② cancels with term ④ ($g_{\alpha\beta}$ is symmetric)
& term ⑤ cancels with term ⑦ (ditto), leaving:

$$\Gamma'_{\nu\mu\sigma} = \frac{\partial^2 x^{\alpha}}{\partial x'^{\sigma}\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} + \frac{1}{2} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial g_{\alpha\beta}}{\partial x'^{\sigma}} \;①$$

$$- \frac{1}{2} \frac{\partial x^{\alpha}}{\partial x'^{\sigma}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial g_{\alpha\beta}}{\partial x'^{\nu}} \;② + \frac{1}{2} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial x^{\beta}}{\partial x'^{\sigma}} \frac{\partial g_{\alpha\beta}}{\partial x'^{\mu}} \;③$$

The derivatives of $g_{\alpha\beta}$ can be converted to derivatives w.r.t. unprimed co-ordinates:

$$\Gamma'_{\nu\mu\sigma} = \frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

$$+ \frac{1}{2}\Big[\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\sigma}\frac{\partial g_{\alpha\beta}}{\partial x^\rho}$$

$$- \frac{\partial x^\alpha}{\partial x'^\sigma}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial g_{\alpha\beta}}{\partial x^\rho}$$

$$+ \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial g_{\alpha\beta}}{\partial x^\rho}\Big]$$

Next we can relabel the dummy indices (α, β, ρ) in the second & third bracketted terms to give

$$\Gamma'_{\nu\mu\sigma} = \frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

$$+ \frac{1}{2}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\sigma}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\rho} - \frac{\partial g_{\rho\alpha}}{\partial x^\beta} + \frac{\partial g_{\beta\rho}}{\partial x^\alpha}\right)$$

i.e. $$\boxed{\Gamma'_{\nu\mu\sigma} = \frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} + \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\sigma}\Gamma_{\beta\alpha\rho},}$$

where $\Gamma_{\beta\alpha\rho} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\rho} - \frac{\partial g_{\rho\alpha}}{\partial x^\beta} + \frac{\partial g_{\beta\rho}}{\partial x^\alpha}\right)$ by the Fundamental theorem of Riemannian geometry.

(b). Using the suggested trick of multiplying by $g'^{\nu\tau}\partial x^\varepsilon/\partial x'^\tau$ we have

$$\frac{\partial x^\varepsilon}{\partial x'^\tau} g'^{\nu\tau}\Gamma'_{\nu\mu\sigma} = \left(g'^{\nu\tau}\frac{\partial x^\varepsilon}{\partial x'^\tau}\frac{\partial x^\beta}{\partial x'^\nu}\right) g_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\mu}$$

$$+ \left(g'^{\nu\tau}\frac{\partial x^\varepsilon}{\partial x'^\tau}\frac{\partial x^\beta}{\partial x'^\nu}\right)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\sigma}\Gamma_{\beta\alpha\rho}$$

From the rules for transformation of co-ordinates we have $g^{\beta\varepsilon} = \frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\varepsilon}}{\partial x'^{\tau}} g'^{\nu\tau}$

& hence we have

$$\frac{\partial x^{\varepsilon}}{\partial x'^{\tau}} \Gamma'^{\tau}_{\mu\sigma} = g^{\beta\varepsilon} g_{\alpha\beta} \frac{\partial^2 x^{\alpha}}{\partial x'^{\sigma}\partial x'^{\mu}}$$

$$+ g^{\beta\varepsilon} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\sigma}} \Gamma_{\beta\alpha\rho} .$$

$$= \delta^{\varepsilon}_{\alpha} \frac{\partial^2 x^{\alpha}}{\partial x'^{\sigma}\partial x'^{\mu}} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\sigma}} \Gamma^{\varepsilon}_{\alpha\rho}$$

$$= \frac{\partial^2 x^{\varepsilon}}{\partial x'^{\sigma}\partial x'^{\mu}} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\sigma}} \Gamma^{\varepsilon}_{\alpha\rho}$$

or $$\boxed{\frac{\partial^2 x^{\varepsilon}}{\partial x'^{\sigma}\partial x'^{\mu}} = \frac{\partial x^{\varepsilon}}{\partial x'^{\tau}} \Gamma'^{\tau}_{\mu\sigma} - \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\sigma}} \Gamma^{\varepsilon}_{\alpha\rho},}$$

as required.

(c). The rule for transforming a derivative is

$$\frac{\partial A'_{\mu}}{\partial x'^{\beta}} = \frac{\partial^2 x^{\nu}}{\partial x'^{\beta}\partial x'^{\mu}} A_{\nu} + \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\beta}} \frac{\partial A_{\nu}}{\partial x^{\gamma}}$$

$$= \left(\frac{\partial x^{\nu}}{\partial x'^{\tau}} \Gamma'^{\tau}_{\mu\beta} - \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\beta}} \Gamma^{\nu}_{\alpha\rho} \right) A_{\nu}$$

$$+ \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\beta}} \frac{\partial A_{\nu}}{\partial x^{\gamma}},$$

using our expression for the 2nd derivative obtained in (b).

Rearranging:

$$\frac{\partial A'_\mu}{\partial x'^\beta} - \Gamma'^\tau{}_{\mu\beta}\left(\frac{\partial x^\nu}{\partial x'^\tau} A_\nu\right)$$
$$= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\beta}\frac{\partial A_\nu}{\partial x^\gamma} - \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\beta}\Gamma^\nu{}_{\alpha\rho} A_\nu$$

i.e. $$\frac{\partial A'_\mu}{\partial x'^\beta} - \Gamma'^\tau{}_{\mu\beta} A'_\tau = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\beta}\left(\frac{\partial A_\nu}{\partial x^\gamma} - \Gamma^\sigma{}_{\nu\gamma} A_\sigma\right)$$

where the dummy indices in the last term have been relabelled. Recalling the definition of the covariant derivative it is clear that we have established that

$$\boxed{A'_{\mu;\beta} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\beta} A_{\nu;\gamma} ,}$$

as required.

"if and only if"

2. We are required to prove $G_{\mu\nu} = 0 \Leftrightarrow R_{\mu\nu} = 0$

First note that the reverse direction is trivial. If $R_{\mu\nu} = 0$ then $R = R^\alpha{}_\alpha = g^{\alpha\mu} R_{\mu\alpha} = 0$, & hence $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$.

For the forward direction, assuming $G_{\mu\nu} = 0$ gives

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \qquad (*)$$

Multiplying by $g^{\alpha\mu}$:

$$R^\alpha{}_\nu = \frac{1}{2} g^\alpha{}_\nu R$$

From lectures we have $g^\alpha{}_\nu = \delta^\alpha{}_\nu$. Hence we

have
$$R^{\alpha}{}_{\nu} = \tfrac{1}{2}\delta^{\alpha}{}_{\nu} R.$$
Setting $\nu = \alpha$ gives
$$R^{\alpha}{}_{\alpha} = \tfrac{1}{2}\delta^{\alpha}{}_{\alpha} R$$
i.e. $R = 2R$ (recall $\delta^{\alpha}{}_{\alpha} = 4$)

i.e. $R = 0$

& substituting this back into ⊛ gives $R_{\mu\nu} = 0$, for all μ & ν, as required.

3. By inspection the components of the metric tensor and its inverse are

$$g_{tt} = 1 \qquad g^{tt} = 1$$
$$g_{zz} = -1 \qquad g^{zz} = -1$$
$$g_{\phi\phi} = -[a(t)]^2 \qquad g^{\phi\phi} = -[a(t)]^{-2}.$$

(a). From the Fundamental theorem,
$$\Gamma^{\alpha}{}_{\mu\sigma} = \tfrac{1}{2} g^{\alpha\nu}(g_{\mu\nu,\sigma} - g_{\sigma\mu,\nu} + g_{\nu\sigma,\mu}).$$
the metric connections $\Gamma^{\alpha}{}_{\mu\sigma}$ are symmetric in μ & σ. Hence, for fixed α there are only six independent choices of $\mu\sigma$, which we can take to be $\mu\sigma = tt, tz, t\phi, \phi\phi, \phi z, zz$. Since there are three choices of α, there are then 18 components of $\Gamma^{\alpha}{}_{\mu\sigma}$ that need to be evaluated.

Evaluating the first of these:
$$\Gamma^{t}{}_{tt} = \tfrac{1}{2} g^{t\nu}(g_{t\nu,t} - g_{tt,\nu} + g_{\nu t,t})$$
$$= \tfrac{1}{2} g^{tt}(g_{tt,t} - g_{tt,t} + g_{tt,t}),$$
since only the diagonal elements are non-zero,

i.e. $\Gamma^t{}_{tt} = 0$.

Similarly tedious calculation gives

$$\Gamma^t{}_{tz} = 0 \qquad \Gamma^z{}_{\phi z} = 0$$
$$\Gamma^t{}_{t\phi} = 0 \qquad \Gamma^\phi{}_{tt} = 0$$
$$\Gamma^t{}_{\phi\phi} = +a(t)\dot{a}(t) \qquad \Gamma^\phi{}_{tz} = 0$$
$$\Gamma^t{}_{\phi z} = 0 \qquad \Gamma^\phi{}_{t\phi} = \frac{\dot{a}(t)}{a(t)}$$
$$\Gamma^t{}_{zz} = 0 \qquad \Gamma^\phi{}_{\phi\phi} = 0$$
$$\Gamma^z{}_{tt} = 0 \qquad \Gamma^\phi{}_{zz} = 0$$
$$\Gamma^z{}_{tz} = 0 \qquad \Gamma^\phi{}_{\phi z} = 0$$
$$\Gamma^z{}_{t\phi} = 0$$
$$\Gamma^z{}_{\phi\phi} = 0$$
$$\Gamma^z{}_{zz} = 0$$

Hence the only non-zero components are

$$\Gamma^t{}_{\phi\phi} = +a(t)\,\dot{a}(t)$$

$$\& \quad \Gamma^\phi{}_{t\phi} = \Gamma^\phi{}_{\phi t} = \frac{\dot{a}(t)}{a(t)}.$$

(b). The Riemann curvature tensor is defined by

$$R^\mu{}_{\rho\beta\alpha} = \Gamma^\mu{}_{\rho\alpha,\beta} - \Gamma^\mu{}_{\rho\beta,\alpha} + \Gamma^\mu{}_{\sigma\beta}\Gamma^\sigma{}_{\rho\alpha} - \Gamma^\mu{}_{\sigma\alpha}\Gamma^\sigma{}_{\rho\beta}$$

First we note that only $\mu = t, \phi$ can lead to non-zero components, because only $\Gamma^t{}_{\phi\phi}$, $\Gamma^\phi{}_{t\phi}$ & $\Gamma^\phi{}_{\phi t}$ are non-zero. Next,

$$R^\mu{}_{\rho\beta\alpha} = -R^\mu{}_{\rho\alpha\beta},$$

by the symmetries of the curvature tensor. In other words, for fixed μ & ρ $R^{\mu}{}_{\rho\beta\alpha}$ is antisymmetric in $\beta\alpha$. An antisymmetric 3×3 matrix has only 3 independent elements (the diagonal values are zero). We can choose to evaluate only $\beta\alpha = tz, t\phi, z\phi$. Hence there are two choices for μ, 3 choices for ρ, & 3 choices for $\beta\alpha$, for a total of $2\times3\times3 = 18$ components of $R^{\mu}{}_{\rho\beta\alpha}$ that need to be evaluated.

Evaluating the first of these:

$$R^{t}{}_{ttz} = \cancel{\Gamma^{t}{}_{tz,t}} - \cancel{\Gamma^{t}{}_{tt,z}} + \cancel{\Gamma^{t}{}_{tt}\Gamma^{t}{}_{tz}} + \cancel{\Gamma^{t}{}_{\phi t}\Gamma^{\phi}{}_{tz}} - \cancel{\Gamma^{t}{}_{tz}\Gamma^{t}{}_{tt}} - \cancel{\Gamma^{t}{}_{\phi z}\Gamma^{\phi}{}_{tt}}$$

$$= 0$$

Extremely tedious calculation gives

$R^{t}{}_{tt\phi} = 0$

$R^{t}{}_{tz\phi} = 0$

$R^{t}{}_{ztz} = 0$

$R^{t}{}_{zt\phi} = 0$

$R^{t}{}_{zz\phi} = 0$

$R^{t}{}_{\phi tz} = 0$

$R^{t}{}_{\phi t\phi} = +a\ddot{a}$

$R^{t}{}_{\phi z\phi} = 0$

$R^{\phi}{}_{ttz} = 0$

$R^{\phi}{}_{tt\phi} = \dfrac{\ddot{a}}{a}$

$R^{\phi}{}_{tz\phi} = 0$

$R^{\phi}{}_{ztz} = 0$

$R^{\phi}{}_{zt\phi} = 0$

$R^{\phi}{}_{zz\phi} = 0$

$R^{\phi}{}_{\phi tz} = 0$

$R^{\phi}{}_{\phi t\phi} = 0$

$R^{\phi}{}_{\phi z\phi} = 0$

Hence the only non-zero components of the curvature tensor are

$$R^{\phi}{}_{tt\phi} = -R^{\phi}{}_{t\phi t} = \frac{\ddot{a}}{a}$$

$$\& \quad R^{t}{}_{\phi t\phi} = -R^{t}{}_{\phi\phi t} = -a\ddot{a}$$

We are required to prove that $R^{\alpha}{}_{\beta\gamma\delta} = 0 \Leftrightarrow \dot{a} = \text{const.}$ Clearly if $\dot{a} = \text{const.}$ then $\ddot{a} = 0$, & hence the curvature tensor is identically zero. Conversely, if $R^{\alpha}{}_{\beta\gamma\delta} = 0$ then

$$\ddot{a}/a = 0$$

$$\& \quad a\ddot{a} = 0.$$

Multiplying these gives $(\ddot{a})^2 = 0$ i.e. $\ddot{a} = 0$, which implies $\dot{a} = \text{const.}$

4. The Schwarzschild metric is

$$ds^2 = c^2\left(1 - \frac{r_0}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{r_0}{r}} - r^2 d\Omega^2 \qquad ①$$

(a). We assume the light propagates in the equator $(\theta = \frac{\pi}{2})$ & the motion is purely radial $(d\phi = 0)$, so that $d\Omega = 0$. A photon describes a null path $(ds^2 = 0)$, so we have

$$0 = c^2\left(1 - \frac{r_0}{r}\right)dt^2 - \frac{dr^2}{1 - r_0/r}$$

i.e. $$\frac{dr}{dt} = \pm c\left(1 - \frac{r_0}{r}\right)$$

For a photon moving towards the origin

the minus sign is the right choice, & so the co-ordinate velocity is

$$\frac{dr}{dt} = -c\left(1-\frac{r_0}{r}\right) \qquad ②$$

(b). The time from r_1 to r_2 is obtained by integrating ②:

$$\Delta t_{12} = -\frac{1}{c}\int_{r_1}^{r_2} \frac{dr}{1-\frac{r_0}{r}}$$

$$= -\frac{1}{c}\int_{r_1}^{r_2} \frac{r\,dr}{r-r_0} = -\frac{1}{c}\int_{r_1}^{r_2}\left(\frac{r-r_0+r_0}{r-r_0}\right)dr$$

$$= -\frac{1}{c}\int_{r_1}^{r_2}\left(1+\frac{r_0}{r-r_0}\right)dr$$

$$= -\frac{1}{c}\left[r_2-r_1+r_0\ln\left(\frac{r_2-r_0}{r_1-r_0}\right)\right]$$

i.e. $$\Delta t_{12} = \frac{1}{c}\left[r_1-r_2+r_0\ln\left(\frac{r_1-r_0}{r_2-r_0}\right)\right]$$

The return journey takes the same time, so the total (co-ordinate) time for the trip is

$$\Delta t = \frac{2}{c}\left[r_1-r_2+r_0\ln\left(\frac{r_1-r_0}{r_2-r_0}\right)\right]$$

(c). The departure & return of the signal to r_1 represent two events at the same location to an observer at r_1. The relationship between proper time τ (time measured by a local observer) & co-ordinate time for events at the same location follows from the metric ① with $ds^2 = c^2 d\tau^2$, $r = r_1$ & $dr^2 = d\phi^2 = d\theta^2 = 0$:

$$c^2 d\tau^2 = c^2 dt^2 \left(1 - \frac{r_0}{r_1}\right).$$

Hence

$$\Delta\tau = \Delta t \left(1 - \frac{r_0}{r_1}\right)^{\frac{1}{2}}$$

is the proper time between the departure & return of the signal at r_1, i.e. the observer at r_1 measures the round-trip time to be

$$\Delta\tau = \frac{2}{c}\left(1 - \frac{r_0}{r_1}\right)^{\frac{1}{2}}\left[r_1 - r_2 + r_0 \ln\left(\frac{r_1 - r_0}{r_2 - r_0}\right)\right]$$

5. (a). The equation for a null geodesic in the Schwarzschild metric is

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2, \qquad ①$$

where $u = 1/r$.

For a circular orbit $u = \text{const.}$, & hence $d^2u/d\phi^2 = 0$. The geodesic equation then reduces to

$$u\left(\frac{3GM}{c^2}u - 1\right) = 0$$

which has the non trivial solution

$u = \dfrac{c^2}{3GM}$, as required.

(b). Consider a slightly perturbed orbit,

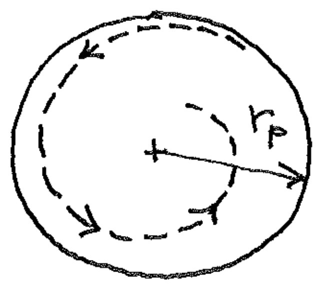
$u = \dfrac{c^2}{3GM} + \varepsilon$, where $|\varepsilon| << \dfrac{c^2}{3GM}$.

Substituting this into ① & keeping only

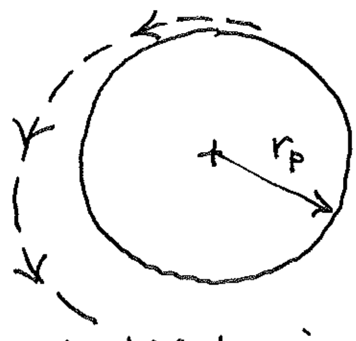
terms of order ε leads to

$$\frac{d^2\varepsilon}{d\phi^2} = \varepsilon . \quad (2)$$

If $\varepsilon > 0$ then u is slightly larger than $c^2/3GM$, & $r = \frac{1}{u}$ is slightly less than the photospheric value $r_p = \frac{3GM}{c^2}$. Equation (2) says that in this case $\frac{d^2\varepsilon}{d\phi^2} > 0$, & hence u will increase with phase angle, which means r will decrease with phase angle. Hence in this case the photon starts just inside the photosphere & spirals in:



If $\varepsilon < 0$ then u is just less than $\frac{c^2}{3GM}$, & $r = \frac{1}{u}$ is just greater than r_p. Eq. (2) says that $d^2\varepsilon/d\phi^2 < 0$, i.e. u will decrease with phase angle, & hence r increases with phase angle. Hence if the photon starts just outside the photosphere it spirals out:



Hence the photospheric orbit is unstable to small perturbations.

PHYS378 General Relativity and Cosmology 2000
Assignment 3 due Monday October 9

1. Some insight into curved space-time may be obtained by "embedding" diagrams. An example is provided by the Schwarzschild metric. The interval for an equatorial ($\theta = \pi/2$) slice of this metric at a fixed co-ordinate time is

$$ds^2 = \frac{-dr^2}{1 - r_0/r} - r^2 d\phi^2, \tag{1}$$

where r_0 is the Schwarzschild radius. We seek a 2-D surface embedded in Euclidean space that has this interval. The Euclidean interval can be written

$$ds^2 = -dz^2 - dr^2 - r^2 d\phi^2. \tag{2}$$

Assuming the required surface has the form $z = z(r)$ we have $dz = (dz/dr)dr$, and hence

$$ds^2 = -\left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 - r^2 d\phi^2. \tag{3}$$

(a) Comparing (1) and (3), determine $z = z(r)$. 2

(b) Sketch the resulting surface, for $r > r_0$. 2

2. A curved space-time has an interval

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)r^2 d\theta^2 - C(r)r^2 \sin^2\theta d\phi^2, \tag{4}$$

where r, θ, ϕ are regarded as spherical co-ordinates, and where $A(r)$, $B(r)$ and $C(r)$ are given functions of r. This metric can be written

$$ds^2 = c^2 d\tau^2 - ds_r^2 - ds_\theta^2 - ds_\phi^2, \tag{5}$$

where $cd\tau = \sqrt{A(r)}dt$, $ds_r = \sqrt{B(r)}dr$, $ds_\theta = \sqrt{C(r)}r d\theta$ and $ds_\phi = \sqrt{C(r)}r \sin\theta d\phi$. The quantity ds_r represents a locally measured increment in distance corresponding to a change dr in the co-ordinate r, made with the other co-ordinates fixed. The quantities $d\tau$, ds_θ and ds_ϕ have analogous meanings. With this knowledge, establish the following results for measured quantities in the given metric.

(a) The circumference of the circle $r = r_1$ is 2

$$2\pi\sqrt{C(r_1)}r_1. \tag{6}$$

(b) The area of the sphere $r = r_1$ is 2

$$4\pi r_1^2 C(r_1). \tag{7}$$

(c) The distance between the points $r = r_1$ and $r = r_2$ on a given radial line is

$$\int_{r_1}^{r_2} \sqrt{B(r)}dr. \tag{8}$$

2

(d) The volume of the spherical shell $r_1 < r < r_2$ is

2

$$4\pi \int_{r_1}^{r_2} r^2 C(r)\sqrt{B(r)}dr. \tag{9}$$

3. Consider two concentric coplanar circles in the Schwarzschild geometry. Suppose the measured lengths of their circumferences are L_1 and L_2. Find an Integral for

(a) What is the radial co-ordinate distance Δr between these circles? ~~What is~~ the measured radial distance between them? 2

(b) Take two circles around the Sun with $L_1 = 2\pi R_\odot$ and $L_2 = 4\pi R_\odot$. By how much does the measured radial distance between them differ from the result in a flat space? [Hint: you may find it convenient to expand the integral involved in r_0/r.] 2

4. The Sun rotates with a period of approximately 25 days.

(a) Idealize it as a solid sphere rotating uniformly. Its moment of inertia is then $\frac{2}{5}M_\odot R_\odot^2$, where $M_\odot = 2 \times 10^{30}$ kg and $R_\odot = 7 \times 10^8$ m. Calulate the angular momentum of the Sun, $J_\odot$. 2

(b) If the entire Sun suddenly collapsed to a black hole, it might be expected to form a Kerr hole of mass $M_\odot$ and angular momentum $J_\odot$. What is the value of the Kerr parameter a in this case? What is the ratio $2a/r_0$? If this ratio is larger than unity, how might a "naked singularity" be avoided? 2

ASSIGNMENT 3 SOLUTIONS

1. (a) comparing (1) & (3) we have

$$1 + \left(\frac{dz}{dr}\right)^2 = \left(1 - \frac{r_0}{r}\right)^{-1}$$

i.e. $$\left(\frac{dz}{dr}\right)^2 = \frac{r}{r-r_0} - 1 = \frac{r_0}{r-r_0}$$

so $$\frac{dz}{dr} = \frac{r_0^{\frac{1}{2}}}{(r-r_0)^{\frac{1}{2}}}$$

i.e. $$z = r_0^{\frac{1}{2}} \int \frac{dr}{(r-r_0)^{\frac{1}{2}}} + c$$

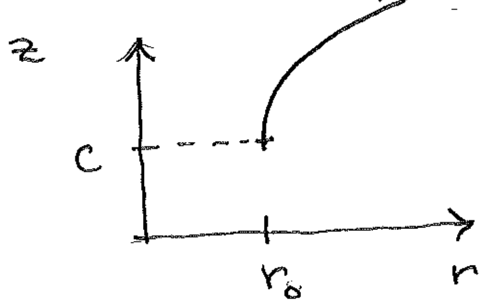
i.e. $$z = 2r_0^{\frac{1}{2}}(r-r_0)^{\frac{1}{2}} + c, \quad (1)$$

which is the required expression for $z = z(r)$. The constant of integration c is arbitrary.

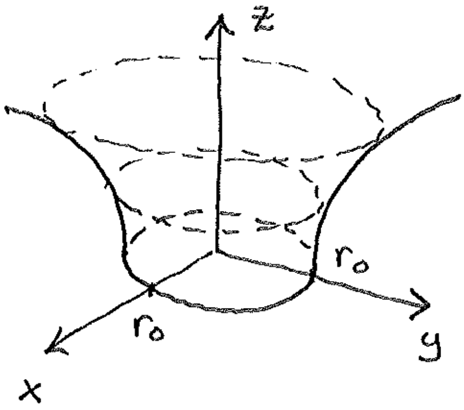
(b). Rearranging (1) gives

$$r = \frac{1}{4r_0}(z-c)^2 + r_0$$

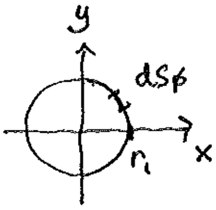
so the surface is half a parabola on its side in the z-r plane:



Choosing $c = 0$ for simplicity, the surface looks like a "paraboloid of revolution":



2(a). Without loss of generality, we can assume the circle is in the equatorial plane ($\theta = \frac{\pi}{2}$). The circle is described by $r = r_1$, $0 \leq \theta \leq 2\pi$. An infinitesimal element of measured length along the circle is given by



$$ds_\phi(r=r_1, \theta=\tfrac{\pi}{2}) = C(r_1)^{\frac{1}{2}} r_1 \, d\phi$$

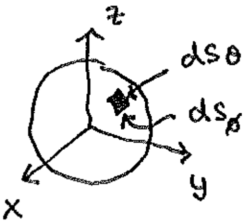
The measured circumference will be

$$L = \int_{0 \leq \phi \leq 2\pi} ds_\phi(r=r_1, \theta=\tfrac{\pi}{2}) = C(r_1)^{\frac{1}{2}} r_1 \int_0^{2\pi} d\phi$$

$$= 2\pi \, C(r_1)^{\frac{1}{2}} r_1,$$

as required.

(b). An infinitesimal measured area on the sphere is given by



$$ds_\theta(r=r_1) \cdot ds_\phi(r=r_1)$$

$$= C(r_1) r_1^2 \sin\theta \, d\theta \, d\phi$$

The total measured area of the sphere is

$$A = \int_{0 \leq \theta \leq \pi} ds_\theta(r=r_1) \, ds_\phi(r=r_1)$$

i.e. $A = r_1^2 C(r_1) \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$

$= 4\pi r_1^2 C(r_1)$, as required.

(c). the measured distance is

$$R = \int_{r_1 \le r \le r_2} ds_r = \int_{r_1}^{r_2} B(r)^{\frac{1}{2}} dr,$$

as required.

(d). The measured volume is

$$V = \int_{\substack{r_1 \le r \le r_2 \\ 0 \le \theta \le \pi \\ 0 \le \phi \le 2\pi}} ds_r . ds_\theta\, ds_\phi$$

$$= \left(\int_{r_1}^{r_2} B(r)^{\frac{1}{2}} C(r) r^2 dr\right).\left(\int_0^{\pi} \sin\theta\, d\theta\right) \times . \left(\int_0^{2\pi} d\phi\right)$$

$$= 4\pi \int_{r_1}^{r_2} r^2 B(r)^{\frac{1}{2}} C(r)\, dr,$$

as required.

3(a). From 2(a) we have the formula for the measured circumference

$$L = 2\pi\, C(r)^{\frac{1}{2}} r.$$

For the Schwarzschild metric $C(r) = 1$, so

$$L = 2\pi r$$

i.e. the same as in flat space-time.

Hence we have $L_1 = 2\pi r_1$ & $L_2 = 2\pi r_2$, where r_1 & r_2 are the radial co-ordinates of the circles, &

$$\Delta r = r_2 - r_1 = \frac{1}{2\pi}(L_2 - L_1)$$

is the radial co-ordinate distance between the circles.

The measured distance between the circles is given by the formula in 2(c), i.e.

$$\Delta R = \int_{r_1}^{r_2} B(r)^{\frac{1}{2}} dr.$$

For the Schwarzschild metric $B(r) = \frac{1}{1 - r_0/r}$, so

$$\Delta R = \int_{L_1/2\pi}^{L_2/2\pi} \frac{dr}{(1 - r_0/r)^{\frac{1}{2}}} . \qquad (1)$$

This integral is a bit tricky to evaluate. (In the question I probably should have said "find an integral for the measured distance.") The exact answer is

$$\Delta R = r_0 \left[\frac{1}{2} \ln \left\{ \left(\frac{1 + \sqrt{1 - \frac{r_0}{r_2}}}{1 - \sqrt{1 - \frac{r_0}{r_2}}} \right) \left(\frac{1 - \sqrt{1 - \frac{r_0}{r_1}}}{1 + \sqrt{1 - \frac{r_0}{r_1}}} \right) \right\} + \left[\left(\frac{r_2}{r_0}\right)\left(\frac{r_2}{r_0} - 1\right) \right]^{\frac{1}{2}} - \left[\left(\frac{r_1}{r_0}\right)\left(\frac{r_1}{r_0} - 1\right) \right]^{\frac{1}{2}} \right]$$

where $r_1 = L_1/2\pi$ & $r_2 = L_2/2\pi$. I would accept the approximate answer for $\frac{r_1}{r_0}, \frac{r_2}{r_0} \gg 1$ (see below), or even just the integral (1).

(b). The measured distance in flat space is $\frac{L_2}{2\pi} - \frac{L_1}{2\pi} = R_\odot$. Hence the difference between the measured distance in the Schwarzschild geometry & that in flat space is

$$\delta = \int_{L_1/2\pi}^{L_2/2\pi} \frac{dr}{\left(1-\frac{r_0}{r}\right)^{\frac{1}{2}}} - R_\odot$$

using the result of 2(a), equ. ①.

Following the hint, we can expand the integrand using the Binomial theorem

$$\left(1-\frac{r_0}{r}\right)^{-\frac{1}{2}} = 1 + \frac{r_0}{2r} + \cdots$$

where the extra terms are order $\left(\frac{r_0}{r}\right)^2$ & higher. Hence

$$\delta = \int_{R_\odot}^{2R_\odot} \left(1 + \frac{r_0}{2r} + \ldots\right) dr - R_\odot$$

$$= \left[r + \frac{r_0}{2r} + \ldots\right]_{R_\odot}^{2R_\odot} - R_\odot$$

$$= R_\odot + \frac{r_0}{2} \ln 2 - R_\odot + \ldots$$

$$= \frac{r_0}{2} \ln 2 + \ldots$$

So to order $r_0/R_\odot$,

$$\delta = \frac{GM_\odot}{c^2} \ln 2 \approx \frac{6.67\times10^{-11} \cdot 2\times10^{30} \ln 2}{9\times10^{16}} \text{ m}$$

$$\simeq 1027 \text{ m} \approx 1\text{km}$$

i.e. the measured distance differs from the flat space-time value by about 1km.

<u>Q4(a)</u>. The angular momentum is

$$J_\odot = I\omega = \frac{2}{5} M_\odot R_\odot^2 \left(\frac{2\pi}{T}\right)$$

where T is the period of rotation. Hence

$$J_\odot \approx \frac{2}{5} \cdot 2\times10^{30} \cdot (7\times10^8)^2 \cdot \frac{2\pi}{25.86400} \quad kg\,m^2 s^{-1}$$

$$\approx 1.14\times10^{42} \; Nms$$

<u>(b)</u>. From the notes, the Kerr parameter a is given by $a = J/Mc$. Hence we have

$$a = \frac{J_\odot}{M_\odot c} \approx \frac{1.1\times10^{42}}{2\times10^{30} \cdot 3\times10^8} m \approx 1900 \; m$$

So the Kerr parameter is about 1.9 km.

The requested ratio is

$$\frac{2a}{r_0} = \frac{2J_\odot}{M_\odot c} \cdot \frac{c^2}{2GM_\odot} = \frac{J_\odot c}{GM_\odot^2}$$

$$\approx \frac{1.14\times10^{42} \cdot 3\times10^8}{6.67\times10^{-11} \cdot 4\times10^{60}}$$

$$\approx 1.28$$

Hence we have $\frac{2a}{r_0} > 1$. From the lecture notes, theory predicts that there is no event horizon in this case, i.e. there is a "naked singularity". This may be avoided if material is expelled during the collapse, taking with it enough angular momentum to produce $\frac{2a}{r_0} < 1$.

(This is one possible answer - I'd accept anything!)

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PHYS378 General Relativity and Cosmology (2000)

Assignment 1 due August 10

1. A reference frame S' passes a frame S with a velocity of $0.6c$ in the X direction. Clocks are adjusted in the two frames so that when $t = t' = 0$ the origins of the two reference frames coincide.

   (a) An event occurs in S with space-time coordinates $x_1 = 50\text{m}$, $t_1 = 2.0 \times 10^{-7}\text{s}$. What are the coordinates of this event in S'?

   (b) If a second event occurs at $x_2 = 10\text{m}$, $t_2 = 3.0 \times 10^{-7}\text{s}$ in S what is the difference in time between the events as measured in S'?

2. A spaceship A of length 100 m in its own rest frame S_A passes spaceship B with rest frame S_B at a relative speed of $\sqrt{3}c/2$ and on a parallel course. When an observer at the centre of spaceship A passes an observer located at the centre of spaceship B, a crew member of A simultaneously fires very short bursts from two lasers mounted perpendicularly at the ends of A so as to leave burn marks on the hull of B. The spaceships pass so close to each other that these laser beams travel negligibly short distances. Assuming that the event of the two observers being adjacent are the reference points $t_A = t_B = 0$, $x_A = x_B = 0$, and that the second spaceship is of sufficient length that the laser beams will strike its hull:

   (a) What are the coordinates of the two laser bursts (considered as events in space-time) in S_A?

   (b) What are the coordinates of these two events as measured in S_B?

   (c) What is the distance between the marks appearing on the hull of S_B? Is this result an example of length contraction?

3. The mean lifetime of a muon in its own rest frame is $2.0 \times 10^{-6}\text{s}$. What average distance would the particle travel in vacuum before decaying when measured in reference frames in which its velocity is $0.1c, 0.6c, 0.99c$? Determine also the distances through which the muon claims it travelled in each case.

4. A fluorescent tube, stationary in a reference frame S, is arranged so as to light up simultaneously (in S) along its entire length. By considering the lighting up of two parts of the tube an infinitesimal distance Δx apart as two simultaneous events in S, determine the temporal and spatial separation of these two events in another frame of reference S' moving with a velocity v parallel to the orientation of the tube. Hence describe what is observed from this other frame of reference.

5. Two identical rods of proper length L_0 move towards each other at the same speed v relative to a reference frame collide end on and stick together. Show that the combined lengths of the rods (which remain intact) must compress to a total length less than or equal to

$$2L_0\sqrt{\frac{c-v}{c+v}}.$$

PHYS378 General Relativity and Cosmology 2000
Assignment 2 due Friday September 8

1. The following steps establish that the covariant derivative transforms tensorially.

(a) Start with the Fundamental Theorem of Riemannian geometry in co-ordinates x'^{μ}:

$$\Gamma'_{\nu\mu\sigma} = \frac{1}{2}\left(\frac{\partial g'_{\mu\nu}}{\partial x'^{\sigma}} - \frac{\partial g'_{\sigma\mu}}{\partial x'^{\nu}} + \frac{\partial g'_{\nu\sigma}}{\partial x'^{\mu}}\right)$$

Replace the primed metric tensors on the RHS by unprimed ones, using the transformation rules $[g'_{\mu\nu} = (\partial x^{\alpha}/\partial x'^{\mu})(\partial x^{\beta}/\partial x'^{\nu})g_{\alpha\beta}$, etc.]. Expand the derivatives and use the symmetry of the metric tensor and relabelling of indices to arrive at

$$\Gamma'_{\nu\mu\sigma} = \frac{\partial^2 x^{\alpha}}{\partial x'^{\sigma}\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}g_{\alpha\beta} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\rho}}{\partial x'^{\sigma}}\Gamma_{\beta\alpha\rho}. \qquad (1)$$ 2

(b) Multiply (1) by $g'^{\nu\tau}\partial x^{\epsilon}/\partial x'^{\tau}$ and simplify terms to obtain

$$\frac{\partial^2 x^{\epsilon}}{\partial x'^{\sigma}\partial x'^{\mu}} = \frac{\partial x^{\epsilon}}{\partial x'^{\tau}}\Gamma'^{\tau}{}_{\mu\sigma} - \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\rho}}{\partial x'^{\sigma}}\Gamma^{\epsilon}{}_{\alpha\rho}. \qquad (2)$$ 2

(c) Next, recall the transformation rule for a derivative:

6

$$\begin{aligned}\frac{\partial A'_{\mu}}{\partial x'^{\beta}} &= \frac{\partial}{\partial x'^{\beta}}\left(\frac{\partial x^{\nu}}{\partial x'^{\mu}}A_{\nu}\right)\\ &= \frac{\partial^2 x^{\nu}}{\partial x'^{\beta}\partial x'^{\mu}}A_{\nu} + \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial x^{\gamma}}{\partial x'^{\beta}}\frac{\partial A_{\nu}}{\partial x^{\gamma}}. \qquad (3)\end{aligned}$$ 2

Use (2) to replace the second partial derivative in (3). Rearrange terms and use the definition of the covariant derivative to arrive at

$$A'_{\mu;\beta} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial x^{\gamma}}{\partial x'^{\beta}}A_{\nu;\gamma},$$

i.e. the covariant derivative of a covariant vector transforms like a second-rank covariant tensor.

2. Starting from the definition of the Einstein tensor $G_{\mu\nu}$:

3

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$ 1, 2

show that $G_{\mu\nu} = 0$ if and only if $R_{\mu\nu} = 0$.

3. Consider the 3-D space-time with metric

2

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = dt^2 - dz^2 - [a(t)]\,d\phi^2,$$

where $a(t)$ is an increasing function of time. The spatial part of this metric looks like a cylinder that expands with time.

4

(a) Find the non-zero components of the metric connections $\Gamma^{\alpha}{}_{\beta\gamma}$. 2

(b) Find the non-zero components of the Riemann curvature tensor $R^{\alpha}{}_{\beta\gamma\delta}$. Show that the space is flat (i.e. the Riemann curvature tensor vanishes) if and only if $\dot{a}(t) = \text{const}$. 2

PHYS 378 GENERAL RELATIVITY & COSMOLOGY 2000

ASSIGNMENT 2 SOLUTIONS

1. (a). We have

$$\Gamma'_{\nu\mu\sigma} = \frac{1}{2}\left(\frac{\partial g'_{\mu\nu}}{\partial x'^{\sigma}} - \frac{\partial g'_{\sigma\mu}}{\partial x'^{\nu}} + \frac{\partial g'_{\nu\sigma}}{\partial x'^{\mu}}\right)$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x'^{\sigma}}\left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}\right) - \frac{\partial}{\partial x'^{\nu}}\left(\frac{\partial x^{\alpha}}{\partial x'^{\sigma}}\frac{\partial x^{\beta}}{\partial x'^{\mu}} g_{\alpha\beta}\right) + \frac{\partial}{\partial x'^{\mu}}\left(\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial x^{\beta}}{\partial x'^{\sigma}} g_{\alpha\beta}\right)\right]$$

$$= \frac{1}{2}\left[\overset{①}{\frac{\partial^2 x^{\alpha}}{\partial x'^{\sigma}\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}} + \overset{②}{\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial^2 x^{\beta}}{\partial x'^{\sigma}\partial x'^{\nu}} g_{\alpha\beta}} + \overset{③}{\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial g_{\alpha\beta}}{\partial x'^{\sigma}}} - \overset{④}{\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\sigma}}\frac{\partial x^{\beta}}{\partial x'^{\mu}} g_{\alpha\beta}} - \overset{⑤}{\frac{\partial x^{\alpha}}{\partial x'^{\sigma}}\frac{\partial^2 x^{\beta}}{\partial x'^{\nu}\partial x'^{\mu}} g_{\alpha\beta}} - \overset{⑥}{\frac{\partial x^{\alpha}}{\partial x'^{\sigma}}\frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial g_{\alpha\beta}}{\partial x'^{\nu}}} + \overset{⑦}{\frac{\partial^2 x^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}}\frac{\partial x^{\beta}}{\partial x'^{\sigma}} g_{\alpha\beta}} + \overset{⑧}{\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial^2 x^{\beta}}{\partial x'^{\mu}\partial x'^{\sigma}} g_{\alpha\beta}} + \overset{⑨}{\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial x^{\beta}}{\partial x'^{\sigma}}\frac{\partial g_{\alpha\beta}}{\partial x'^{\mu}}}\right]$$

Term ② cancels with term ④ ($g_{\alpha\beta}$ is symmetric) & term ⑤ cancels with term ⑦ (ditto), leaving:

$$\Gamma'_{\nu\mu\sigma} = \frac{\partial^2 x^{\alpha}}{\partial x'^{\sigma}\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} + \overset{①}{\frac{1}{2}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial g_{\alpha\beta}}{\partial x'^{\sigma}}} - \overset{②}{\frac{1}{2}\frac{\partial x^{\alpha}}{\partial x'^{\sigma}}\frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial g_{\alpha\beta}}{\partial x'^{\nu}}} + \overset{③}{\frac{1}{2}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial x^{\beta}}{\partial x'^{\sigma}}\frac{\partial g_{\alpha\beta}}{\partial x'^{\mu}}}$$

The derivatives of $g_{\alpha\beta}$ can be converted to derivatives w.r.t. unprimed co-ordinates:

$$\Gamma'_{\nu\mu\sigma} = \frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

$$+ \frac{1}{2}\left[\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial g_{\alpha\beta}}{\partial x^\rho} \right.$$

$$- \frac{\partial x^\alpha}{\partial x'^\sigma} \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial g_{\alpha\beta}}{\partial x^\rho}$$

$$\left. + \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial g_{\alpha\beta}}{\partial x^\rho} \right]$$

Next we can relabel the dummy indices (α, β, ρ) in the second & third bracketted terms to give

$$\Gamma'_{\nu\mu\sigma} = \frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

$$+ \frac{1}{2} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \left(\frac{\partial g_{\alpha\beta}}{\partial x^\rho} - \frac{\partial g_{\rho\alpha}}{\partial x^\beta} + \frac{\partial g_{\beta\rho}}{\partial x^\alpha} \right)$$

i.e. $$\boxed{\Gamma'_{\nu\mu\sigma} = \frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} + \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \Gamma_{\beta\alpha\rho},}$$

where $\Gamma_{\beta\alpha\rho} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\rho} - \frac{\partial g_{\rho\alpha}}{\partial x^\beta} + \frac{\partial g_{\beta\rho}}{\partial x^\alpha} \right)$ by the Fundamental theorem of Riemannian geometry.

(b). Using the suggested trick of multiplying by $g'^{\nu\tau} \partial x^\varepsilon / \partial x'^\tau$ we have

$$\frac{\partial x^{\varepsilon}}{\partial x'^{\tau}} g'^{\nu\tau} \Gamma'_{\nu\mu\sigma} = \left(g'^{\nu\tau} \frac{\partial x^\varepsilon}{\partial x'^\tau} \frac{\partial x^\beta}{\partial x'^\nu} \right) g_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\mu}$$

$$+ \left(g'^{\nu\tau} \frac{\partial x^\varepsilon}{\partial x'^\tau} \frac{\partial x^\beta}{\partial x'^\nu} \right) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\sigma} \Gamma_{\beta\alpha\rho}$$

From the rules for transformation of co-ordinates we have $g^{\beta\varepsilon} = \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\varepsilon}{\partial x'^\tau} g'^{\nu\tau}$

& hence we have

$$\frac{\partial x^\varepsilon}{\partial x'^\tau}\Gamma'^\tau{}_{\mu\sigma} = g^{\beta\varepsilon} g_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\mu}$$

$$+ g^{\beta\varepsilon}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\sigma}\Gamma_{\beta\alpha\rho}.$$

$$= \delta^\varepsilon_\alpha \frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\mu} + \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\sigma}\Gamma^\varepsilon{}_{\alpha\rho}$$

$$= \frac{\partial^2 x^\varepsilon}{\partial x'^\sigma \partial x'^\mu} + \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\sigma}\Gamma^\varepsilon{}_{\alpha\rho}$$

or $$\boxed{\frac{\partial^2 x^\varepsilon}{\partial x'^\sigma \partial x'^\mu} = \frac{\partial x^\varepsilon}{\partial x'^\tau}\Gamma'^\tau{}_{\mu\sigma} - \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\sigma}\Gamma^\varepsilon{}_{\alpha\rho},}$$

as required.

(c). The rule for transforming a derivative is

$$\frac{\partial A'_\mu}{\partial x'^\beta} = \frac{\partial^2 x^\nu}{\partial x'^\beta \partial x'^\mu} A_\nu + \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\beta}\frac{\partial A_\nu}{\partial x^\gamma}$$

$$= \left(\frac{\partial x^\nu}{\partial x'^\tau}\Gamma'^\tau{}_{\mu\beta} - \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\beta}\Gamma^\nu{}_{\alpha\rho}\right) A_\nu$$

$$+ \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\beta}\frac{\partial A_\nu}{\partial x^\gamma},$$

using our expression for the 2nd derivative obtained in (b).

Rearranging:

$$\frac{\partial A'_\mu}{\partial x'^\beta} - \Gamma'^\tau{}_{\mu\beta}\left(\frac{\partial x^\nu}{\partial x'^\tau} A_\nu\right)$$

$$= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\beta}\frac{\partial A_\nu}{\partial x^\gamma} - \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\beta}\Gamma^\nu{}_{\alpha\rho} A_\nu$$

i.e. $$\frac{\partial A'_\mu}{\partial x'^\beta} - \Gamma'^\tau{}_{\mu\beta} A'_\tau = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\beta}\left(\frac{\partial A_\nu}{\partial x^\gamma} - \Gamma^\sigma{}_{\nu\gamma} A_\sigma\right)$$

where the dummy indices in the last term have been relabelled. Recalling the definition of the covariant derivative it is clear that we have established that

$$\boxed{A'_{\mu;\beta} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\beta} A_{\nu;\gamma} \quad ,}$$

as required.

"if and only if"

2. We are required to prove $G_{\mu\nu} = 0 \Leftrightarrow R_{\mu\nu} = 0$

First note that the reverse direction is trivial. If $R_{\mu\nu} = 0$ then $R = R^\alpha{}_\alpha = g^{\alpha\mu} R_{\mu\alpha} = 0$, & hence $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$.

For the forward direction, assuming $G_{\mu\nu} = 0$ gives

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \quad (*)$$

Multiplying by $g^{\alpha\mu}$:

$$R^\alpha{}_\nu = \frac{1}{2} g^\alpha{}_\nu R$$

From lectures we have $g^\alpha{}_\nu = \delta^\alpha{}_\nu$. Hence we

have
$$R^{\alpha}{}_{\nu} = \tfrac{1}{2}\delta^{\alpha}{}_{\nu}R.$$
Setting $\nu = \alpha$ gives
$$R^{\alpha}{}_{\alpha} = \tfrac{1}{2}\delta^{\alpha}{}_{\alpha}R$$
i.e. $R = 2R$ (recall $\delta^{\alpha}{}_{\alpha} = 4$)

i.e. $R = 0$

& substituting this back into ⊛ gives $R_{\mu\nu} = 0$, for all μ & ν, as required.

3. By inspection the components of the metric tensor and its inverse are

$$g_{tt} = 1 \qquad g^{tt} = 1$$
$$g_{zz} = -1 \qquad g^{zz} = -1$$
$$g_{\phi\phi} = -[a(t)]^2 \qquad g^{\phi\phi} = -[a(t)]^{-2}.$$

(a). From the Fundamental theorem,
$$\Gamma^{\alpha}{}_{\mu\sigma} = \tfrac{1}{2}g^{\alpha\nu}(g_{\mu\nu,\sigma} - g_{\sigma\mu,\nu} + g_{\nu\sigma,\mu}).$$
The metric connections $\Gamma^{\alpha}{}_{\mu\sigma}$ are symmetric in μ & σ. Hence, for fixed α there are only six independent choices of $\mu\sigma$, which we can take to be $\mu\sigma = tt, tz, t\phi, \phi\phi, \phi z, zz$. Since there are three choices of α, there are then 18 components of $\Gamma^{\alpha}{}_{\mu\sigma}$ that need to be evaluated.

Evaluating the first of these:
$$\Gamma^{t}{}_{tt} = \tfrac{1}{2}g^{t\nu}(g_{t\nu,t} - g_{tt,\nu} + g_{\nu t,t})$$
$$= \tfrac{1}{2}g^{tt}(g_{tt,t} - g_{tt,t} + g_{tt,t}),$$
since only the diagonal elements are non-zero,

i.e. $\Gamma^t{}_{tt} = 0$.

Similarly tedious calculation gives

$$\Gamma^t{}_{tz} = 0 \qquad \Gamma^z{}_{\phi z} = 0$$
$$\Gamma^t{}_{t\phi} = 0 \qquad \Gamma^\phi{}_{tt} = 0$$
$$\Gamma^t{}_{\phi\phi} = +a(t)\dot{a}(t) \qquad \Gamma^\phi{}_{tz} = 0$$
$$\Gamma^t{}_{\phi z} = 0 \qquad \Gamma^\phi{}_{t\phi} = \frac{\dot{a}(t)}{a(t)}$$
$$\Gamma^t{}_{zz} = 0 \qquad \Gamma^\phi{}_{\phi\phi} = 0$$
$$\Gamma^z{}_{tt} = 0 \qquad \Gamma^\phi{}_{zz} = 0$$
$$\Gamma^z{}_{tz} = 0 \qquad \Gamma^\phi{}_{\phi z} = 0$$
$$\Gamma^z{}_{t\phi} = 0$$
$$\Gamma^z{}_{\phi\phi} = 0$$
$$\Gamma^z{}_{zz} = 0$$

Hence the only non-zero components are

$$\Gamma^t{}_{\phi\phi} = +a(t)\,\dot{a}(t)$$

$$\& \quad \Gamma^\phi{}_{t\phi} = \Gamma^\phi{}_{\phi t} = \frac{\dot{a}(t)}{a(t)} .$$

(b). The Riemann curvature tensor is defined by

$$R^\mu{}_{\rho\beta\alpha} = \Gamma^\mu{}_{\rho\alpha,\beta} - \Gamma^\mu{}_{\rho\beta,\alpha} + \Gamma^\mu{}_{\sigma\beta}\Gamma^\sigma{}_{\rho\alpha} - \Gamma^\mu{}_{\sigma\alpha}\Gamma^\sigma{}_{\rho\beta}$$

First we note that only $\mu = t, \phi$ can lead to non-zero components, because only $\Gamma^t{}_{\phi\phi}$, $\Gamma^\phi{}_{t\phi}$ & $\Gamma^\phi{}_{\phi t}$ are non-zero. Next,

$$R^\mu{}_{\rho\beta\alpha} = -R^\mu{}_{\rho\alpha\beta} ,$$

by the symmetries of the curvature tensor. In other words, for fixed μ & ρ $R^{\mu}{}_{\rho\beta\alpha}$ is antisymmetric in $\beta\alpha$. An antisymmetric 3×3 matrix has only 3 independent elements (the diagonal values are zero). We can choose to evaluate only $\beta\alpha = tz, t\phi, z\phi$. Hence there are two choices for μ, 3 choices for ρ, & 3 choices for $\beta\alpha$, for a total of $2 \times 3 \times 3 = 18$ components of $R^{\mu}{}_{\rho\beta\alpha}$ that need to be evaluated.

Evaluating the first of these:

$$R^{t}{}_{ttz} = \cancel{\Gamma^{t}{}_{tz,t}} - \cancel{\Gamma^{t}{}_{tt,z}} + \cancel{\Gamma^{t}{}_{tt}}\Gamma^{t}{}_{tz} + \cancel{\Gamma^{t}{}_{\phi t}}\Gamma^{\phi}{}_{tz} - \cancel{\Gamma^{t}{}_{tz}}\Gamma^{t}{}_{tt} - \cancel{\Gamma^{t}{}_{\phi z}}\Gamma^{\phi}{}_{tt}$$

$$= 0$$

Extremely tedious calculation gives

$R^{t}{}_{tt\phi} = 0$

$R^{t}{}_{tz\phi} = 0$

$R^{t}{}_{ztz} = 0$

$R^{t}{}_{zt\phi} = 0$

$R^{t}{}_{zz\phi} = 0$

$R^{t}{}_{\phi tz} = 0$

$R^{t}{}_{\phi t\phi} = +a\ddot{a}$

$R^{t}{}_{\phi z\phi} = 0$

$R^{\phi}{}_{ttz} = 0$

$R^{\phi}{}_{tt\phi} = \dfrac{\ddot{a}}{a}$

$R^{\phi}{}_{tz\phi} = 0$

$R^{\phi}{}_{ztz} = 0$

$R^{\phi}{}_{zt\phi} = 0$

$R^{\phi}{}_{zz\phi} = 0$

$R^{\phi}{}_{\phi tz} = 0$

$R^{\phi}{}_{\phi t\phi} = 0$

$R^{\phi}{}_{\phi z\phi} = 0$

Hence the only non-zero components of the curvature tensor are

$$R^{\phi}{}_{tt\phi} = -R^{\phi}{}_{t\phi t} = \frac{\ddot{a}}{a}$$

$$\& \quad R^{t}{}_{\phi t\phi} = -R^{t}{}_{\phi\phi t} = -a\ddot{a}$$

We are required to prove that $R^{\alpha}{}_{\beta\gamma\delta} = 0 \Leftrightarrow \dot{a} = \text{const.}$ Clearly if $\dot{a} = \text{const.}$ then $\ddot{a} = 0$, & hence the curvature tensor is identically zero. Conversely, if $R^{\alpha}{}_{\beta\gamma\delta} = 0$ then

$$\ddot{a}/a = 0$$

$$\& \quad a\ddot{a} = 0.$$

Multiplying these gives $(\ddot{a})^2 = 0$ i.e. $\ddot{a} = 0$, which implies $\dot{a} = \text{const.}$

4. The Schwarzschild metric is

$$ds^2 = c^2\left(1 - \frac{r_0}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{r_0}{r}} - r^2 d\Omega^2 \qquad (1)$$

(a). We assume the light propagates in the equator ($\theta = \frac{\pi}{2}$) & the motion is purely radial ($d\phi = 0$), so that $d\Omega = 0$. A photon describes a null path ($ds^2 = 0$), so we have

$$0 = c^2\left(1 - \frac{r_0}{r}\right)dt^2 - \frac{dr^2}{1 - r_0/r}$$

$$\text{i.e.} \quad \frac{dr}{dt} = \pm c\left(1 - \frac{r_0}{r}\right)$$

For a photon moving towards the origin

the minus sign is the right choice, & so the co-ordinate velocity is

$$\frac{dr}{dt} = -c\left(1-\frac{r_0}{r}\right) \qquad (2)$$

(b). The time from r_1 to r_2 is obtained by integrating (2):

$$\Delta t_{12} = -\frac{1}{c}\int_{r_1}^{r_2} \frac{dr}{1-\frac{r_0}{r}}$$

$$= -\frac{1}{c}\int_{r_1}^{r_2}\frac{r\,dr}{r-r_0} = -\frac{1}{c}\int_{r_1}^{r_2}\left(\frac{r-r_0+r_0}{r-r_0}\right)dr$$

$$= -\frac{1}{c}\int_{r_1}^{r_2}\left(1+\frac{r_0}{r-r_0}\right)dr$$

$$= -\frac{1}{c}\left[r_2-r_1+r_0\ln\left(\frac{r_2-r_0}{r_1-r_0}\right)\right]$$

i.e. $$\Delta t_{12} = \frac{1}{c}\left[r_1-r_2+r_0\ln\left(\frac{r_1-r_0}{r_2-r_0}\right)\right]$$

The return journey takes the same time, so the total (co-ordinate) time for the trip is

$$\Delta t = \frac{2}{c}\left[r_1-r_2+r_0\ln\left(\frac{r_1-r_0}{r_2-r_0}\right)\right]$$

(c). The departure & return of the signal to r_1 represent two events at the same location to an observer at r_1. The relationship between proper time τ (time measured by a local observer) & co-ordinate time for events at the same location follows from the metric ① with $ds^2 = c^2 d\tau^2$, $r = r_1$ & $dr^2 = d\phi^2 = d\theta^2 = 0$:

$$c^2 d\tau^2 = c^2 dt^2 \left(1 - \frac{r_0}{r_1}\right).$$

Hence

$$\Delta\tau = \Delta t \left(1 - \frac{r_0}{r_1}\right)^{\frac{1}{2}}$$

is the proper time between the departure & return of the signal at r_1, i.e. the observer at r_1 measures the round-trip time to be

$$\Delta\tau = \frac{2}{c}\left(1 - \frac{r_0}{r_1}\right)^{\frac{1}{2}} \left[r_1 - r_2 + r_0 \ln\left(\frac{r_1 - r_0}{r_2 - r_0}\right)\right]$$

5. (a). The equation for a null geodesic in the Schwarzschild metric is

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2} u^2, \qquad ①$$

where $u = 1/r$.

For a circular orbit $u = $ const., & hence $d^2u/d\phi^2 = 0$. The geodesic equation then reduces to

$$u\left(\frac{3GM}{c^2} u - 1\right) = 0$$

which has the non trivial solution

$u = \dfrac{c^2}{3GM}$, as required.

(b). Consider a slightly perturbed orbit,

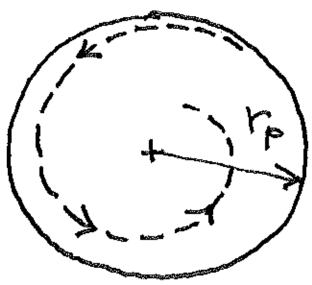
$u = \dfrac{c^2}{3GM} + \varepsilon$, where $|\varepsilon| \ll \dfrac{c^2}{3GM}$.

Substituting this into ① & keeping only

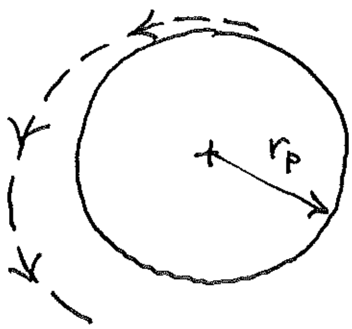
terms of order ε leads to

$$\frac{d^2\varepsilon}{d\phi^2} = \varepsilon . \qquad (2)$$

<u>If $\varepsilon > 0$</u> then u is slightly larger than $c^2/3GM$, & $r = \frac{1}{u}$ is slightly less than the photospheric value $r_p = \frac{3GM}{c^2}$. Equation (2) says that in this case $\frac{d^2\varepsilon}{d\phi^2} > 0$, & hence u will increase with phase angle, which means r will decrease with phase angle. Hence in this case the photon starts just inside the photosphere & spirals in:



<u>If $\varepsilon < 0$</u> then u is just less than $\frac{c^2}{3GM}$, & $r = \frac{1}{u}$ is just greater than r_p. Eq. (2) says that $d^2\varepsilon/d\phi^2 < 0$, i.e. u will decrease with phase angle, & hence r increases with phase angle. Hence if the photon starts just outside the photosphere it spirals out:



Hence the photospheric orbit is unstable to small perturbations.

PHYS378 General Relativity and Cosmology 2000
Assignment 3 due Monday October 9

1. Some insight into curved space-time may be obtained by "embedding" diagrams. An example is provided by the Schwarzschild metric. The interval for an equatorial ($\theta = \pi/2$) slice of this metric at a fixed co-ordinate time is

$$ds^2 = \frac{-dr^2}{1 - r_0/r} - r^2 d\phi^2, \tag{1}$$

where r_0 is the Schwarzschild radius. We seek a 2-D surface embedded in Euclidean space that has this interval. The Euclidean interval can be written

$$ds^2 = -dz^2 - dr^2 - r^2 d\phi^2. \tag{2}$$

Assuming the required surface has the form $z = z(r)$ we have $dz = (dz/dr)dr$, and hence

$$ds^2 = -\left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 - r^2 d\phi^2. \tag{3}$$

(a) Comparing (1) and (3), determine $z = z(r)$. 2

(b) Sketch the resulting surface, for $r > r_0$. 2

2. A curved space-time has an interval

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)r^2 d\theta^2 - C(r)r^2 \sin^2\theta d\phi^2, \tag{4}$$

where r, θ, ϕ are regarded as spherical co-ordinates, and where $A(r)$, $B(r)$ and $C(r)$ are given functions of r. This metric can be written

$$ds^2 = c^2 d\tau^2 - ds_r^2 - ds_\theta^2 - ds_\phi^2, \tag{5}$$

where $c d\tau = \sqrt{A(r)}dt$, $ds_r = \sqrt{B(r)}dr$, $ds_\theta = \sqrt{C(r)}r d\theta$ and $ds_\phi = \sqrt{C(r)}r \sin\theta d\phi$. The quantity ds_r represents a locally measured increment in distance corresponding to a change dr in the co-ordinate r, made with the other co-ordinates fixed. The quantities $d\tau$, ds_θ and ds_ϕ have analogous meanings. With this knowledge, establish the following results for measured quantities in the given metric.

(a) The circumference of the circle $r = r_1$ is 2

$$2\pi\sqrt{C(r_1)}r_1. \tag{6}$$

(b) The area of the sphere $r = r_1$ is 2

$$4\pi r_1^2 C(r_1). \tag{7}$$

(c) The distance between the points $r = r_1$ and $r = r_2$ on a given radial line is

$$\int_{r_1}^{r_2} \sqrt{B(r)}dr. \tag{8}$$

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(d) The volume of the spherical shell $r_1 < r < r_2$ is

2

$$4\pi \int_{r_1}^{r_2} r^2 C(r)\sqrt{B(r)}dr. \tag{9}$$

3. Consider two concentric coplanar circles in the Schwarzschild geometry. Suppose the measured lengths of their circumferences are L_1 and L_2.

Find an ~~eq~~ integral for

(a) What is the radial co-ordinate distance Δr between these circles? ~~What~~ is the measured radial distance between them?

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(b) Take two circles around the Sun with $L_1 = 2\pi R_\odot$ and $L_2 = 4\pi R_\odot$. By how much does the measured radial distance between them differ from the result in a flat space? [Hint: you may find it convenient to expand the integral involved in r_0/r.]

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4. The Sun rotates with a period of approximately 25 days.

(a) Idealize it as a solid sphere rotating uniformly. Its moment of inertia is then $\frac{2}{5}M_\odot R_\odot^2$, where $M_\odot = 2 \times 10^{30}\,\mathrm{kg}$ and $R_\odot = 7 \times 10^8\,\mathrm{m}$. Calulate the angular momentum of the Sun, $J_\odot$.

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(b) If the entire Sun suddenly collapsed to a black hole, it might be expected to form a Kerr hole of mass $M_\odot$ and angular momentum $J_\odot$. What is the value of the Kerr parameter a in this case? What is the ratio $2a/r_0$? If this ratio is larger than unity, how might a "naked singularity" be avoided?

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PHYS 378 GENERAL RELATIVITY & COSMOLOGY 2000

ASSIGNMENT 3 SOLUTIONS

1. (a) comparing (1) & (3) we have

$$1 + \left(\frac{dz}{dr}\right)^2 = \left(1 - \frac{r_0}{r}\right)^{-1}$$

i.e. $$\left(\frac{dz}{dr}\right)^2 = \frac{r}{r-r_0} - 1 = \frac{r_0}{r-r_0}$$

so $$\frac{dz}{dr} = \frac{r_0^{\frac{1}{2}}}{(r-r_0)^{\frac{1}{2}}}$$

i.e. $$z = r_0^{\frac{1}{2}} \int \frac{dr}{(r-r_0)^{\frac{1}{2}}} + c$$

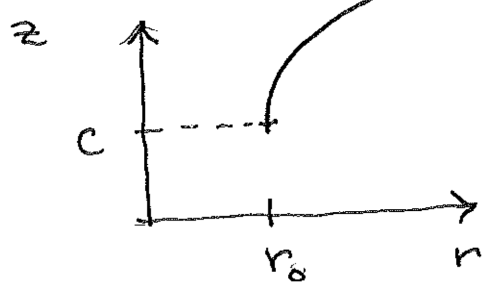
i.e. $$z = 2r_0^{\frac{1}{2}}(r-r_0)^{\frac{1}{2}} + c, \qquad (1)$$

which is the required expression for $z = z(r)$. The constant of integration c is arbitrary.

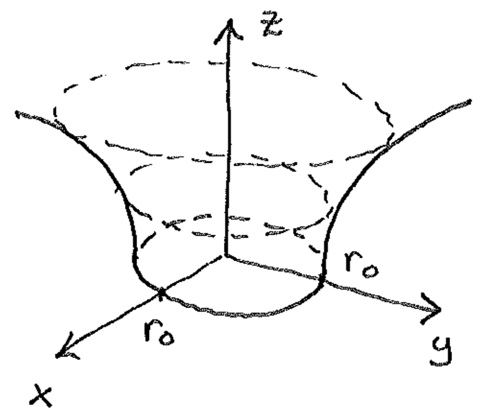
(b). Rearranging (1) gives

$$r = \frac{1}{4r_0}(z-c)^2 + r_0$$

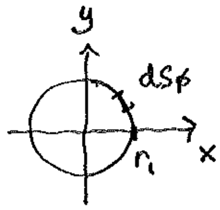
so the surface is half a parabola on its side in the z-r plane:



Choosing $c = 0$ for simplicity, the surface looks like a "paraboloid of revolution":



2(a). Without loss of generality, we can assume the circle is in the equatorial plane ($\theta = \frac{\pi}{2}$). The circle is described by $r = r_1$, $0 \leq \theta \leq 2\pi$. An infinitesimal element of <u>measured</u> length along the circle is given by



$$ds_\phi (r = r_1, \theta = \tfrac{\pi}{2}) = C(r_1)^{\frac{1}{2}} r_1 \, d\phi$$

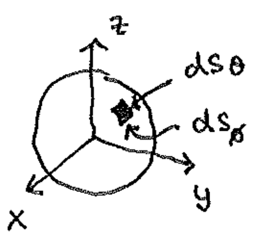
The measured circumference will be

$$L = \int_{0 \leq \phi \leq 2\pi} ds_\phi (r = r_1, \theta = \tfrac{\pi}{2}) = C(r_1)^{\frac{1}{2}} r_1 \int_0^{2\pi} d\phi$$

$$= 2\pi \, C(r_1)^{\frac{1}{2}} r_1,$$

as required.

(b). An infinitesimal measured area on the sphere is given by



$$ds_\theta (r = r_1) . ds_\phi (r = r_1)$$

$$= C(r_1) r_1^2 \sin\theta \, d\theta \, d\phi$$

The total measured area of the sphere is

$$A = \int_{0 \leq \theta \leq \pi} ds_\theta (r = r_1) \, ds_\phi (r = r_1)$$

i.e. $A = r_1^2 C(r_1) \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$

$= 4\pi r_1^2 C(r_1)$, as required.

(c). the measured distance is

$$R = \int_{r_1 \leq r \leq r_2} ds_r = \int_{r_1}^{r_2} B(r)^{\frac{1}{2}} dr,$$

as required.

(d). The measured volume is

$$V = \int_{\substack{r_1 \leq r \leq r_2 \\ 0 \leq \theta \leq \pi \\ 0 \leq \phi \leq 2\pi}} ds_r . ds_\theta \, ds_\phi$$

$$= \left(\int_{r_1}^{r_2} B(r)^{\frac{1}{2}} C(r) r^2 dr \right) . \left(\int_0^{\pi} \sin\theta \, d\theta \right) \times . \left(\int_0^{2\pi} d\phi \right)$$

$$= 4\pi \int_{r_1}^{r_2} r^2 B(r)^{\frac{1}{2}} C(r) dr,$$

as required.

3(a). From 2(a) we have the formula for the measured circumference

$$L = 2\pi \, C(r)^{\frac{1}{2}} r.$$

For the Schwarzschild metric $C(r) = 1$, so

$$L = 2\pi r$$

i.e. the same as in flat space-time.

Hence we have $L_1 = 2\pi r_1$ & $L_2 = 2\pi r_2$, where r_1 & r_2 are the radial co-ordinates of the circles, &

$$\Delta r = r_2 - r_1 = \frac{1}{2\pi}(L_2 - L_1)$$

is the radial co-ordinate distance between the circles.

The measured distance between the circles is given by the formula in 2(c), i.e.

$$\Delta R = \int_{r_1}^{r_2} B(r)^{\frac{1}{2}}\,dr.$$

For the Schwarzschild metric $B(r) = \frac{1}{1 - r_0/r}$, so

$$\Delta R = \int_{L_1/2\pi}^{L_2/2\pi} \frac{dr}{(1 - r_0/r)^{\frac{1}{2}}} . \qquad (1)$$

This integral is a bit tricky to evaluate. (In the question I probably should have said "find an integral for the measured distance.") The exact answer is

$$\Delta R = r_0 \left[\frac{1}{2} \ln \left\{ \left(\frac{1 + \sqrt{1 - \frac{r_0}{r_2}}}{1 - \sqrt{1 - \frac{r_0}{r_2}}} \right) \left(\frac{1 - \sqrt{1 - \frac{r_0}{r_1}}}{1 + \sqrt{1 - \frac{r_0}{r_1}}} \right) \right\} + \left[\left(\frac{r_2}{r_0}\right)\left(\frac{r_2}{r_0} - 1\right) \right]^{\frac{1}{2}} - \left[\left(\frac{r_1}{r_0}\right)\left(\frac{r_1}{r_0} - 1\right) \right]^{\frac{1}{2}} \right]$$

where $r_1 = L_1/2\pi$ & $r_2 = L_2/2\pi$. I would accept the approximate answer for $\frac{r_1}{r_0}, \frac{r_2}{r_0} \gg 1$ (see below), or even just the integral (1).

(b). The measured distance in flat space is $\frac{L_2}{2\pi} - \frac{L_1}{2\pi} = R_\odot$. Hence the difference between the measured distance in the Schwarzschild geometry & that in flat space is

$$\delta = \int_{L_1/2\pi}^{L_2/2\pi} \frac{dr}{\left(1-\frac{r_0}{r}\right)^{\frac{1}{2}}} - R_\odot$$

using the result of 2(a), equ. ①.

Following the hint, we can expand the integrand using the Binomial theorem

$$\left(1-\frac{r_0}{r}\right)^{-\frac{1}{2}} = 1 + \frac{r_0}{2r} + \cdots$$

where the extra terms are order $\left(\frac{r_0}{r}\right)^2$ & higher. Hence

$$\delta = \int_{R_\odot}^{2R_\odot} \left(1 + \frac{r_0}{2r} + \ldots\right) dr - R_\odot$$

$$= \left[r + \frac{r_0}{2r} + \ldots \right]_{R_\odot}^{2R_\odot} - R_\odot$$

$$= R_\odot + \frac{r_0}{2} \ln 2 - R_\odot + \ldots$$

$$= \frac{r_0}{2} \ln 2 + \ldots$$

So to order $r_0/R_\odot$,

$$\delta = \frac{GM_\odot}{c^2} \ln 2 \approx \frac{6.67\times10^{-11} \cdot 2\times10^{30} \ln 2}{9\times10^{16}} \, m$$

$$\approx 1027 \, m \approx 1 km$$

i.e. the measured distance differs from the flat space-time value by about 1km.

Q4(a). The angular momentum is

$$J_\odot = I\omega = \frac{2}{5} M_\odot R_\odot^2 \left(\frac{2\pi}{T}\right)$$

where T is the period of rotation. Hence

$$J_\odot \approx \frac{2}{5} \cdot 2\times10^{30} \cdot (7\times10^8)^2 \cdot \frac{2\pi}{25.86400} \ \mathrm{kg\,m^2 s^{-1}}$$

$$\approx 1.14\times10^{42} \ \mathrm{Nms}$$

(b). From the notes, the Kerr parameter a is given by $a = J/Mc$. Hence we have

$$a = \frac{J_\odot}{M_\odot c} \approx \frac{1.1\times10^{42}}{2\times10^{30} \cdot 3\times10^8} \mathrm{m} \approx 1900 \ \mathrm{m}$$

So the Kerr parameter is about 1.9 km.

The requested ratio is

$$\frac{2a}{r_\odot} = \frac{2J_\odot}{M_\odot c} \cdot \frac{c^2}{2GM_\odot} = \frac{J_\odot c}{GM_\odot^2}$$

$$\approx \frac{1.14\times10^{42} \cdot 3\times10^8}{6.67\times10^{-11} \cdot 4\times10^{60}}$$

$$\approx 1.28$$

Hence we have $\frac{2a}{r_\odot} > 1$. From the lecture notes, theory predicts that there is no event horizon in this case, i.e. there is a "naked singularity". This may be avoided if material is expelled during the collapse, taking with it enough angular momentum to produce $\frac{2a}{r_\odot} < 1$.

(This is one possible answer - I'd accept anything!)

MACQUARIE UNIVERSITY

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PHYS378 General Relativity and Cosmology (2000)

Assignment 1 due August 10

1. A reference frame S' passes a frame S with a velocity of $0.6c$ in the X direction. Clocks are adjusted in the two frames so that when $t = t' = 0$ the origins of the two reference frames coincide.
   (a) An event occurs in S with space-time coordinates $x_1 = 50\text{m}$, $t_1 = 2.0 \times 10^{-7}\text{s}$. What are the coordinates of this event in S'?
   (b) If a second event occurs at $x_2 = 10\text{m}$, $t_2 = 3.0 \times 10^{-7}\text{s}$ in S what is the difference in time between the events as measured in S'?

2. A spaceship A of length 100 m in its own rest frame S_A passes spaceship B with rest frame S_B at a relative speed of $\sqrt{3}c/2$ and on a parallel course. When an observer at the centre of spaceship A passes an observer located at the centre of spaceship B, a crew member of A simultaneously fires very short bursts from two lasers mounted perpendicularly at the ends of A so as to leave burn marks on the hull of B. The spaceships pass so close to each other that these laser beams travel negligibly short distances. Assuming that the event of the two observers being adjacent are the reference points $t_A = t_B = 0$, $x_A = x_B = 0$, and that the second spaceship is of sufficient length that the laser beams will strike its hull:
   (a) What are the coordinates of the two laser bursts (considered as events in space-time) in S_A?
   (b) What are the coordinates of these two events as measured in S_B?
   (c) What is the distance between the marks appearing on the hull of S_B? Is this result an example of length contraction?

3. The mean lifetime of a muon in its own rest frame is $2.0 \times 10^{-6}\text{s}$. What average distance would the particle travel in vacuum before decaying when measured in reference frames in which its velocity is $0.1c, 0.6c, 0.99c$? Determine also the distances through which the muon claims it travelled in each case.

4. A fluorescent tube, stationary in a reference frame S, is arranged so as to light up simultaneously (in S) along its entire length. By considering the lighting up of two parts of the tube an infinitesimal distance Δx apart as two simultaneous events in S, determine the temporal and spatial separation of these two events in another frame of reference S' moving with a velocity v parallel to the orientation of the tube. Hence describe what is observed from this other frame of reference.

5. Two identical rods of proper length L_0 move towards each other at the same speed v relative to a reference frame collide end on and stick together. Show that the combined lengths of the rods (which remain intact) must compress to a total length less than or equal to

$$2L_0\sqrt{\frac{c-v}{c+v}}.$$

MACQUARIE UNIVERSITY

End of Year Examination 2000

GR & COSMOLOGY

Unit:	PHYS 378 – ~~PHYSICS III~~
Date:	Friday 24 November 1.50 pm
Time Allowed:	THREE (3) hours plus TEN (10) minutes reading time.
Total Number of Questions:	Eight (8).
Instructions:	Answer question any TWO (2) questions from Part A and any TWO (2) questions from Part B. you should complete FOUR (4) questions in total.
	Answer questions from Parts A and B in separate books.
	Electronic calculators may be used, excepting those with a full alphabetic keyboard.

You may find the following information useful

Fundamental theorem of Riemannian geometry:

$$\Gamma_{\nu\mu\sigma} = \frac{1}{2}\left(g_{\mu\nu,\sigma} - g_{\sigma\mu,\nu} + g_{\nu\sigma,\mu}\right)$$

Covariant derivative of a first rank contravariant vector:

$$A^{\mu}{}_{;\nu} = A^{\mu}{}_{,\nu} + \Gamma^{\mu}{}_{\sigma\nu}A^{\sigma}$$

Definition of Riemann curvature tensor:

$$R^{\mu}{}_{\nu\sigma\tau} = \Gamma^{\mu}{}_{\nu\tau,\sigma} - \Gamma^{\mu}{}_{\nu\sigma,\tau} + \Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\alpha}{}_{\nu\tau} + \Gamma^{\mu}{}_{\alpha\tau}\Gamma^{\alpha}{}_{\nu\sigma}$$

Minkowski metric:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

Robertson Walker metric:

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2\right]$$

Friedmann equations:

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \end{aligned}$$

Definitions:

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

$$q(t) = -\frac{\ddot{a}(t)}{a(t)H^2(t)}$$

Part A: General Relativity

Attempt TWO (2) questions from Part A
(50 marks in total, all questions are of equal value)
Answer questions from Part A in a separate book

1. (a) (4 marks)
Explain how Newtonian gravity is incompatible with special relativity.

(b) (4 marks)
Briefly describe the "weak" and "strong" equivalence principles.

(c) (4 marks)
Based on the strong equivalence principle, present an argument that light must be deflected by a gravitational field.

(d) (4 marks)
Briefly explain why tensors are important in special relativity.

(e) (4 marks)
Write down the transformation rule under change of co-ordinates for a mixed tensor of the second rank, $A^{\mu}{}_{\nu}$.

(f) (5 marks)
Show that if A^{μ} is a contravariant tensor then (in general) the quantity $D^{\mu}{}_{\nu} = \partial A^{\mu}/\partial x^{\nu}$ does not transform like a tensor.

2. (a) (5 marks)
The relativistically correct (valid in any inertial reference frame) version of Newton's second law for the motion of a particle is

$$\frac{dp^{\mu}}{d\tau} = F^{\mu},$$

where

$$p^{\mu} = m_0 \frac{dx^{\mu}}{d\tau}$$

is the four momentum. The various quantities in these equations are defined as follows: m_0 is the rest mass of the particle, $x^{\mu} = (ct, \mathbf{x})$ is the four vector describing the particle's position, $d\tau = dt/\gamma$ is the proper time, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor of the particle, and F^{μ} describes the action of external forces.

Write down a version of this equation that is valid in *all* reference frames.

(b) (5 marks)
For a body in free-fall $F^\mu = 0$. Hence show that a body in free-fall satisfies the geodesic equation,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0. \tag{1}$$

Equation (1) does not depend on m_0. What principle does this represent?

(c) (5 marks)
Demonstrate that in flat space time the falling particle follows a straight line path.

(d) (10 marks)
A spherical surface is described by the metric

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2,$$

where R is a constant.

Write down the components of the metric tensor, $g_{\mu\nu}$ for this surface, and the components of the inverse of the metric tensor, $g^{\mu\nu}$. Hence work out the metric connections, $\Gamma^\mu{}_{\nu\sigma}$.

3. (a) (5 marks)
The Einstein equations may be written

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{2}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. Briefly explain each term in this tensor equation, and its physical significance.

(b) (4 marks)
State four physically significant properties of the tensor $T_{\mu\nu}$.

(c) (6 marks)
Contract (2) with $g^{\mu\nu}$ to arrive at an expression for $R = R^\mu{}_\mu$. Use this together with (2) to establish the alternative form for the Einstein equations,

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\mu{}_\mu\right) - \Lambda g_{\mu\nu}.$$

(d) (4 marks)
Show that the Minkowski metric satisfies the Einstein equations in vacuum (for $\Lambda = 0$).

(e) (6 marks)
Describe in about half a page one test of General Relativity. It is not necessary to use equations.

4. (a) (5 marks)
What does the Schwarzschild metric describe? Define the Schwarzschild radius. What is the significance of the Schwarzschild radius?

(b) (5 marks)
What is a black hole? How are stellar-mass black holes believed to be formed?

(c) (5 marks)
An observer at $r = r_1$ in a Schwarzschild field transmits a light signal in the radial direction to $r = r_2$ ($r_1 > r_2$).

What is the co-ordinate velocity dr/dt of the signal?

(d) (5 marks)
Suppose the signal is reflected at $r = r_2$ and returns to $r = r_1$. How long does the round trip take in co-ordinate time t?

(e) (5 marks)
How long does the round trip take according to the observer?

Part B: Cosmology

Attempt TWO (2) questions from Part B
(50 marks in total, all questions are of equal value)
Answer questions from Part B in a separate book

5. (a) (5 marks)
   What is the Cosmological Principle?

   (b) (5 marks)
   Describe some observational evidence for accepting the Cosmological Principle.

   (c) (5 marks)
   What is peculiar velocity?

   (d) (10 marks)
   Show that peculiar velocities tend to zero as the universe expands.

6. (a) (5 marks)
   Show that in a universe described by the Robertson Walker metric that the distance between two points is given by

$$d = a(t) \int_0^r (1 - kr'^2)^{-\frac{1}{2}} dr' \tag{3}$$

   (b) (5 marks)
   What are the units of $a(t)$ and r in (3)? Explain the significance of k.

   (c) (15 marks)
   Show that the red shift of a photon is given by

$$z = \frac{\Delta\lambda}{\lambda_e} = \frac{a(t_o)}{a(t_e)} - 1$$

   where the subscripts refer to observed and emitted wavelength and epoch.

7. (a) (8 marks)
   Show that for a matter dominated Friedmann model of the universe with zero curvature the density is given by

$$\rho_c = \frac{3H_0^2}{8\pi G}$$

   and that

$$q_0 = \frac{1}{2}.$$

(b) (5 marks)
Show that for a matter dominated Friedmann model of the universe with positive curvature

$$\Omega = \frac{\rho}{\rho_c} > 1$$

while for a negative curvature

$$\Omega < 1.$$

(c) (4 marks)
Draw a diagram illustrating the evolution of the scale factor for the three cases in (a) and (b) above.

(d) (8 marks)
Describe briefly some of the problems of the Friedmann models and in particular that Ω must be very close to unity.

8. (a) 5 marks)
Show that in a universe containing radiation and matter, and having a non-zero cosmological constant, that eventually the vacuum energy will dominate.

(b) (5 marks)
Show that a Λ dominated universe expands exponentially and will eventually have no significant radiation or matter density.

(c) (5 marks)
How does a non-zero cosmological constant solve the problems of the Friedmann models?

(d) (5 marks)
How does the modern inflationary model avoid the difficulties in (b) above?

(e) (5 marks)
What observational support is there for inflation?

Q1. ANSWERS + MARKING SCHEME

1. (a). Newtonian gravity is described by
(4)

$$F = \frac{G m_1 m_2}{r^2}. \quad 1$$

If the separation r between the masses m_1 & m_2 changes, then F changes instantaneously.1 Hence this equation implies that a signal (representing gravitational influence) can propagate faster than light.1 This is inconsistent with special relativity, which implies that signals must propagate with speeds less than or equal to that of light.1

(b). The weak equivalence principle is
(4) the statement that the free-fall of a body is independent of its composition.1 the strong equivalence principle states:

1. That the results of all local
2 experiments in a frame in free-fall are independent of the motion,

&

2. The results of the local expts are 1
1 in accord with special relativity.

(c): consider a room in free-fall
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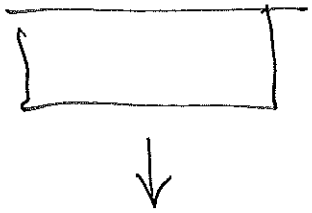
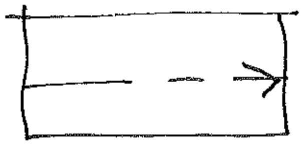


A person inside shines a light horizontally across the room. ~~To the pers~~ The result of this "experiment" must be in accord with usual laws of physics; according to the SEP, so light moves in a straightline to far end:



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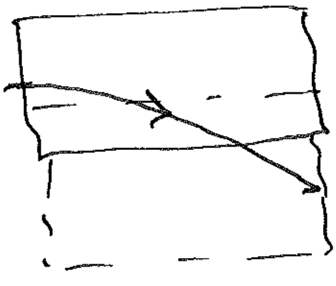
To an ~~distant~~ external observer, however, the path of the light is as follows:



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i.e. it follows a parabola. So light is deflected by the gravitational field.

(d). General relativity describes
(4) physical processes in all reference frames, & in particular must describe the physics of processes in accelerated reference frames. Tensor equations ~~are~~ ~~vati~~ retain the same form in all co-ordinate system. Hence tensor equations provide an appropriate language for formulating general relativistic laws.

The transformation rule of 2

(e). (4)

$$A'^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} A^{\alpha}{}_{\beta} \quad 3$$

where $x'^{\mu} = x'^{\mu}(x^1, x^2, \dots)$ 1
describes the co-ordinate transformation

(f). (5)

$$D^{\mu}{}_{\nu} = \frac{\partial A^{\mu}}{\partial x^{\nu}}$$

consider this quantity under transformation:

$$D'^{\mu}{}_{\nu} = \frac{\partial A'^{\mu}}{\partial x'^{\nu}} = \frac{\partial}{\partial x'^{\nu}}\left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha}\right) \quad 1$$

$$= \frac{\partial^2 x'^{\mu}}{\partial x'^{\nu}\partial x^{\alpha}} A^{\alpha} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial A^{\alpha}}{\partial x'^{\nu}} \quad 1$$

$$= \frac{\partial^2 x'^{\mu}}{\partial x'^{\nu}\partial x^{\alpha}} A^{\alpha} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial A^{\alpha}}{\partial x^{\beta}} \quad 1$$

$$= \frac{\partial^2 x'^{\mu}}{\partial x'^{\nu}\partial x^{\alpha}} A^{\alpha} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} D^{\alpha}{}_{\beta}$$

The second term on the RHS is the usual transformation rule for a second rank mixed tensor but the first term is different : hence in general there is an "extra term" & $D^{\mu}{}_{\nu}$ does not transform like a tensor.

16 min

Q2. (a). The version valid in
5 all frames is obtained by replacing derivatives by covariant derivatives, 2

so

$$F^{\mu} = \frac{Dp^{\mu}}{D\tau} \quad 3$$

too many marks!

is the appropriate law.

5 (b). $F^{\mu} = 0 \Rightarrow \dfrac{Dp^{\mu}}{D\tau} = 0$

i.e. $$\frac{dp^{\mu}}{d\tau} + \Gamma^{\mu}{}_{\sigma\nu}\, p^{\sigma} \frac{dx^{\nu}}{d\tau} = 0$$

using the expression for the covariant derivative.

Now $p^{\mu} = m_0 \dfrac{dx^{\mu}}{d\tau}$, so we have

$$\cancel{m_0} \frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\sigma\nu}\, \cancel{m_0} \frac{dx^{\sigma}}{d\tau} \frac{dx^{\nu}}{d\tau} = 0$$

i.e. $$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\sigma\nu} \frac{dx^{\sigma}}{d\tau} \frac{dx^{\nu}}{d\tau} = 0$$

which is the geodesic equation.

The cancellation of the m_0's represents the weak equivalence principle: all masses fall the same way, irrespective of mass.

(c) (5) Flat space time $\Rightarrow$ $\Gamma^{\mu}{}_{\nu\sigma} = 0$

so $$\frac{d^2 x^\mu}{d\tau^2} = 0 \quad \circledast$$

& $d\tau = dt/\gamma$

But $x^\mu = (ct, \underset{\sim}{x})$. So zeroth (time) component of $\circledast$ $\Rightarrow$

$$\gamma \frac{d}{dt} \gamma \frac{d}{dt}(ct) = 0$$

$$\Rightarrow \quad \frac{d\gamma}{dt} = 0 \quad \Rightarrow \quad \gamma = \text{const.}$$

So we have (spatial parts of $\circledast$)

$$\gamma \frac{d^2 \underset{\sim}{x}}{dt^2} = 0$$

$$\Rightarrow \quad \frac{d^2 \underset{\sim}{x}}{dt^2} = 0$$

$$\Rightarrow \quad \underset{\sim}{x} = \underset{\sim}{A} t + \underset{\sim}{B}$$

which is a straight line. So in flat space-time geodesics are straight lines (Newton's 1st law!)

(d). (10)

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2$$

$$= g_{\mu\nu} dx^\mu dx^\nu$$

$$= g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2$$

so $g_{\theta\theta} = R^2$

$g_{\phi\phi} = R^2 \sin^2\theta$ are the nonzero elements of the metric tensor.
The inverse of a diagonal matrix is ~~a di~~ easy:

$$g^{\theta\theta} = R^{-2}$$

$$g^{\phi\phi} = R^{-2} \sin^{-2}\theta$$

are the nonzero elements of the inverse of the metric tensor.

For the metric connections we need

$$\Gamma_{\nu\mu\sigma} = \frac{1}{2}\left(g_{\mu\nu,\sigma} - g_{\sigma\mu,\nu} + g_{\nu\sigma,\mu}\right)$$

so $$\Gamma^\alpha_{\ \mu\sigma} = \frac{1}{2} g^{\alpha\nu}\left(g_{\mu\nu,\sigma} - g_{\sigma\mu,\nu} + g_{\nu\sigma,\mu}\right)$$

The metric connections are symmetric. Hence there are three independent combinations of $\mu\sigma$, & overall $2 \times 3 = 6$ independent $\Gamma^\alpha_{\ \mu\sigma}$'s:

$$\Gamma^{\theta}_{\theta\theta} = \tfrac{1}{2} g^{\theta\theta} \left(g_{\theta\theta,\theta} - \cancel{g_{\theta\theta,\theta}} + \cancel{g_{\theta\theta,\theta}} \right)$$

$$= 0 \qquad \tfrac{1}{2}$$

$$\Gamma^{\theta}_{\theta\phi} = \Gamma_{\phi\theta} = \tfrac{1}{2} g^{\theta\theta} \left(g_{\theta\theta,\phi} - \cancel{g_{\phi\theta,\theta}} + \cancel{g_{\theta\phi,\theta}} \right)$$

$$= 0 \qquad \tfrac{1}{2}$$

$$\cancel{\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta}}$$

$$\Gamma^{\theta}_{\phi\phi} = \tfrac{1}{2} g^{\theta\theta} \left(\cancel{g_{\phi\theta,\phi}} - g_{\phi\phi,\theta} + \cancel{g_{\theta\phi,\phi}} \right)$$

$$= -\tfrac{1}{2} \cancel{R^{-2}} \frac{\partial}{\partial\theta} \cdot \cancel{R^{2}} \sin^2\theta$$

$$= -\tfrac{1}{2} \cdot 2 \sin\theta \cos\theta$$

$$= -\sin\theta\cos\theta \quad \checkmark \quad 1$$

$$\Gamma^{\phi}_{\phi\phi} = \tfrac{1}{2} g^{\phi\phi} \left(g_{\phi\phi,\phi} - \cancel{g_{\phi\phi,\phi}} + \cancel{g_{\phi\phi,\phi}} \right)$$

$$= 0 \qquad \tfrac{1}{2}$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \tfrac{1}{2} g^{\phi\phi} \left(\cancel{g_{\theta\phi,\phi}} - \cancel{g_{\phi\theta,\phi}} + g_{\phi\phi,\theta} \right)$$

$$= \cancel{\tfrac{1}{2}} \cdot \cancel{R^{-2}} \sin^{-2}\theta \cdot \cancel{R^{2}} \cdot \cancel{2} \sin\theta\cos\theta \quad 1$$

$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$ ✓

& $\Gamma^{\phi}_{\theta\theta} = \frac{1}{2} g^{\phi\phi} (\cancel{g_{\theta\phi,\theta}} - g_{\theta\theta,\phi} + \cancel{g_{\phi\theta,\theta}})$

$= 0$ $\frac{1}{2}$

~ 15 mins
got interrupted!

Q3. (a). $\overset{5}{}$ $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu}$

the terms are:

$G_{\mu\nu}$: the Einstein tensor. This describes the curvature of space-time. at a point
It is constructed from contractions of the Riemann tensor, namely $R_{\mu\nu}$ (the Ricci tensor) & R (the Ricci scalar).

2

$T_{\mu\nu}$: ~~that~~ the stress-energy tensor. This describes the mass-energy at a point. The factor $\frac{8\pi G}{c^4}$ describes how much a certain amount of mass energy distorts space-time. In vacuum $T_{\mu\nu} = 0$.

2

$\Lambda g_{\mu\nu}$: this is the cosmological term, which permits the curvature of space-time in the absence of matter / energy.

(b). $T_{\mu\nu}$ is
(4)

- symmetric
- divergenceless
- a second rank tensor
- vanishes in the absence of matter

(c). We have
(6)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu} \qquad (1)$$

Contracting with $g^{\mu\nu}$:

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu} + \Lambda g^{\mu\nu} g_{\mu\nu}$$

i.e. $$R^{\nu}{}_{\nu} - \frac{1}{2} \delta^{\nu}{}_{\nu} R = \frac{8\pi G}{c^4} T^{\nu}{}_{\nu} + \Lambda \delta^{\nu}{}_{\nu}$$

$$\delta^{\nu}{}_{\nu} = \delta^0{}_0 + \delta^1{}_1 + \delta^2{}_2 + \delta^3{}_3 = 4$$

$$R^{\nu}{}_{\nu} = R$$

So

$$R - 2R = -R = \frac{8\pi G}{c^4} T^{\mu}{}_{\mu} + 4\Lambda$$

$$\Rightarrow R = -\frac{8\pi G}{c^4} T^{\mu}{}_{\mu} - 4\Lambda \quad 4$$

& substituting this back into ①

$$R_{\mu\nu} = +\frac{1}{2} g_{\mu\nu}\left(-\frac{8\pi G}{c^4} T^{\alpha}{}_{\alpha} - 4\Lambda\right) + \Lambda g_{\mu\nu} + \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\cancel{= \frac{4\pi G}{c^4} T_{\mu\nu} - 2\Lambda g_{\mu\nu} + \Lambda g_{\mu\nu}}$$

$$\cancel{R_{\mu\nu} = \frac{4\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu}}$$

$$\boxed{R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha}\right) - \Lambda g_{\mu\nu}} \quad ②$$

2

which is the alternative form.

~~(d). If $R_{\mu\nu} = 0$ then $R = g^{\mu\nu} R_{\mu\nu}$~~
$= 0$

~~& so $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$~~

~~In the other direction, consider~~

~~(b): For $G_{\mu\nu} = 0$ we must have~~

~~$T_{\mu\nu} = 0$ & $\Lambda = 0$. Putting these~~

~~into (2) gives~~

~~$R_{\mu\nu} = 0$.~~

(d). The Minkowski metric has
(4)

$$g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$$

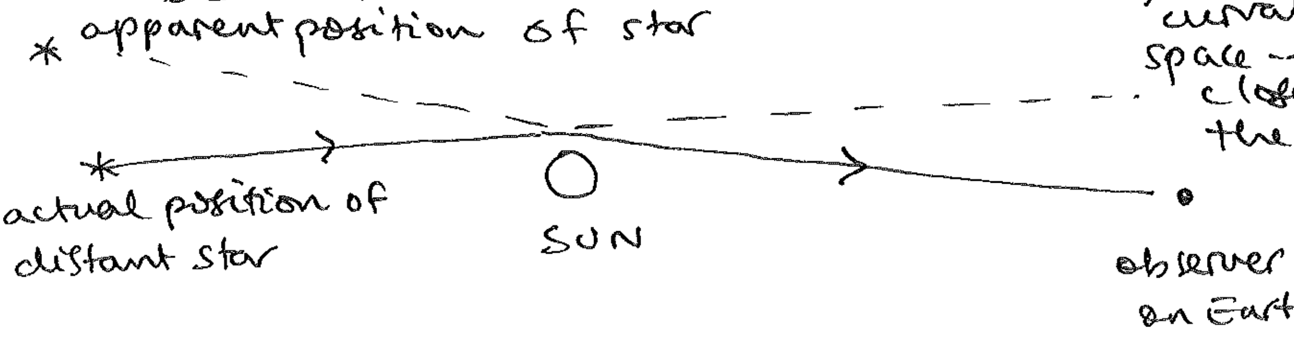
Hence $\Gamma^{\mu}{}_{\alpha\beta} = 0$ for all μ, α, β

& $R^{\mu}{}_{\nu\sigma\tau} = 0$

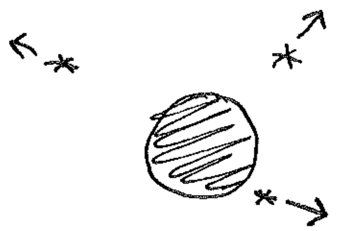
Hence $G_{\mu\nu} = 0$

& the Einstein equations in vacuum are satisfied.

(f). The deflection of starlight by
(6) the Sun provides a test of GR.
General relativity predicts that light passing close to the Sun will be deflected towards the Sun; due to the curvature of space-time close to the Sun.



The deflection for starlight grazing the limb is 1·75". This prediction can be tested by measuring the positions of stars in a photograph taken during an eclipse. GR says they should be ~~do~~ radially displaced away from their expected positions (by a maximum of 1·75"):



occulted disk
of Sun

In 1919 an eclipse expedition lead by Eddington tested this prediction. they found stars were deflected, by amounts consistent with Einstein's prediction.

19 mins.

4. (a). The Schwarzschild metric (SM) describes a static, spherically symmetric gravitational field with no matter. As such it is appropriate to describe the gravitational field outside the sun, outside a non-rotating black hole, etc.

5

The Schwarzschild radius is

$$r_0 = \frac{2GM}{c^2}$$

This radius defines the departure from flat space time. For an object with $r \gg r_0$, the space-time around the object is essentially flat. When an object has a radius r such that $r \approx r_0$, space-time around the object is very curved. If $r < r_0$ then ~~the~~ space-time is so warped that even light cannot escape the object. 2

(b). When a compact object becomes smaller than the Schwarzschild radius, not even light can escape from the object. We say that the object becomes a black hole.

5

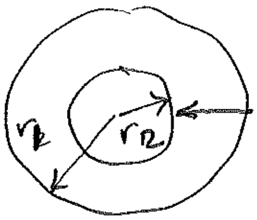
A plausible scenario for the formation of a stellar-mass black hole is the gravitational collapse of the iron core of a massive ($M \gtrsim 20 M_\odot$)

star. At the end of the thermonuclear burning cycle, such a star has a predominantly iron core. When thermal pressure is no longer able to support the core it begins to collapse under its own gravity. It continues to collapse, & forms a black hole. 3

(c). The SM is
(5)

$$ds^2 = \left(1 - \frac{r_0}{r}\right)c^2 dt^2 - \frac{dr^2}{1 - \frac{r_0}{r}} - r^2 d\Omega^2$$

where r_0 is the Schwarzschild radius.



Assume the light is transmitted in the equator, so $\theta = \frac{\pi}{2}$ & $d\phi = 0$. Then

$$ds^2 = \left(1 - \frac{r_0}{r}\right)c^2 dt^2 - \frac{dr^2}{1 - r_0/r}$$

Also light follows a null path, so $ds^2 = 0$. Hence

$$\left(1 - \frac{r_0}{r}\right)c^2 dt^2 = \frac{dr^2}{1 - r_0/r}$$

i.e. $$\frac{dr}{dt} = \pm c\left(1 - \frac{r_0}{r}\right)$$

& since the light is going in the direction of decreasing r,

$$\frac{dr}{dt} = -c\left(1 - r_0/r\right)$$

describes the co-ordinate velocity.

(d). The time to go from $r_1 \to r_2$ is
(5) found by integrating:

$$\Delta t_{12} = -\frac{1}{c}\int_{r_1}^{r_2} \frac{dr}{1 - r_0/r}$$

$$= -\frac{1}{c}\int_{r_1}^{r_2} \frac{r}{r-r_0}\, dr$$

$$= -\frac{1}{c}\int_{r_1}^{r_2} \frac{r-r_0+r_0}{r-r_0}\, dr$$

$$= -\frac{1}{c}\int_{r_1}^{r_2} \left(1 + \frac{r_0}{r-r_0}\right) dr$$

$$= -\frac{1}{c}\left\{ r_2 - r_1 + r_0 \ln \frac{r_2-r_0}{r_1-r_0} \right\}$$

$$\Delta t_{12} = \frac{r_1-r_2}{c} + \frac{r_0}{c} \ln \frac{r_1-r_0}{r_2-r_0}$$

The time for the round trip $(r_1 \to r_2 \to r_1)$ is twice this:

$$2\Delta t_{12} = \Delta t = 2\,\frac{r_1-r_2}{c} + \frac{2r_0}{c} \ln \frac{r_1-r_0}{r_2-r_0}$$

~~step~~.

(e). The emission & receipt of
(5) the signal at r_1 are two events
at the same location so
setting $ds^2 = c^2 d\tau^2$ in the metric & $r = r_1$
together with $dr = d\phi = d\theta = 0$
obtain

$$c^2 d\tau^2 = c^2 dt^2 \left(1 - \frac{r_0}{r_1}\right)$$

Hence $$d\tau^2 = dt^2 \left(1 - \frac{r_0}{r_1}\right)$$

or $$d\tau = dt \left(1 - \frac{r_0}{r_1}\right)^{\frac{1}{2}}$$ 2

describes the relation between
proper time intervals $(d\tau)$ & co-ordinate
time intervals (dt) for events at r_1.

Hence the time for the round
trip according to the observer at
r_1 is

$$\Delta\tau = \Delta t \left(1 - \frac{r_0}{r_1}\right)^{\frac{1}{2}}$$

2 $$= 2\left(1 - \frac{r_0}{r_1}\right)^{\frac{1}{2}} \left[\frac{r_1 - r_2}{c} + \frac{r_0}{c} \ln \frac{r_1 - r_0}{r_2 - r_0}\right]$$

16 min