SURFACE WAVES IN DEGENERATE PLASMA

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1. Introduction

In his 1968 paper [2], R.L. Guernsey derived the dispersion relation for surface waves on a semi infinite plasma confined by a perfect reflecting wall. The aim of the current project is to extend these results to the case of a fully degenerate electron gas described by the unit step function (a limiting case of the Fermi-Dirac distribution). By only considering the effects of the distribution function we are able to use the results derived by Guernsey from the "classical" formalism of the Vlasov equation. However the conclusions we ultimately draw are limited by the fact that we are using a semi-classical description which neglects some "quantum" effects.

2. Background

Surface wave behaviour is an important factor in many practical applications of plasma. Although the electromagnetic field arising due to surface waves is highly localised in space to the boundary of the plasma (decaying exponentially or faster with distance from the boundary) this field will determine how the plasma responds to an applied electromagnetic field. The effect of such external electromagnetic waves on the surface of a plasma (whether or not they occur due to surface waves of another plasma) will be determined by nonlinear interactions with the plasma surface waves.

The results on dispersion relation derived in [2] are established for a classical Maxwellian plasma, an appropriate model to describe the behaviour of an electron gas under standard conditions. By extending these results to the case of a degenerate Fermi-Dirac plasma, we have an appropriate model to apply to the case of an electron plasma inside a volume of metal under standard conditions.

3. The Vlasov-Poisson Equation

Consider a plasma composed of electrons characterized by the electron charge e, mass m and unperturbed density n against an immobile background of ions. In addition to this we will make a simplifying assumption about the electronic distribution function, specifically that it can be written as a sum of an unperturbed equilibrium function $f_0(v)$ and a time dependent perturbation $f(\mathbf{v}, \mathbf{r}, t)$ small in comparison to f_0.

$$\mathbf{F}(\mathbf{v}, \mathbf{r}, t) = f_0(v) + f(\mathbf{v}, \mathbf{r}, t) \qquad (3.1)$$

Working in three dimensional Euclidean space the plasma is confined to the region $x > 0$ by a perfect reflecting wall on the boundary formed by the yz plane. With the condition for reflection at the boundary given by:

$$f(0, -u, \mathbf{R}_\parallel, \mathbf{v}_\parallel, t) = f(0, u, \mathbf{R}_\parallel, \mathbf{v}_\parallel, t). \qquad (3.2)$$

We will refer to the velocity component perpendicular to the boundary by u or $v_\perp$ and the two velocity components in the plane parallel to the boundary by $\mathbf{v}_\parallel$. Similarly we refer to the distance component perpendicular to the boundary by x and the two distance components in the plane of the boundary by $\mathbf{R}_\parallel$. The linearized Vlasov-Poisson equation (assuming no applied electromagnetic field and

Date: October, 2010.

neglecting relativistic effects) for this geometry is given by:

$$\left(\frac{\partial}{\partial t}+v_\perp\frac{\partial}{\partial x}+\mathbf{v}_\parallel\cdot\frac{\partial}{\partial\mathbf{R}_\parallel}\right)f(x,u,\mathbf{R}_\parallel,\mathbf{v}_\parallel,t)$$
$$=\frac{e^2}{m}\int_{-\infty}^{\infty}du'\int d^2\mathbf{v}_\parallel{}'$$
$$\cdot\int_0^\infty dx'\int d^2\mathbf{R}_\parallel{}'f(x',u',\mathbf{R}_\parallel',\mathbf{v}_\parallel',t)$$
$$\cdot\left[\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{(\mathbf{R}_\parallel-\mathbf{R}_\parallel')^2+(x-x')^2}}\right)\frac{\partial}{\partial u}\right.$$
$$\left.+\frac{\partial}{\partial\mathbf{R}_\parallel}\left(\frac{1}{\sqrt{(\mathbf{R}_\parallel-\mathbf{R}_\parallel')^2+(x-x')^2}}\right)\cdot\frac{\partial}{\partial\mathbf{v}_\parallel}\right]$$
$$\cdot f_0(u,\mathbf{v}_\parallel). \tag{3.3}$$

The subtle difference between this and the normal Vlasov-Poisson equation is the fact that the integral involving the spacial components is taken over the half space from $x=0$ to $x=\infty$. As pointed out by Guernsey in [2] it is this integral that leads to surface wave solutions.

Following the procedure in [2] we will apply a Laplace transform with respect to time since we are concerned with the initial value problem for a given initial perturbation to F. In the spacial components $\mathbf{R}_\parallel$ parallel to the boundary we can clearly proceed with a double Fourier transform since f is defined in the whole $\mathbf{R}_\parallel$ plane. To this end we define:

$$\bar{f}(x,u,\mathbf{v}_\parallel,\mathbf{k}_\parallel,\omega)=\int_0^\infty dt\exp(-i\omega t)\int d^2\mathbf{R}_\parallel$$
$$\cdot\exp(-i\mathbf{k}_\parallel\cdot\mathbf{R}_\parallel)\ f(x,u,\mathbf{v}_\parallel,\mathbf{R}_\parallel,t) \tag{3.4}$$

for the region $x>0$ as a consequence of the definition of f. The subtlety arises in the treatment of $\bar{f}$; while it would seem obvious at first glance to take a Laplace transform of $\bar{f}$ with respect to x this neglects the important fact that we do not know the value of f for $x=0$. We do however have the condition for reflection at the boundary given by equation 3.2. In light of these conditions we define the function $\tilde{f}$ as the continuation of $\bar{f}$ into the left half-plane $x<0$ such that it is continuous, satisfies the condition 3.2 at $x=0$ and satisfies equation [2, Equation (8)][1] which reduces to equation [2, Equation (4)] (the transform of equation 3.3) in the region $x>0$. Such a function $\tilde{f}$ will now be defined for all x and accordingly we may now take the Fourier transform of $\tilde{f}$:

$$\hat{f}(u,\mathbf{v}_\parallel,k_\perp,\mathbf{k}_\parallel)=\int_{-\infty}^{\infty}dx\exp(-ik_\perp x)\tilde{f}(x,u,\mathbf{v}_\parallel,\mathbf{k}_\parallel). \tag{3.5}$$

This leads naturally to the definition of the transformed charge density, explicitly:

$$\rho\,'(\mathbf{k})=e\int du\ d^2\mathbf{v}_\parallel\hat{f}(u,\mathbf{v}_\parallel,k_\perp,\mathbf{k}_\parallel). \tag{3.6}$$

Next we define two functions of interest which will form part of the denominator of $\rho\,'$ (the context in which these functions arise is explicitly given in [2]). The first being the Landau denominator given by:

$$\Delta(\omega,k)=1-\frac{4\pi e^2}{k^2m}\int d^3v\frac{\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}f_0(\mathbf{v})}{\mathbf{k}\cdot\mathbf{v}-\omega}. \tag{3.7}$$

The second being the function whose roots give rise to surface wave solutions given by:

$$\epsilon(\omega,k_\parallel)=\ 1-\frac{k_\parallel}{2\pi}\int\frac{dk_\perp}{k^2}\left(1-\frac{1}{\Delta(\omega,k)}\right). \tag{3.8}$$

4. Degenerate Plasma

To model the statistical behaviour of a fully degenerate electron gas we consider the Fermi-Dirac distribution. This is based on the semi classical description of electrons as wave functions which obey the Pauli exclusion principle. This leads to degenerate behaviour as temperature approaches zero since

[1]Guernsey has used the subscript notation $\perp$ for components parallel to the boundary and $\parallel$ for components perpendicular to the boundary, the opposite of what has been used here.

only two electrons of opposite spins can occupy any single energy state. The equilibrium distribution function is given by:

$$f_0(v) = \frac{1}{\exp\left(m(v^2 - v_F^2)/2k_BT\right) + 1} \quad (4.1)$$

where T is the temperature, k_B is the Boltzmann constant and v_F is the Fermi velocity. In the limit $T \to 0$ or when $T << v_F$ this becomes the degenerate distribution:

$$f_0(v) = \begin{cases} \frac{n}{\frac{4}{3}\pi v_F^3} & \text{if } v < v_F \\ 0 & \text{if } v > v_F \end{cases}$$

$$= \frac{n}{\frac{4}{3}\pi v_F^3}(1 - H(v - v_F)) \quad (4.2)$$

where $H(x)$ is the Heaviside step function. This step function, constant up to the Fermi velocity corresponds to the phenomenon of Pauli blocking of electrons in a degenerate electron gas.

Substituting this function for f_0 in 3.7 we obtain:

$$1 + \frac{12\pi ne^2}{k^2 mv_f^2} + \frac{6\pi ne^2\omega}{k^3 mv_F^3}\ \log\left(\frac{\omega - kv_f}{kv_F + \omega}\right) \quad (4.3)$$

applying the change of variables $\tilde{\omega} = \frac{\omega}{\omega_p}$ and $\tilde{k} = k\lambda_d$, where

$$\omega_p = \sqrt{\frac{4\pi ne^2}{m}}, \qquad \lambda_d = \frac{v_F}{\omega_p\sqrt{3}} \quad (4.4)$$

we obtain a dimensionless expression for Δ in terms of the scaled variables:

$$\Delta(\tilde{\omega}, \tilde{k}) = 1 + \frac{1}{\tilde{k}^2} + \frac{\tilde{\omega}}{\tilde{k}^3 2\sqrt{3}} \log\left(\frac{\tilde{\omega} - \sqrt{3}\tilde{k}}{\tilde{\omega} + \sqrt{3}\tilde{k}}\right) \quad (4.5)$$

5. Maxwellian Plasma

To facilitate a comparison between the classical and the semi-classical cases we consider the Maxwell-Boltzman distribution given by:

$$f_0(v) = n\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} \exp\left(\frac{-mv^2}{2T}\right) \quad (5.1)$$

where T is the temperature of the plasma. Which describes a system of classical particles close to thermal equilibrium.

We apply a change of variables as above where ω_p is identical and λ_m is given by:

$$\lambda_m = \frac{\sqrt{2T}}{\omega_p\sqrt{m}} \quad (5.2)$$

to obtain Δ in a dimensionless form.

$$\Delta(\tilde{\omega}, \tilde{k}) = 1 + \frac{2}{\tilde{k}^2} + \frac{2\tilde{\omega}}{\tilde{k}^3\sqrt{\pi}} \int dv_x \frac{\exp(-v_x^2)}{v_x - \tilde{\omega}/\tilde{k}} \quad (5.3)$$

6. Surface Wave Solutions

We are particularly interested in how the surface wave components of the solution evolve with time. The time behavior of the solution is given by the inverse Laplace transform which is an integral along the line $(i\gamma - \infty, i\gamma + \infty)$ in the complex $\tilde{\omega}$ plane such that all poles ω_n of the transformed function satisfy $Re(\omega_n) < \gamma$.

$$\tilde{\rho}(\tilde{k}, t) = -\frac{1}{2\pi}\int_{i\gamma-\infty}^{i\gamma+\infty} \rho\,'(\tilde{k}, \tilde{\omega})\, e^{-i\tilde{\omega}t}\, d\tilde{\omega} \quad (6.1)$$

where $\rho\,'$ is the transformed charge density function. It is important to note that a pole at $\tilde{\omega} = a + ib$ corresponds to a solution which behaves like $e^{bt}e^{-iat}$.

By applying a method similar to that used in [3] we deform the contour of integration such that $\gamma \to -\infty$ but the contour still passes above each pole of $\rho\,'$. This allows us to consider the dominant behaviour determined by the pole with the largest real component, as all other poles will correspond to terms in the solution that become exponentially small compared to this dominant behaviour as they are more highly damped with time.

Following the reasoning laid out in [2] we expect all singularities of $\rho\,'$ to be due to one of the following:

(i) the singularities arising due to the transform of the initial perturbation, these will be damped out in a few plasma periods.

(ii) the roots of $\Delta(\tilde{\omega}, \tilde{k}) = 0$ which correspond to dispersion relation for bulk waves in the plasma.

(iii) the roots of $\epsilon(\tilde{\omega}, \tilde{k}_{\|}) = 0$ which correspond to surface waves on the boundary of the plasma.

As a preliminary exploration of the dispersion relation for degenerate plasma we consider the case of real wave vector k and angular frequency $\omega + i0$ with infinitesimal imaginary part (this is used to avoid a divergent integral for Δ in 3.7). By considering the *maximum modulus principle* we know that all maxima and minima of analytic functions defined on an open domain occur at the boundary of said domain. We expect the function $1/\epsilon(\tilde{\omega}, \tilde{k}_{\|})$ to be meromorphic on the complex ω plane, that is analytic on the open domain defined by the ω plane excluding the singular points. This reasoning allows us to conclude that any maxima of $1/\epsilon(\tilde{\omega}, \tilde{k}_{\|})$ along the line of real ω is caused by a pole 'close' to the real axis (although we cannot say how close). Furthermore we can associate the real component of the singular point with the position of the maximum on the real ω axis as an approximation of angular frequency when we neglect the damping effects.

7. Results

Using numerical integration of the ϵ function for the Δ functions derived in Equations 4.5 and 5.3 we look for the minima of $|\epsilon|$ along the real axis of the complex $\tilde{\omega}$ plane for various values of $\tilde{k}$.

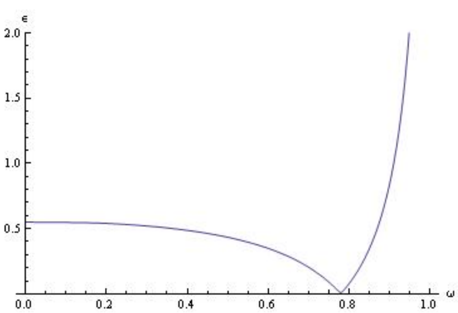

Fig. 1: $|\epsilon|$ vs $\tilde{\omega}$ for $\tilde{k} = 0.1$, degenerate plasma

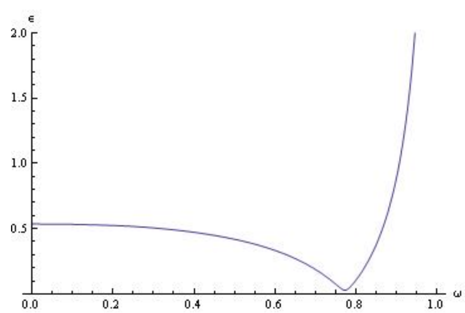

Fig. 2: $|\epsilon|$ vs $\tilde{\omega}$ for $\tilde{k} = 0.1$, classical plasma

For small $\tilde{k} = 0.1$ we see a clear minimum of $|\epsilon|$ for both the degenerate (Fig.1) and Maxwellian plasma (Fig.2) which also appear very close on the $\tilde{\omega}$ axis since both dispersion curves approach $\tilde{\omega} = \frac{1}{\sqrt{2}}$ as $\tilde{k} \to 0$. Even in this case it is evident that the minimum of $|\epsilon|$ for the Maxwellian plasma is slightly less prominent than that of the degenerate plasma.

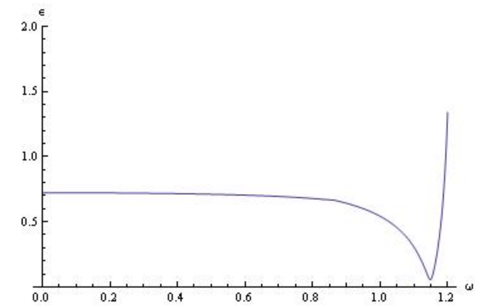

Fig. 3: $|\epsilon|$ vs $\tilde{\omega}$ for $\tilde{k} = 0.5$, degenerate plasma

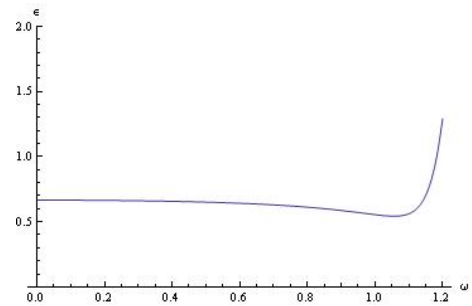

Fig. 4: $|\epsilon|$ vs $\tilde{\omega}$ for $\tilde{k} = 0.5$, classical plasma

For a larger $\tilde{k} = 0.5$ we observe in Fig.3 that $|\epsilon|$ for degenerate plasma retains a clear minimum, while in Fig.4 it is clear that the minimum of $|\epsilon|$ for Maxwellian plasma becomes much less distinct.

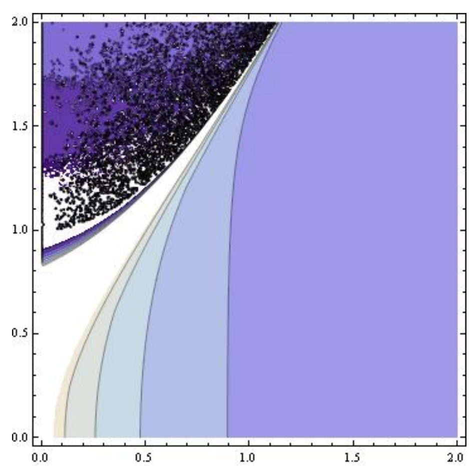

Fig. 5: $\tilde{\omega}$ vs $\tilde{k}$ dispersion plot, degenerate plasma

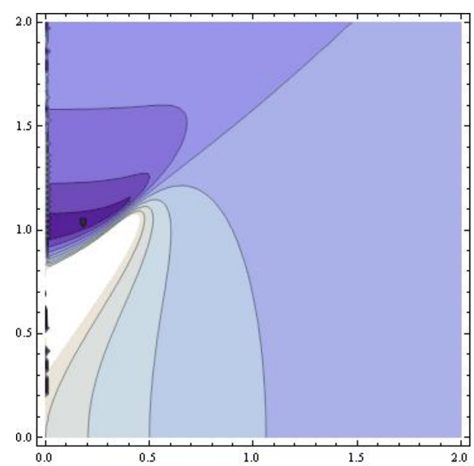

Fig. 6: $\tilde{\omega}$ vs $\tilde{k}$ dispersion plot, classical plasma

In Fig.6 and Fig.7 we present contour plots of $1/|\epsilon|$ in the real $\tilde{\omega}\tilde{k}$ plane for degenerate and Maxwellian plasma respectively. Very close contour lines correspond to the maximum of $1/|\epsilon|$; indicating a minimum of $|\epsilon|$ and hence an approximate dispersion curve for the roots of $|\epsilon|$ in the $\tilde{\omega}$ vs $\tilde{k}$ plane.

8. Discussion & Conclusion

While the plots above describe the approximate relationship between the angular frequency and wave vector along the surface of the plasma, we can also extract some quantitative information about the damping rates of surface waves. From Fig.4 and Fig.6 it is clear that for values of $\tilde{k}$ above 0.6 there are no clear minima of $|\epsilon|$ on the real $\tilde{\omega}$ line, this implies that surface waves corresponding to these wave numbers will be highly damped. From Fig.3 and Fig.5 we can see that the surface waves on a degenerate plasma are less heavily damped since the plots retain the appearance of clear maxima/minima. Similarly for degenerate plasma we have observed that for values of $\tilde{k}$ above 2 there are no clear minima of $|\epsilon|$ on the real $\tilde{\omega}$ line and hence we expect the surface waves corresponding to these wave numbers to be heavily damped. This differs significantly from bulk waves in degenerate plasma which are virtually undamped.

In the discussion given thus far we have only compared the ϵ function of the degenerate and Maxwellian plasma in terms of their dimensionless units. however we have used different normalisations for the two different dimensionless $\tilde{k}$ with the ratio of normalisation factors given by:

$$\frac{\lambda_m}{\lambda_d} = \frac{\sqrt{6T}}{v_F\sqrt{m}}, \quad = \frac{\sqrt{6mT}}{\hbar(3\pi^2 n)^{1/3}} \tag{8.1}$$

where the Fermi Velocity is $v_F = \hbar(3\pi^2 n)^{1/3}/m$. Substituting some characteristic numerical values for metal plasma into this expression we obtain:

$$\begin{aligned} &h = 1.055\times 10^{-27}\ erg\cdot s,\ m = 9.11\times 10^{-28}\ g \\ &T = 293\ K, \quad n = 10^{22}\ cm^{-3} \\ &\frac{\lambda_m}{\lambda_d} = 1.8\times 10^{7}\ K^{1/2} erg^{-1/2} \end{aligned} \tag{8.2}$$

which implies that a direct comparison of the dispersion curves for the degenerate and Maxwellian is not practical since they are of vastly different magnitudes. From this we expect that surface waves on a Maxwellian plasma will oscillate with a much higher angular frequency than those on degenerate plasma in response to the same wave vector.

A natural extension of this work would be to look at the explicit damping co-efficients for each wave vector rather than the qualitative suggestions we have provided currently.

9. Acknowledgements

Several specific thanks must be given, to my supervisors Prof. Sergey Vladimirov, Dr. Yuriy Tyshetskiy and Dr. Roman Kompaneets for answering all my questions and more and Prof Dick Hunstead and Dr. Mike Biercuk for organising and marking these projects once again.

All numerical integration and plotting was done using Wolfram Mathematica 7.0

References

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