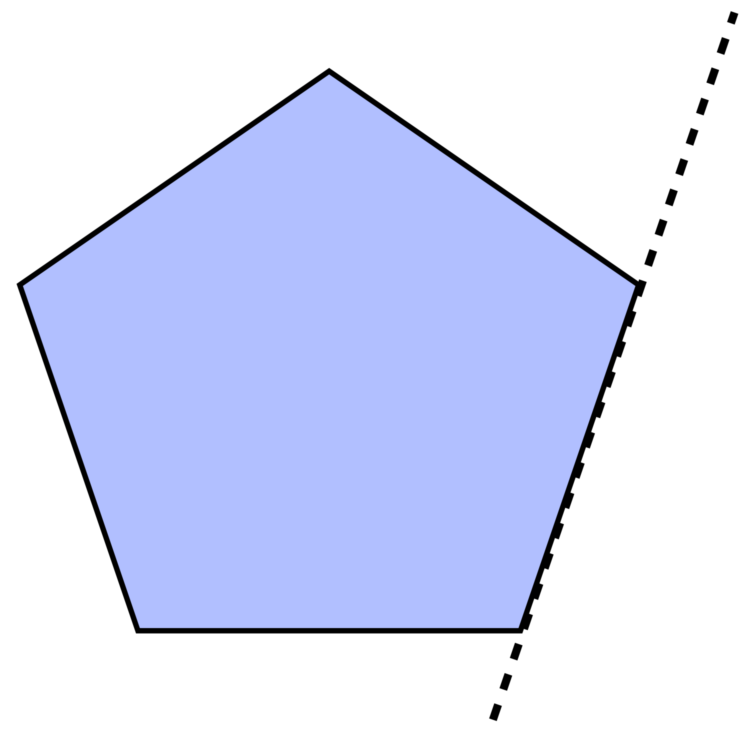
Bell inequalities made simple(r):

Linear functions, enhanced quantum violations, post-selection loopholes (and how to avoid them)



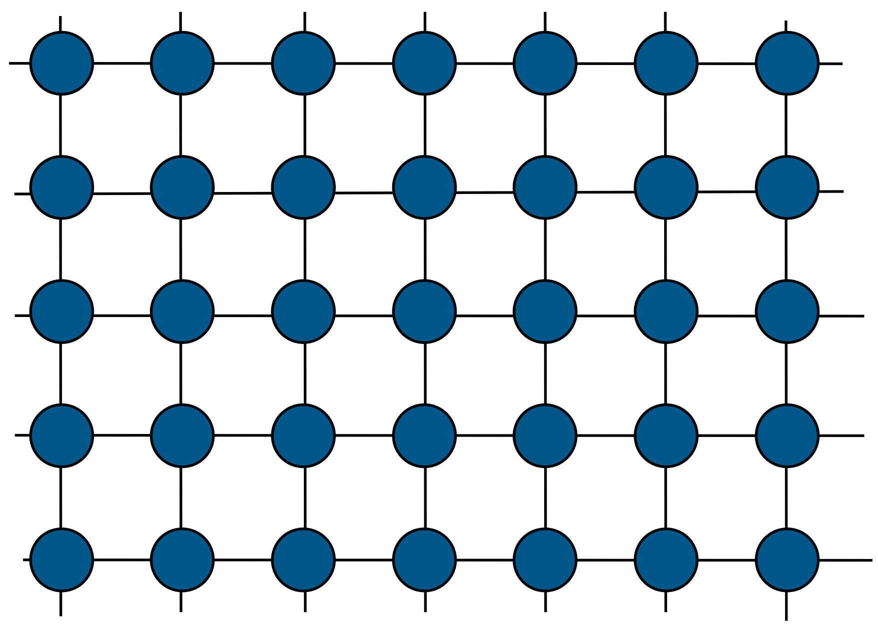
Dan Browne: joint work with **Matty Hoban**

University College London

Arxiv: Next week (after Matty gets back from his holiday in Bali).

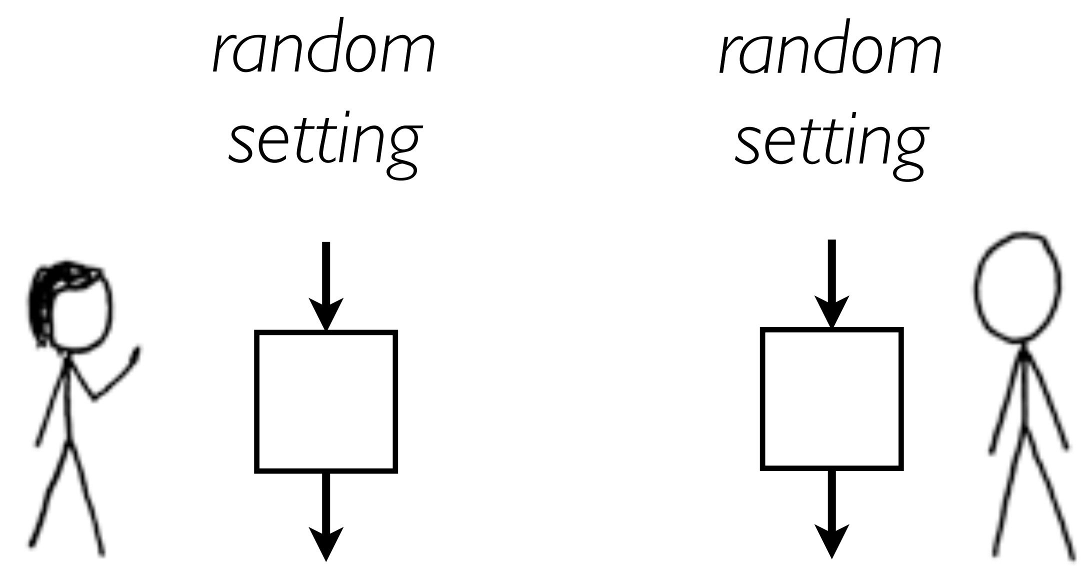
In this talk

MBQC



vs

Bell Inequalities



- I'll try to convince you that **Bell inequalities** and **measurement-based quantum computation** are related...
- ...in ways which are "trivial but interesting".

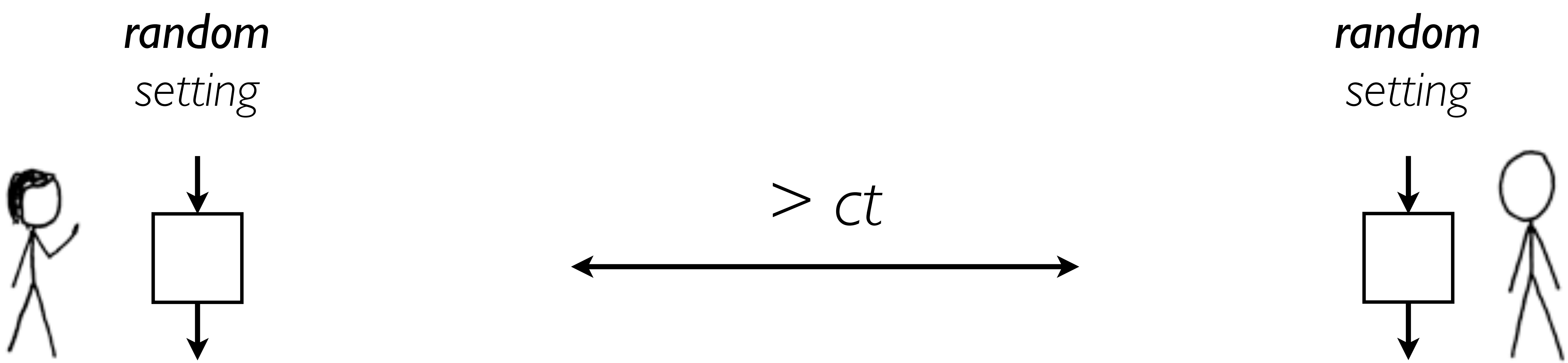
Talk outline

- A (MBQC-inspired) **very simple** derivation / characterisation of CHSH-type **Bell inequalities** and **loopholes**.
- Understand **post-selection** loopholes.
- Develop methods of **post-selection** without loopholes.
- Applications:
  - Bell inequalities for **Measurement-based Quantum Computing**.
  - Implications for the range of CHSH **quantum correlations**.

Bell inequalities

Bell inequalities

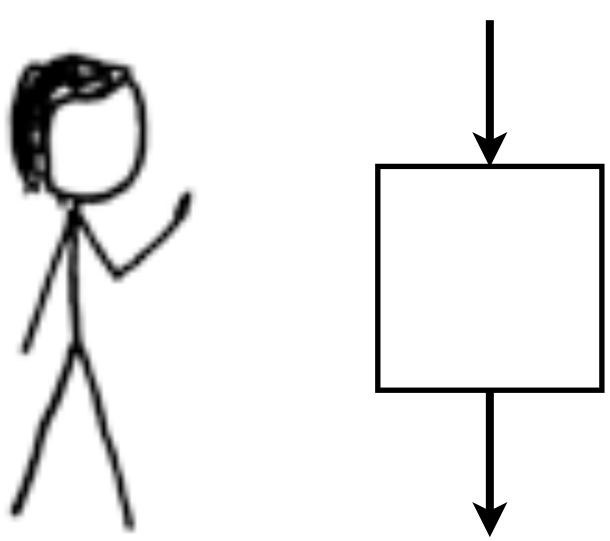
- Bell inequalities (BIs) express **bounds** on the **statistics** of spatially separated measurements in **local hidden variable (LHV) theories**.



Bell inequalities

random setting

A **choice** of different **measurements** chosen "at random".



A **number** of different **outcomes**

Bell inequalities

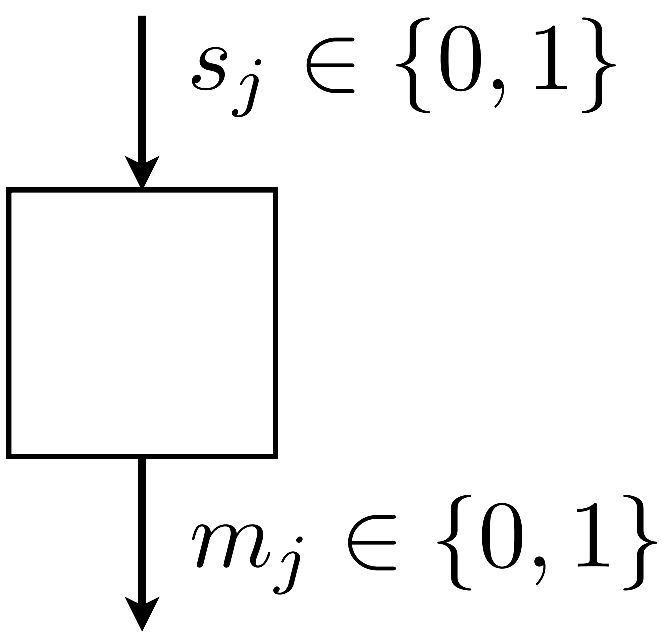
- They **repeat** their experiment many times, and compute **statistics**.
- In a **local hidden variable** (LHV) universe, their statistics are constrained by **Bell inequalities**.
- In a **quantum** universe, the BIs can be **violated**.

CHSH inequality

In this talk, we will only consider the simplest type of Bell experiment (**Clauser-Horne-Shimony-Holt).**
Each measurement has **2 settings** and **2 outcomes**.

Boxes

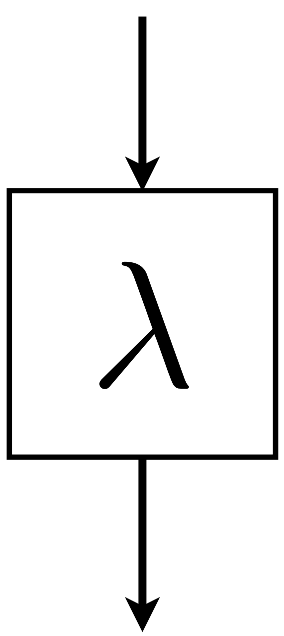
We will illustrate measurements as "**boxes**".



In the **2 setting**, **2 outcome** case we can use **bit values 0/1** to label settings and outcomes.

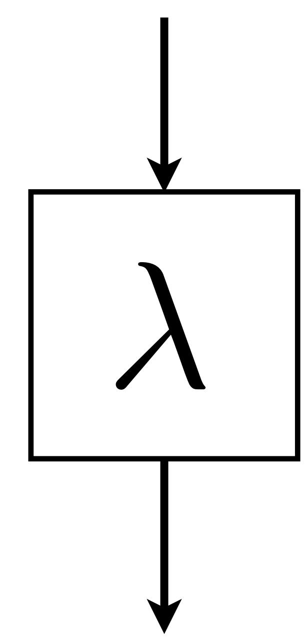
Local realism

- **Realism**: Measurement outcome depends **deterministically** on **setting** and **hidden variables λ**.



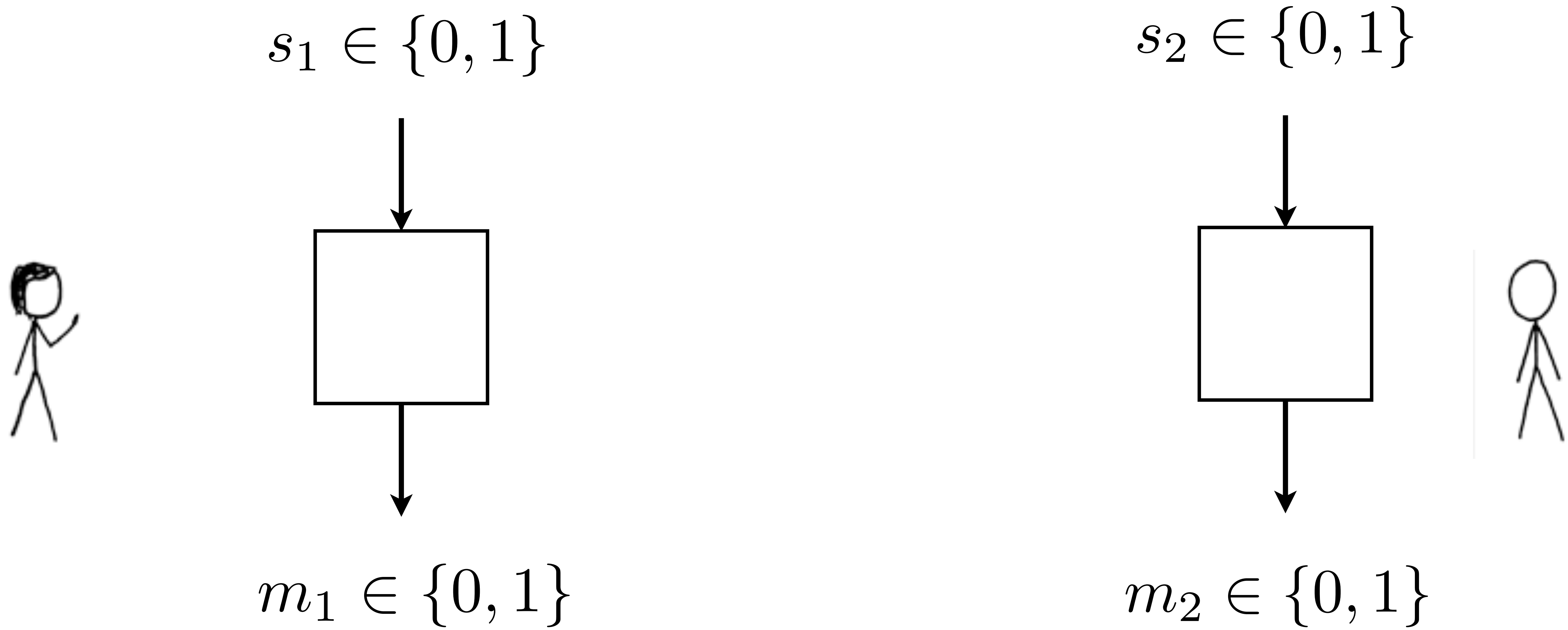
- You can think of **λ** as a long list of values, or as a **stochastic variable** (shared randomness).

- **Locality**: Outcome does not depend on the settings of the other measurement.



- **No other restriction**s are made on the "boxes", we want the "worst case scenario".

CHSH inequality



- In the classical CHSH inequality, we study the statistics of the **parity** of the measurement outcomes via the quantity:

$$E_{s_1,s_2} = p(m_1 \oplus m_2 = 0|s_1,s_2) - \; p(m_1 \oplus m_2 = 1|s_1,s_2)$$

Depends on measurement settings

same

opposite

CHSH inequality

- The range of correlations depends on underlying theory:

LHV (classical) - The CHSH inequality

$$E_{0,0} + E_{0,1} + E_{1,0} - E_{1,1} \leq 2$$

Quantum

$$E_{0,0} + E_{0,1} + E_{1,0} - E_{1,1} \leq 2\sqrt{2}$$

General non-signalling theory (PR Box)

$$E_{0,0} + E_{0,1} + E_{1,0} - E_{1,1} = 4$$

B. S. Tsirelson, Lett. Math. Phys. (1980). **S. Popescu** and **D. Rohrlich,** Found. Phys. (1994)

CHSH inequality

- The range of correlations depends on underlying theory:

LHV (classical) - The CHSH inequality

$$|E_{0,0} \pm E_{0,1}| + |E_{1,0} \mp E_{1,1}| \leq 2$$

Quantum

$$|E_{0,0} \pm E_{0,1}| + |E_{1,0} \mp E_{1,1}| \leq 2\sqrt{2}$$

General non-signalling theory (PR Box)

$$|E_{0,0} \pm E_{0,1}| + |E_{1,0} \mp E_{1,1}| \leq 4$$

B. S. Tsirelson, Lett. Math. Phys. (1980). **S. Popescu** and **D. Rohrlich,** Found. Phys. (1994)

GHZ paradox

$$|\psi\rangle = |001\rangle + |110\rangle$$

(uniquely) satisfies:

$$X \otimes X \otimes X|\psi\rangle = |\psi\rangle$$
$$X \otimes Y \otimes Y|\psi\rangle = |\psi\rangle$$
$$Y \otimes X \otimes Y|\psi\rangle = |\psi\rangle$$

which also imply:

$$Y \otimes Y \otimes X|\psi\rangle = -|\psi\rangle$$

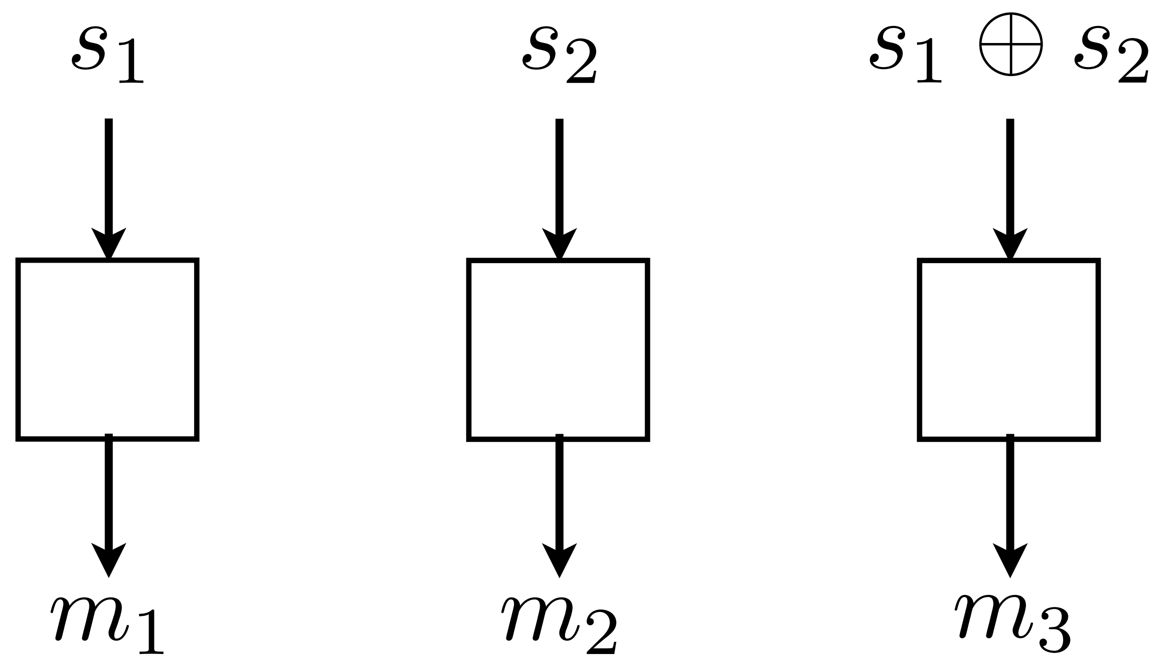
Correlations in outcomes of local measurements

GHZ "Paradox": No real number assignment of ***X*** and ***Y*** can satisfy all these equations.

N. D. Mermin (1990), building on **Greenberger**, et al. (1989)

GHZ paradox

- In the **binary box notation** these correlations can be expressed in a very clean way.



$$m_1 \oplus m_2 \oplus m_3 = s_1 s_2$$

- This looks a bit like a **computation**.

Geometric approach to Bell inequalities

Geometric interpretation of BIs

- Rather than describing the correlation in terms of E_s it is convenient to switch to the equivalent picture of **conditional probabilities.**

$$E_{s_1,s_2} = p(m_1 \oplus m_2 = 0|s_1, s_2) - p(m_1 \oplus m_2 = 1|s_1, s_2)$$

$$= 1 - 2p(m_1 \oplus m_2 = 1|s_1, s_2)$$

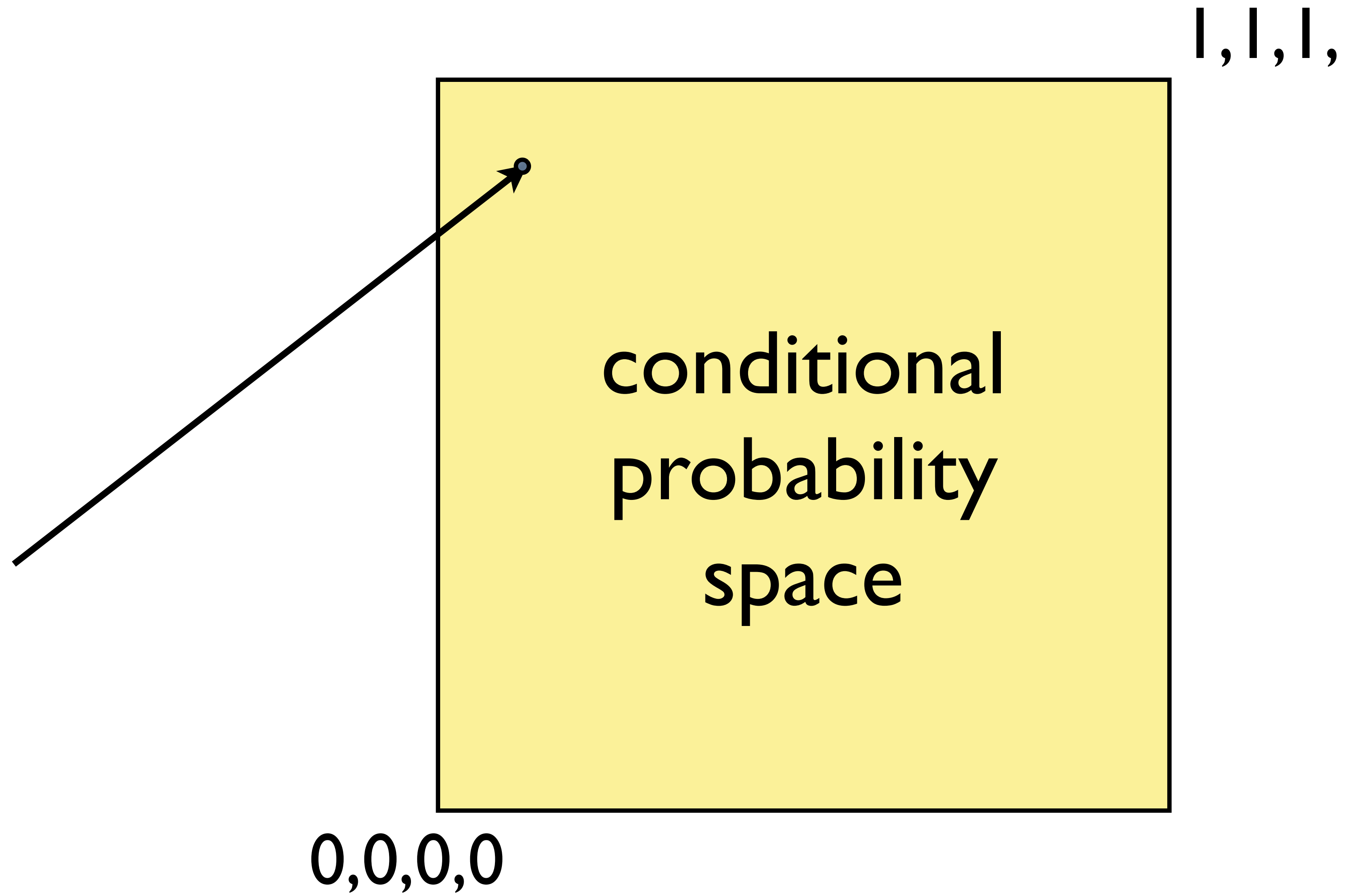
Probability that outputs have **odd parity** conditional on input settings ***s***

Geometric interpretation of BIs

- These conditional probabilities can be combined to form a real vector.

$$p(s_1, s_2) \equiv p(m_1 \oplus m_2 = 1 | s_1, s_2)$$

$$\vec{p} = \begin{bmatrix} p(0,0) \\ p(0,1) \\ p(1,0) \\ p(1,1) \end{bmatrix}$$



- Each possible set of conditional probabilities is represented a **point** in a **unit hypercube**.

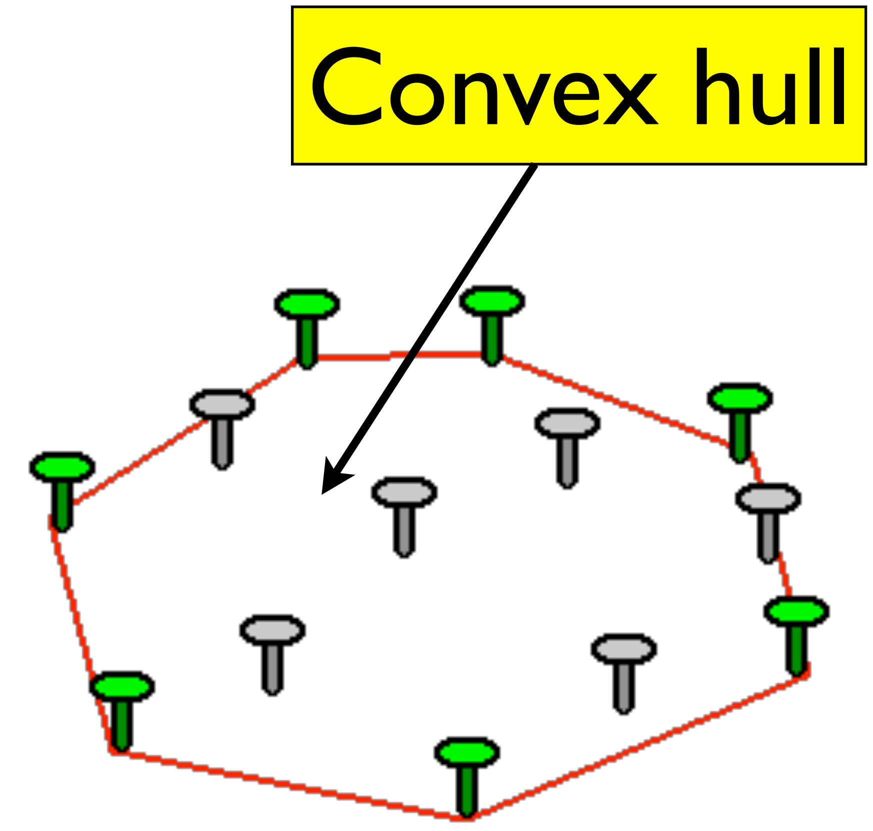
LHV Polytope

- In a **local hidden variable** model, we assume:
  - Outputs depend deterministically on the settings and the shared hidden variable **λ**.
  - Thus for a given value of **λ**

$$p(s) = f(\lambda, s)$$

  - Treating **λ** stochastically,

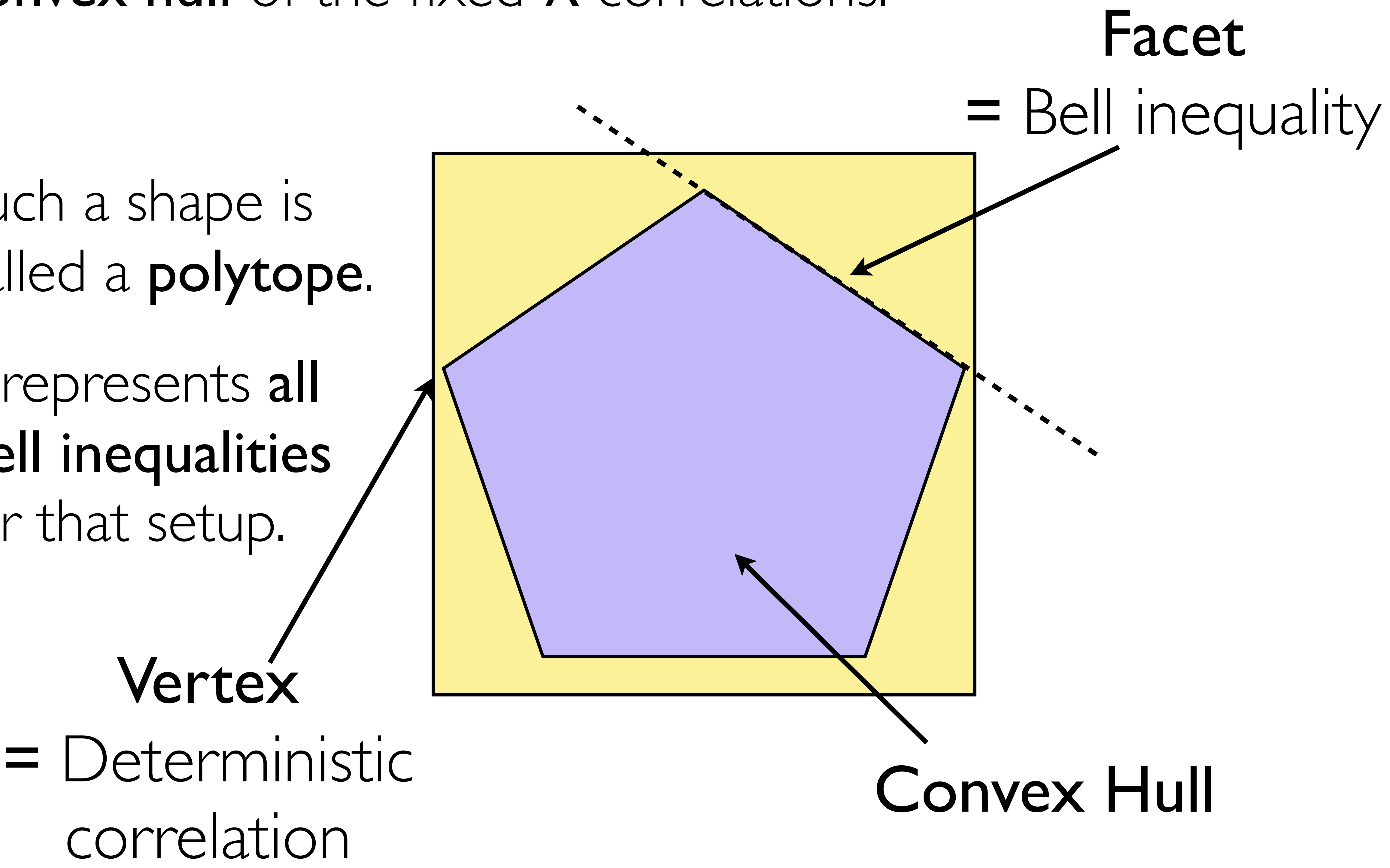
$$p(s) = \sum_{\lambda} p(\lambda) f(\lambda, s)$$



Convex combination

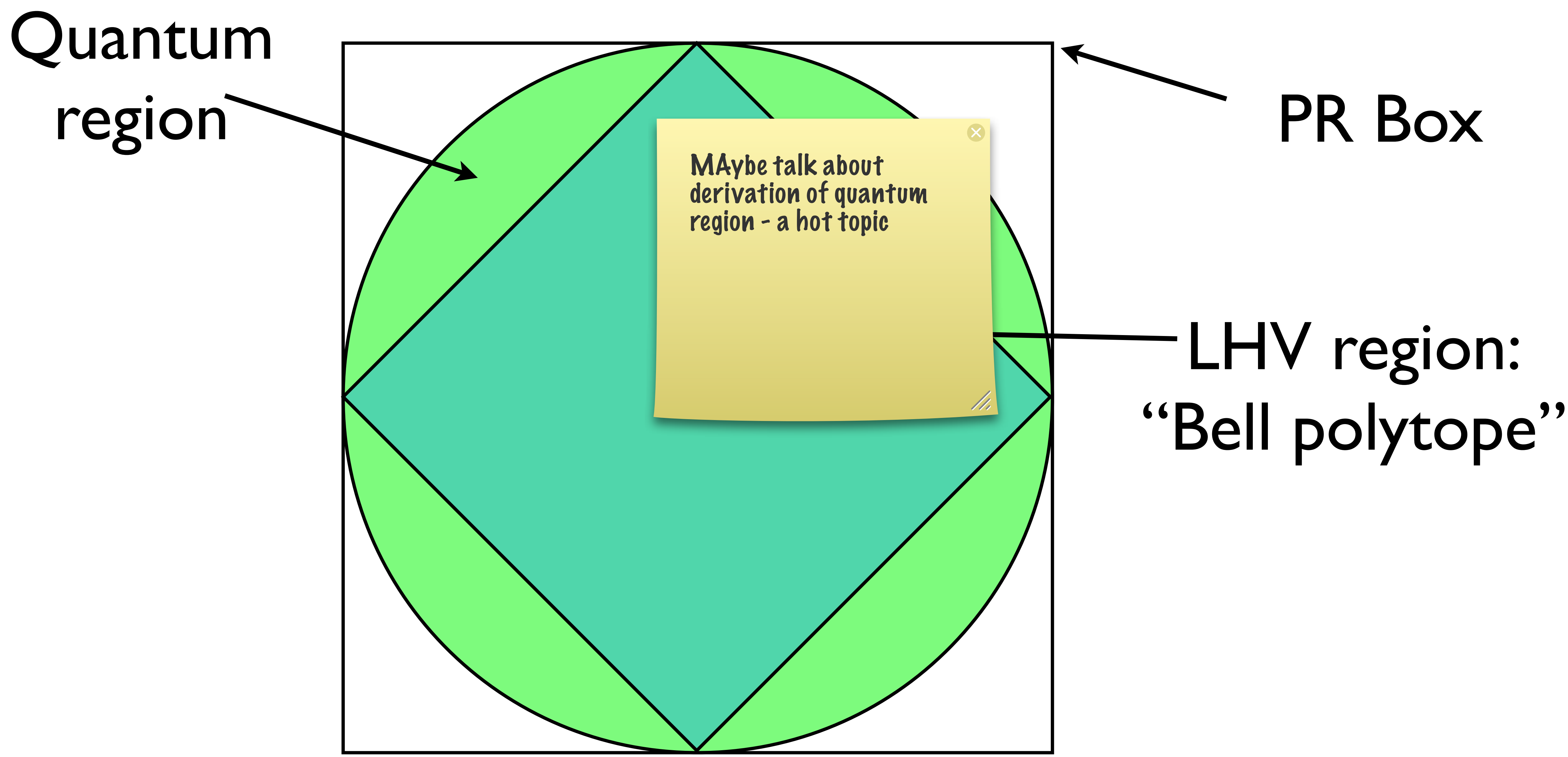
LHV Polytope

- This means that all LHV correlations inhabit the **convex hull** of the fixed-λ correlations.
- Such a shape is called a **polytope**.
- It represents **all Bell inequalities** for that setup.



LHV vs Quantum Regions

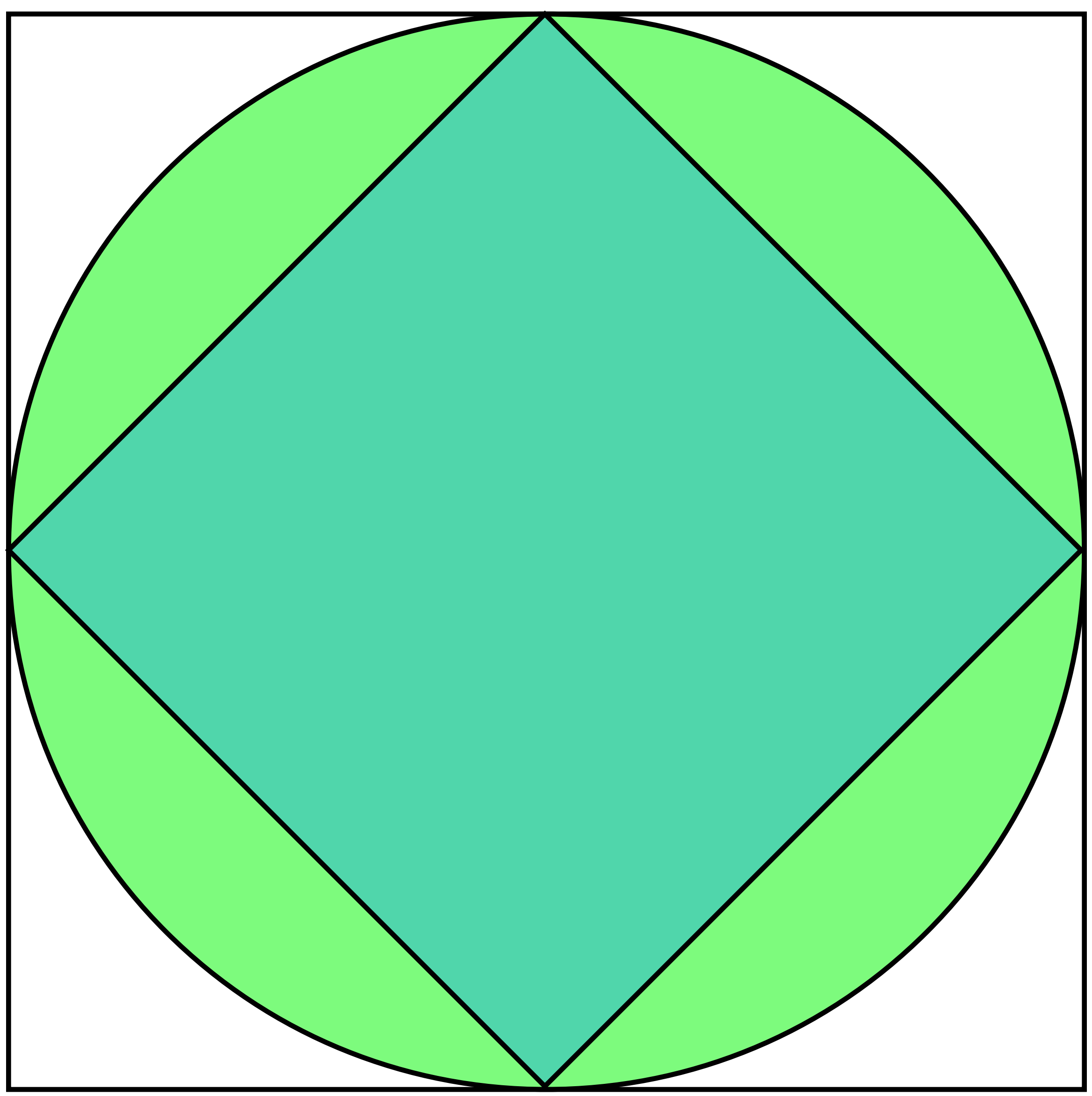
Quantum correlations **violate** Bell inequalities, but **do not** span the whole of correlation space.



Marcel Froissart: Nouvo Cimento (1981), **B.S. Tsirelson**, J. Sov. Math. (1987)

LHV vs Quantum Regions

Current hot topic: **Why** is the quantum region the shape it is?

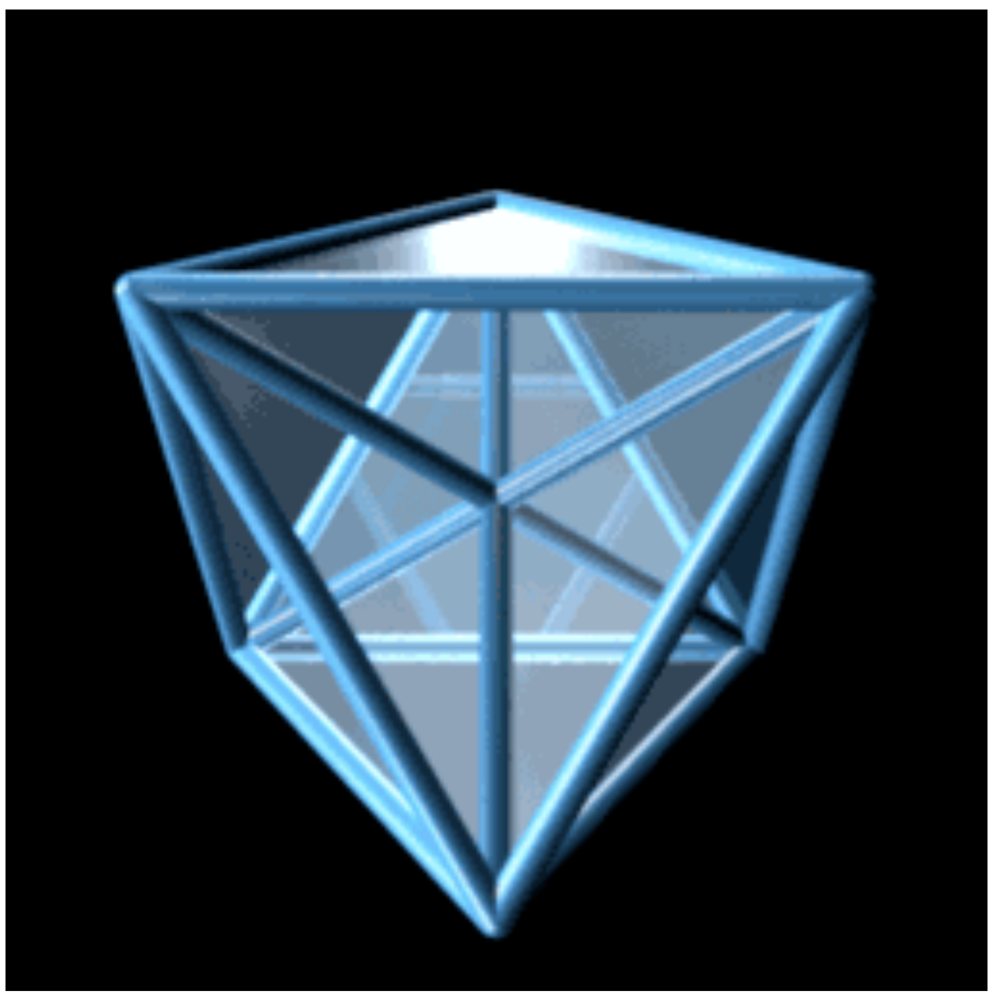


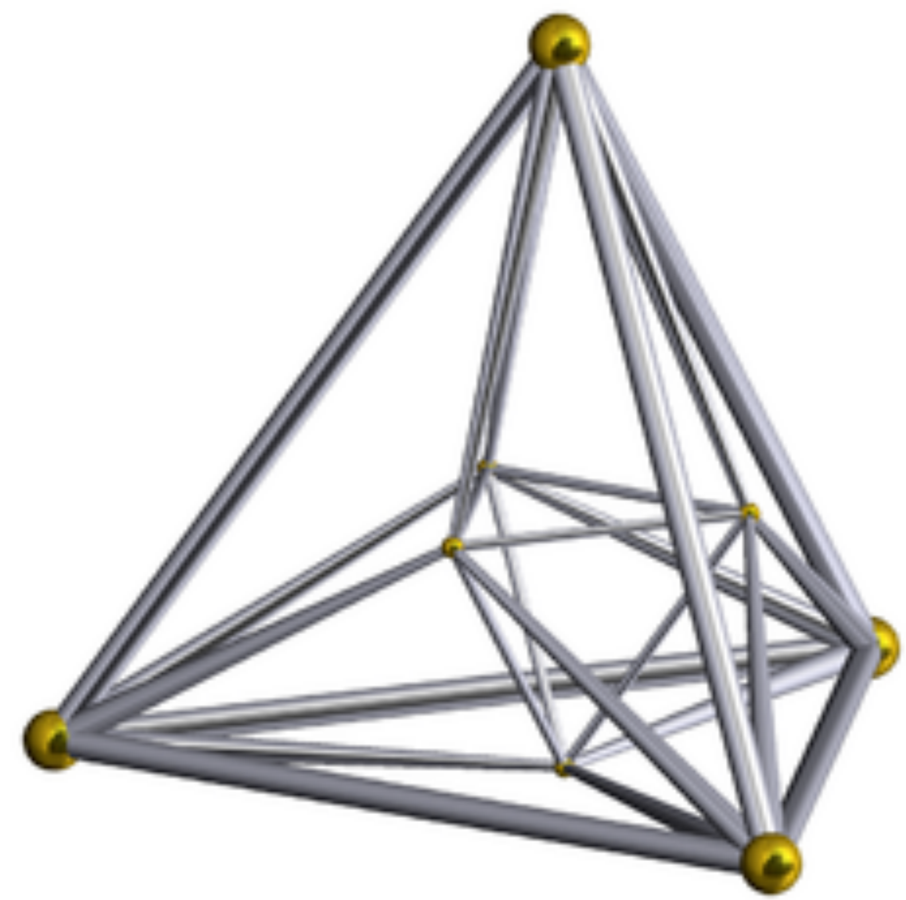
- No-signalling? (Popescu-Rohrlich)
- Information causality.
- Communication complexity.
- Uncertainty principle?

Varying degrees of success, although mostly only the bi-partite setting is investigated.

Geometric interpretation of BIs

- The LHV polytope for the CHSH experiment was first derived by Froissart in 1981.
- The polytope a **hyper-octahedron.** The facets represent the CHSH inequalities (and normalisation conditions).

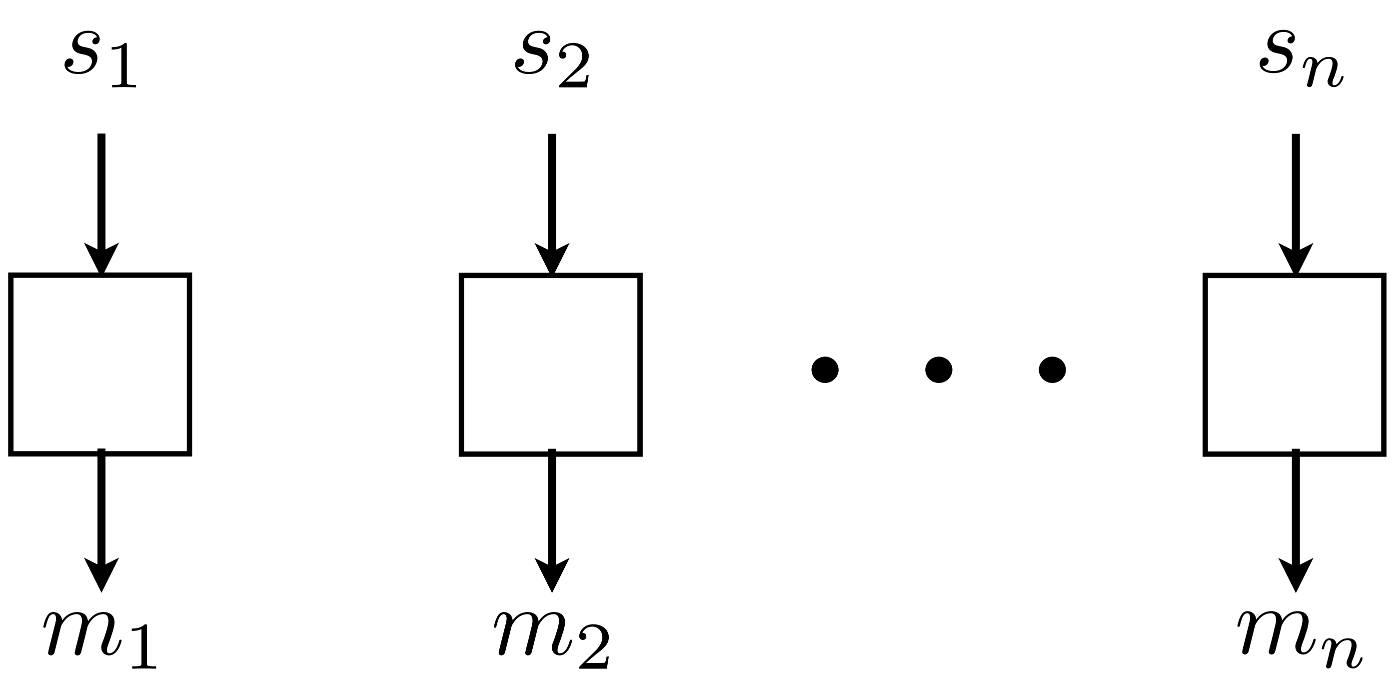


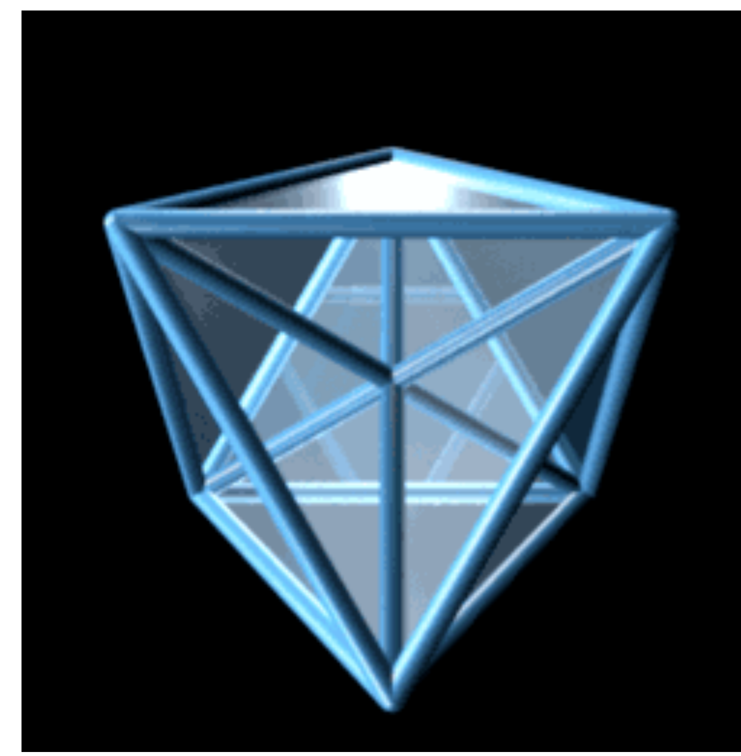


Many-party Bell inequalities

Many-party Bell-inequalities

- Werner and Wolf (2001) **generalised** the CHSH setting to **n-parties**.
    - They keep 2-settings, 2-outputs per measurement and consider conditional probs for the **parity** of **all outputs**.

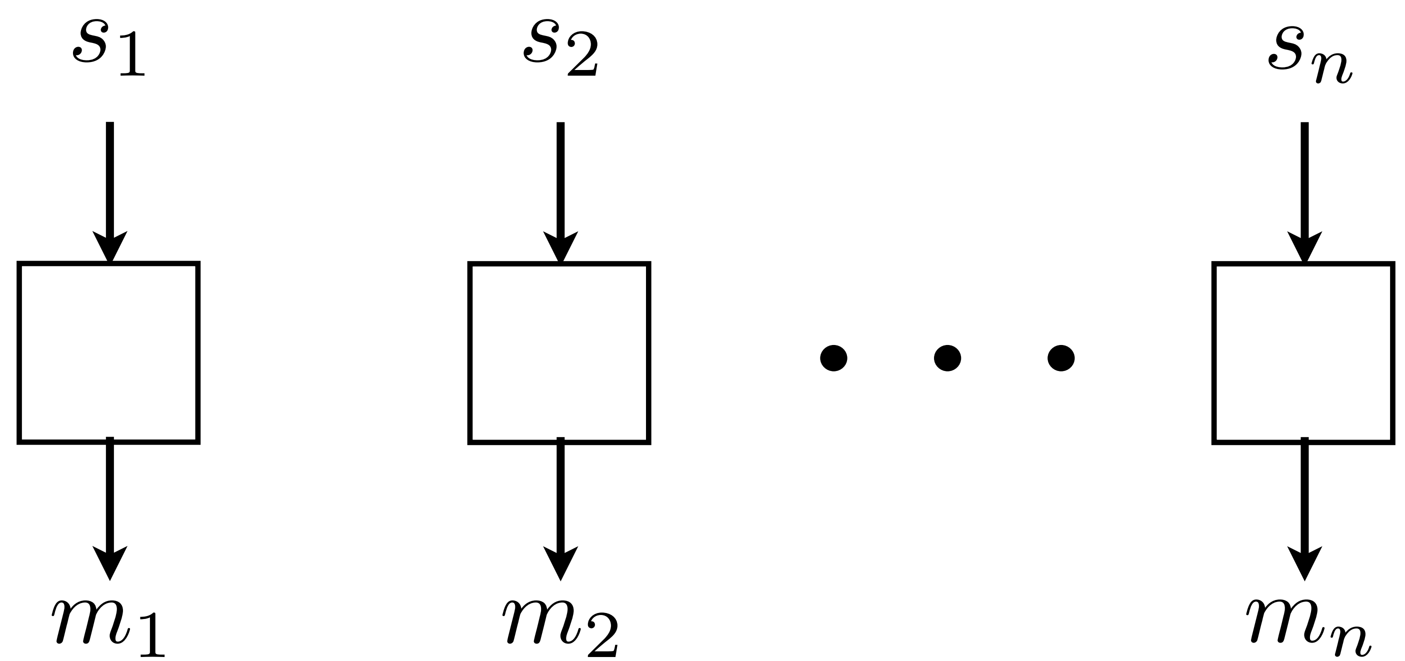




- They showed that the full **n-party Bell polytope** - for **any n**, is a **hyper-octahedron** in **2^n** dimensions.

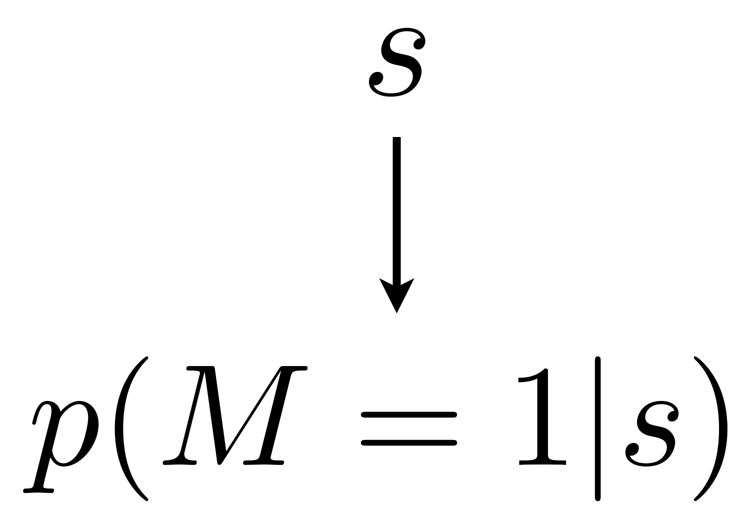
A simple characterisation of LHV correlations

Changing the lens

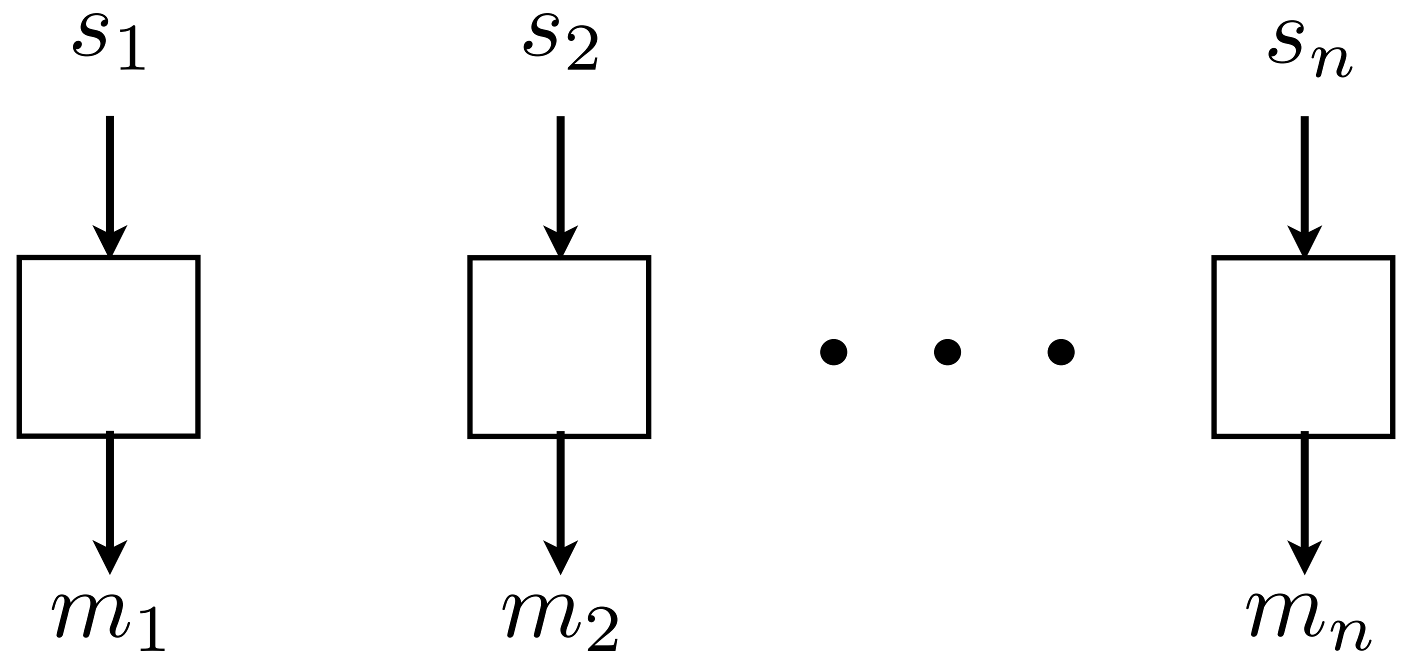


$$M = \bigoplus_j m_j$$

- A **conditional probability**
  - is a map from a **bit string**
  - to a **probability distribution**

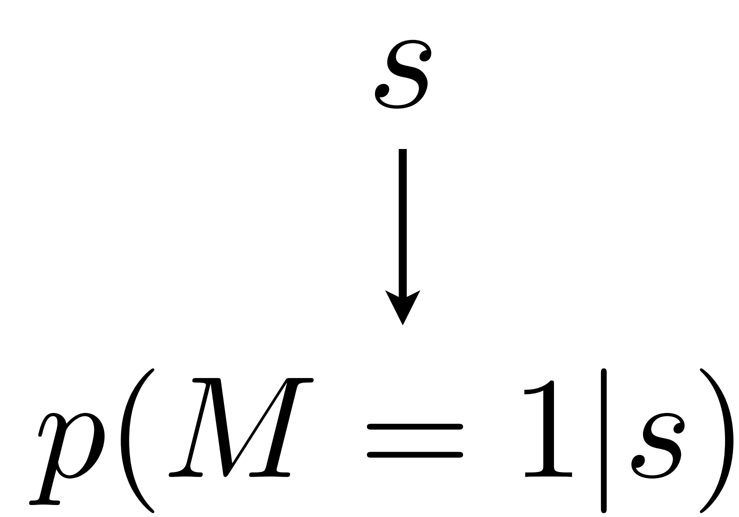


Changing the lens

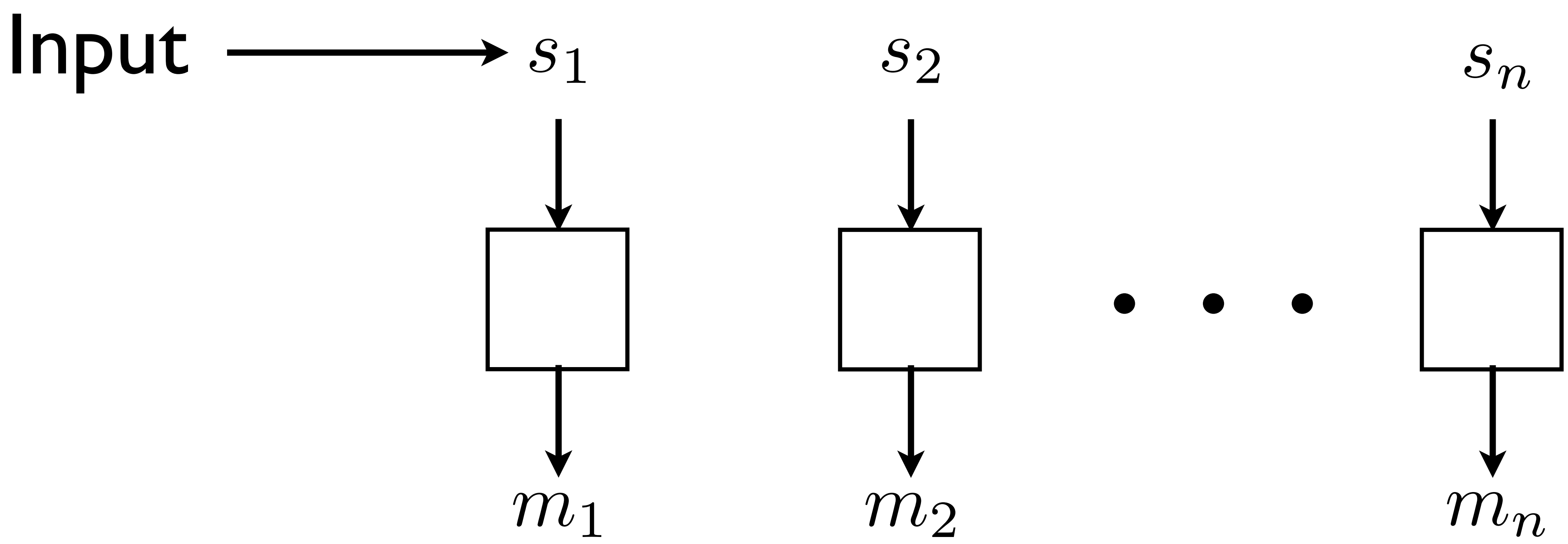


$$M = \bigoplus_j m_j$$

- A **stochastic Boolean map**
  - is a map from a **bit string**
  - to a **probability distribution**



Changing the lens



Output ⟶ $M = \bigoplus_j m_j$

- We can think of this as a **computation**.
- The structure is (a bit!) **reminiscent** of **measurement-based quantum computation**.

LHV region

- Standard approach to deriving Bell inequality region:
    - What **conditional probabilities** can we achieve under LHV?
- This approach:
    - What **stochastic maps** (computations) can we achieve under LHV?

LHV Polytope

- We said, in the LHV model, outcomes depend **deterministically** on **s** and **λ**,

$$p(s) = f(\lambda, s)$$

  and these probabilities form the **vertices** of the **polytope**.

  - If these outcomes are deterministic, given **λ** and **s,**

$$p(s) = f(\lambda, s) \in \{0, 1\}$$

  - i.e. **f(λ,s)** is a **Boolean function**.

- To **characterise the polytope,** we only need to **characterise these functions.**

Boolean functions

- A **Boolean function** maps **n** bits to **1** bit.
- Any Boolean function can be expressed as a **polynomial**.
- The **linear Boolean functions** are degree 1;

$$f(\vec{s}) = \bigoplus_{j=1}^{n} a_j s_j \oplus a_0$$

- In other words they are just **bit-wise sums, (parity**, XOR**)**.

What do we find?

- For the CHSH experiment, the functions are easy to characterise.
- In this case, the LHV region is **simply**:

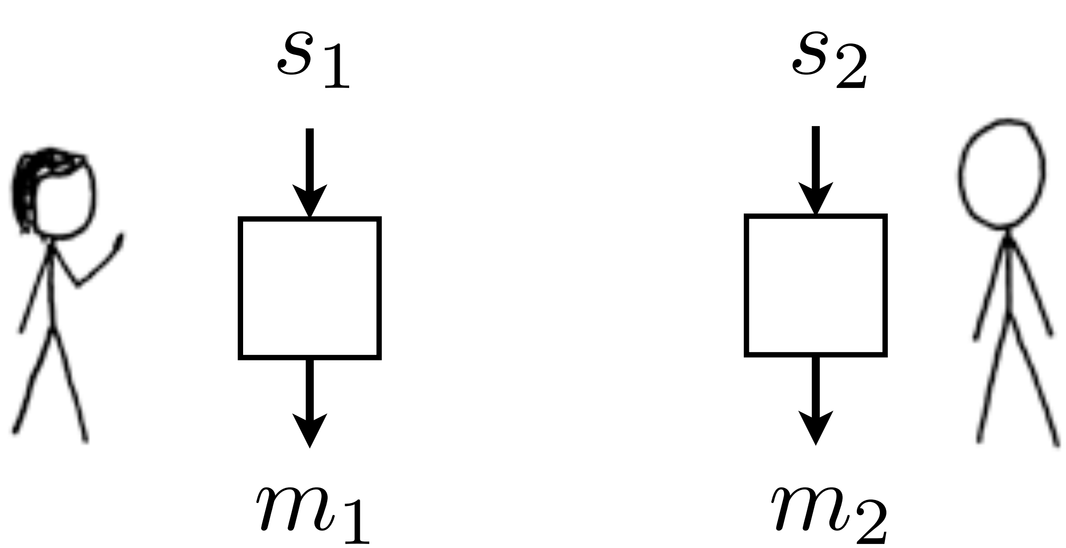
- the convex hull of **all linear functions** on ***s***.

$$M(\vec{s}) = \bigoplus_j b_j s_j \oplus a$$

- This statement defines a **4^n** facet polytope. (A mathematically equivalent polytope was derived by Werner and Wolf.)

Why this shouldn't be surprising

- It is well known in QIP that **CHSH** inequality, **GHZ** paradox, Popescu Rohrlich **non-local box**
  - can all be cast as a **computational XOR game** where the **goal** is to non-locally compute the **AND-function** on input settings.



Goal:

$$m_1 \oplus m_2 = s_1 s_2$$

See e.g. Cleve, Hoyer, Toner and Watrous (2004), Anders and Browne (2009)

What this explains

- It is well known in QIP that **CHSH** inequality, **GHZ** paradox, Popescu Rohrlich **non-local box**
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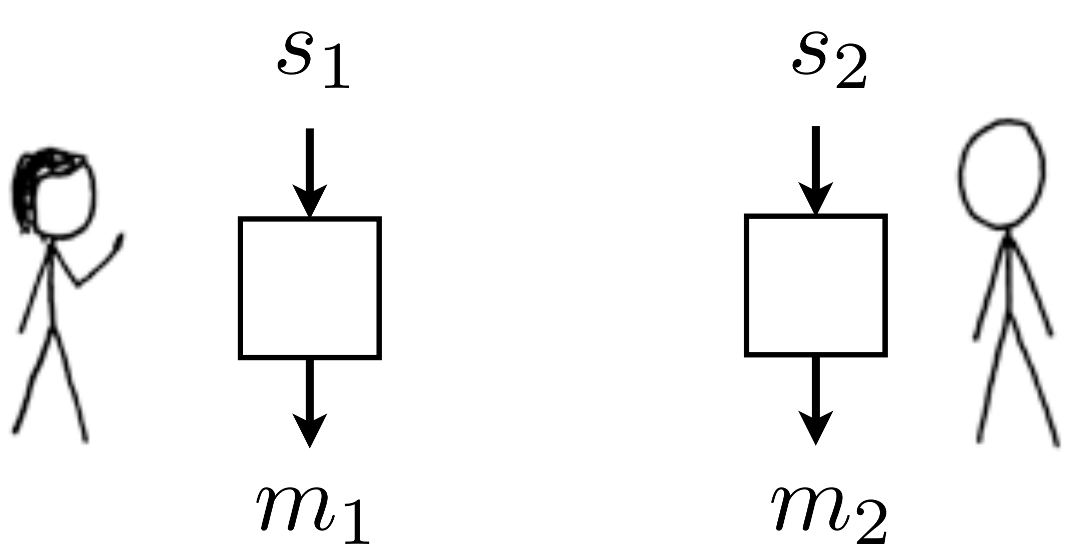


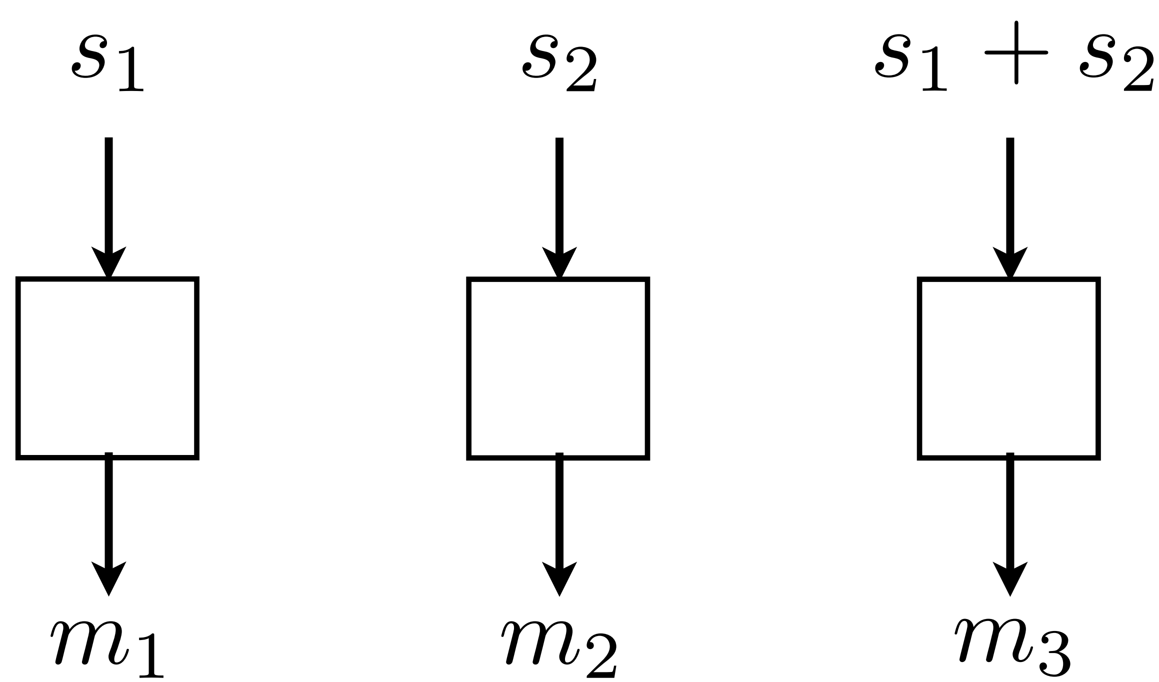
Goal:

$$m_1 \oplus m_2 = s_1 s_2$$

See e.g. Cleve, Hoyer, Toner and Watrous (2004), Anders and Browne (2009)

What else this explains

- **GHZ paradox** can be generalised. Every **non-linear** function, generates a family of GHZ-like paradoxes.



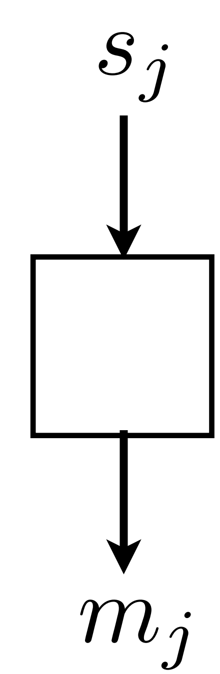
$$m_1 \oplus m_2 \oplus m_3 = s_1 s_2$$

Anders and Browne (2009), Raussendorf (2010), Hoban, et al (2010)

Simple derivation of the LHV region

Proof sketch

- We need to identify **deterministic maps** and then take the convex hull (i.e. allow LHVs to be randomly correlated.)
- First let us consider a **single box**.
  - Due to **locality** and **independence of measurements,** $\boldsymbol{m_j}$ can **only** depend on $\mathbf{s_j}$ and the local hidden variables.



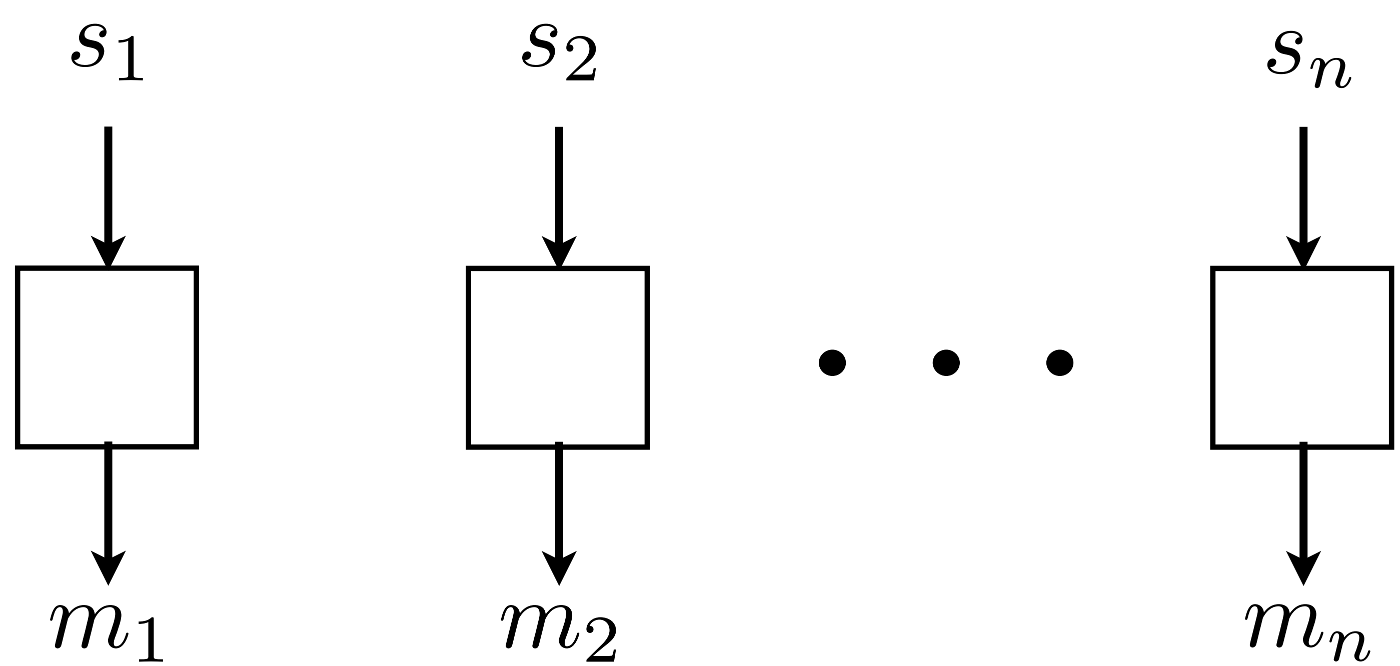
- The **most general** deterministic relationships between output and input can be written:

$$m_j = a_j + b_j m_j \qquad a_j \in \{0,1\} \qquad b_j \in \{0,1\}$$

- I.e. there are **only 4 1-bit** to **1-bit** functions - all **linear**.
- $\boldsymbol{a_j}$ and $\boldsymbol{b_j}$ depend only on the LHV **λ**.

Simple derivation of the LHV region

- Now, we consider the output of many such boxes, and consider their **parity**, whose statistics we are studying.



$$M = \bigoplus_j m_j = \bigoplus_j a_j \oplus \bigoplus_j b_j s_j$$

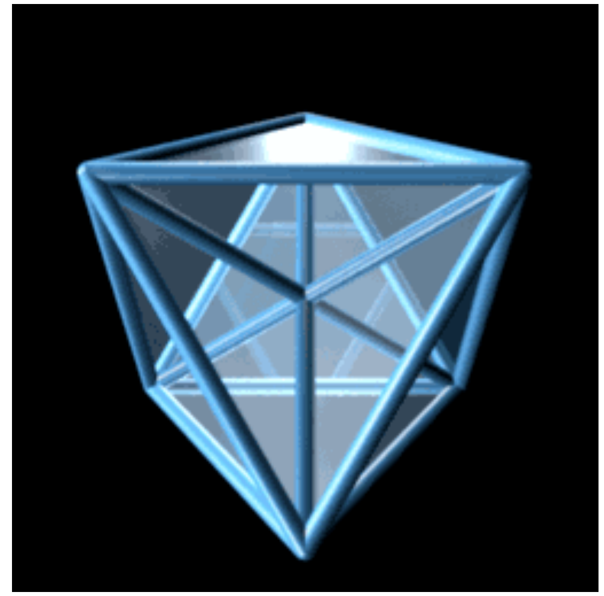
$$M(\vec{s}) = \bigoplus_j b_j s_j \oplus a$$

All linear functions on s

What do we do with this?

Werner-Wolf-Zuchowski-Brukner (2000)

Hyper-octahedron



Us (2010)

Linear functions

- Standard approach:
  - **Compute facets** of the polytope (4^n tight Bell inequalities - e.g. experimental non-locality tests).
    Straightforward, but inefficient
- Alternative approach:
  - Remain in a **vertex picture** and use the simple characterisation to prove some general results *without the need for computing facets*.
- Particularly good for studying **loopholes** and **post-selection**.

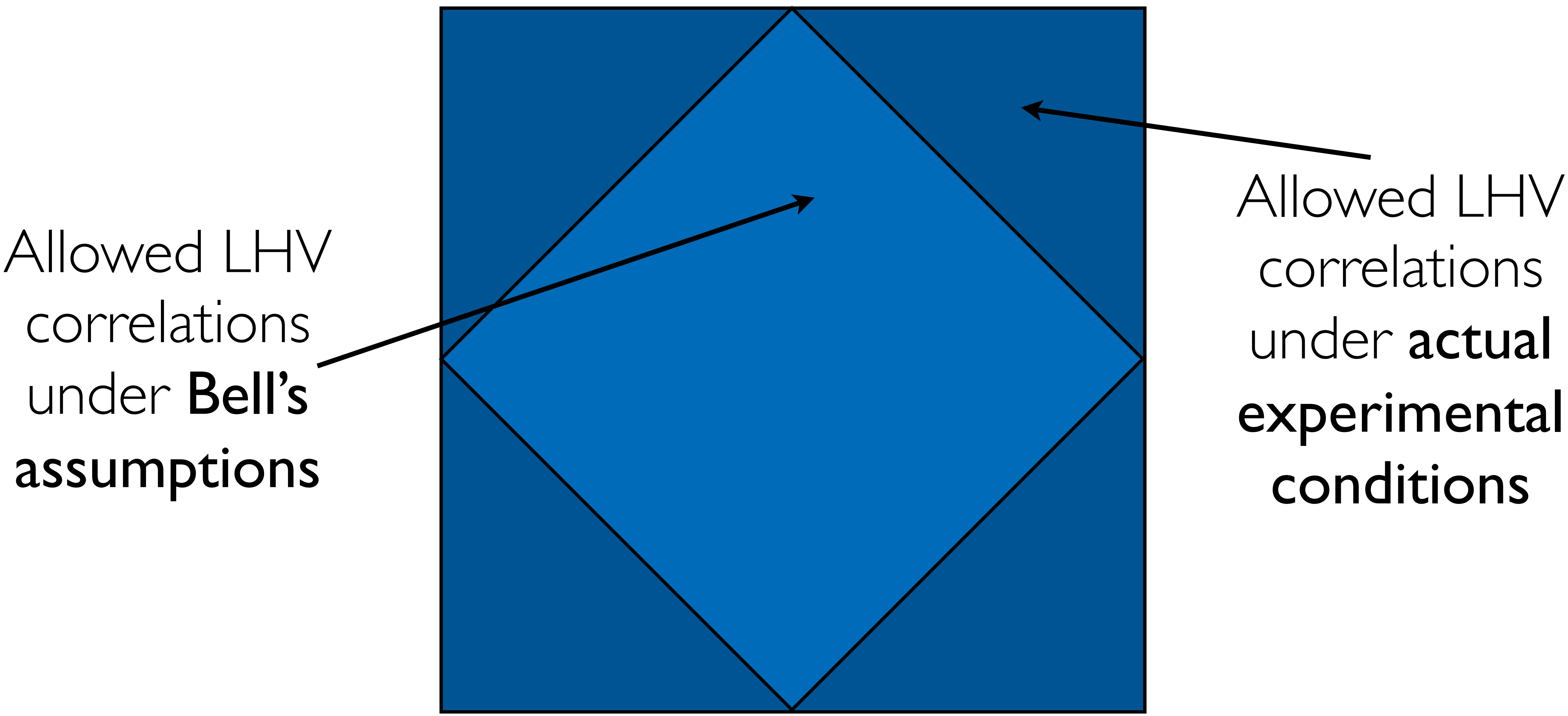
Loopholes in
Bell inequality Experiments

Loopholes in Bell Inequality experiments

- The beauty of Bell inequalities is that they are **experimentally testable.**
- However, Bell's assumptions are **strict**.
  - **Space-like separated** measurements
  - **Perfect detection efficiency**
  - Measurement settings chosen at **random** (free-will).
- If these **do not hold**, then an apparent BI **violation** may be explainable via a **LHV** theory.
- In other word - there may be **loopholes**.

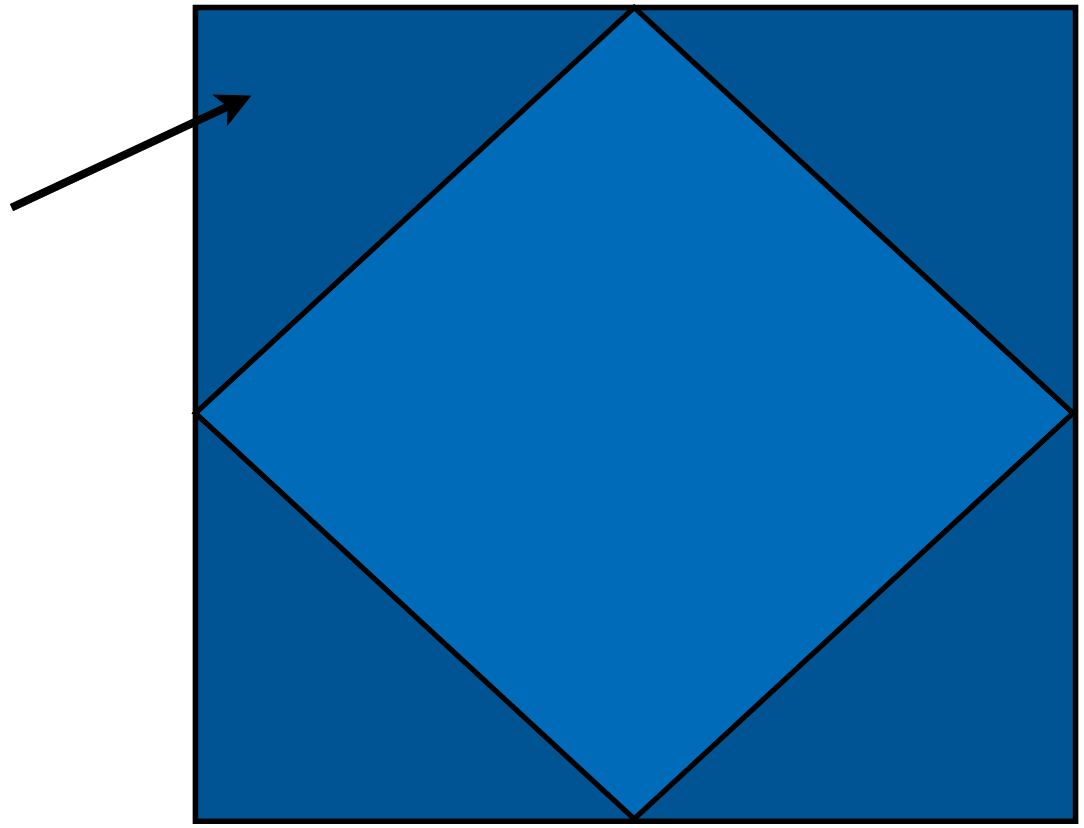
Loopholes in Bell Inequality experiments

Loopholes make the LHV region **larger**.



Loopholes

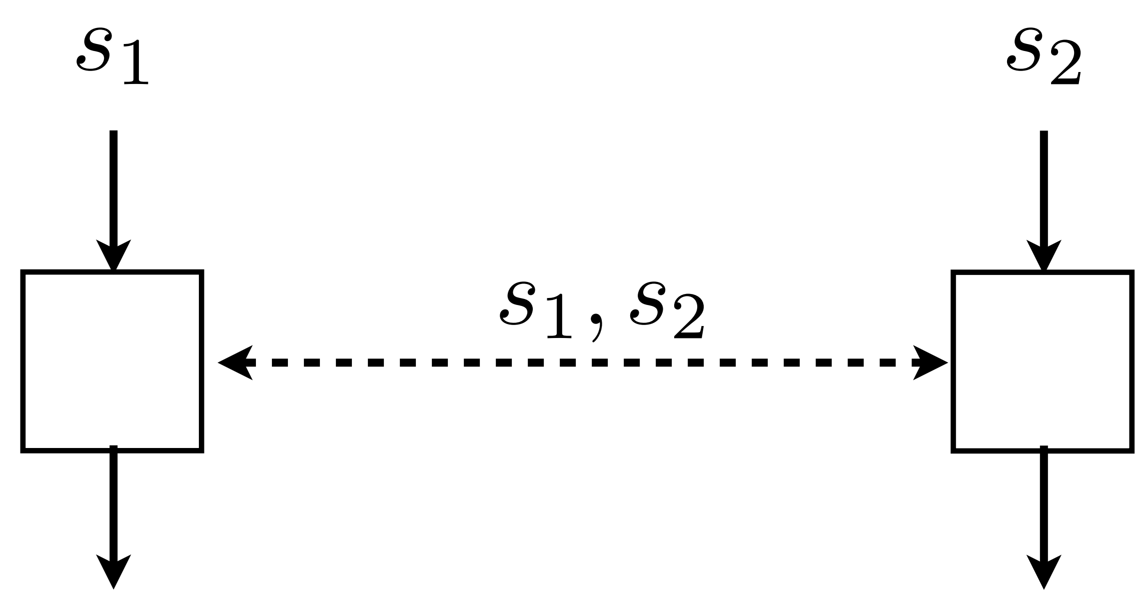
Convex sum contains a **non-linear function!**



- Since **LHV** region corresponds to linear functions, **loopholes** can only arise when there is a mechanism to compute **non-linear functions.**

Loopholes

- E.g. Locality Loophole



- If one measurement site "learns" the value of **any other input** it has the capability to output a non-linear function.

Loopholes

- E.g. Detector Loophole

- **Garg, Mermin (1987):** LHV models can fake inefficient detectors of efficiency η while violating Bell inequalities up to the bound:

$$E_{0,0} + E_{0,1} + E_{1,0} - E_{1,1} \leq \frac{4}{\eta} - 2$$

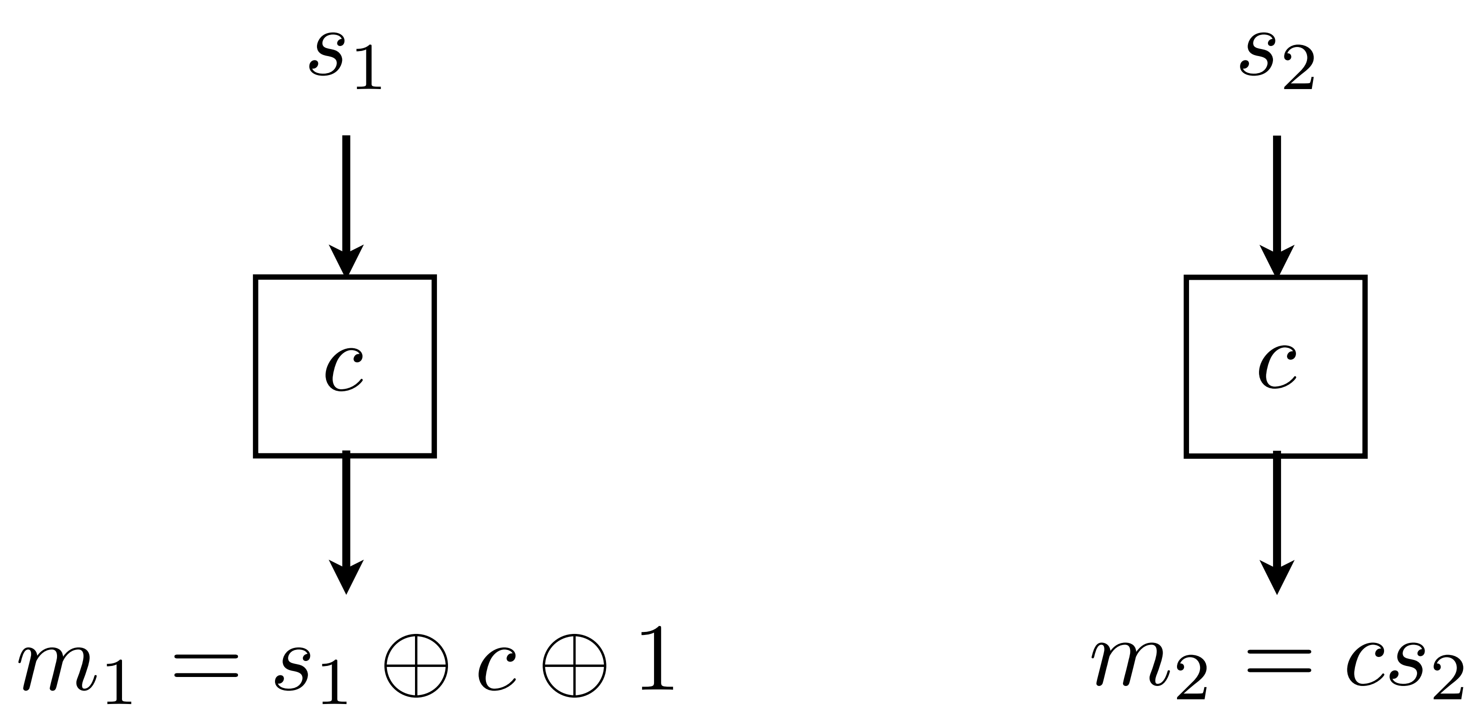
Loopholes

- E.g. Detector Loophole

- Due to the need to **post-select** the data where both detectors fire.
- Post-selection can renormalise the statistics - "**boosting**" certain conditional probabilities relative to each other.
- Here we can give an **explicit and simple model** of how post-selection can introduce a non-linearity.

Post-selection loopholes: A toy example

Consider the following LHV correlation. Bit ***c*** is a random variable shared by the boxes.



Non-linear! Loophole!

Now we **post-select** on $m_1 = 1$.

This implies $c = s_1$ and hence $m_2 = s_1 s_2$.

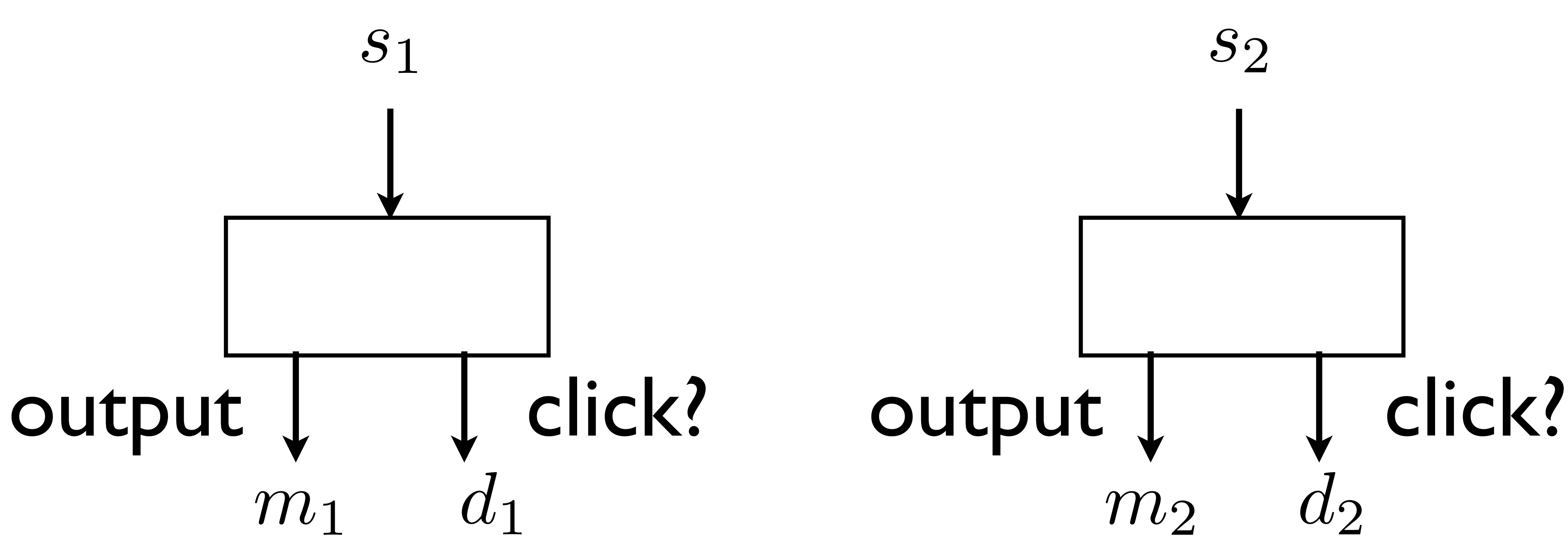
Example: A post-selection loophole

- What is the **source** of **non-linearity**?

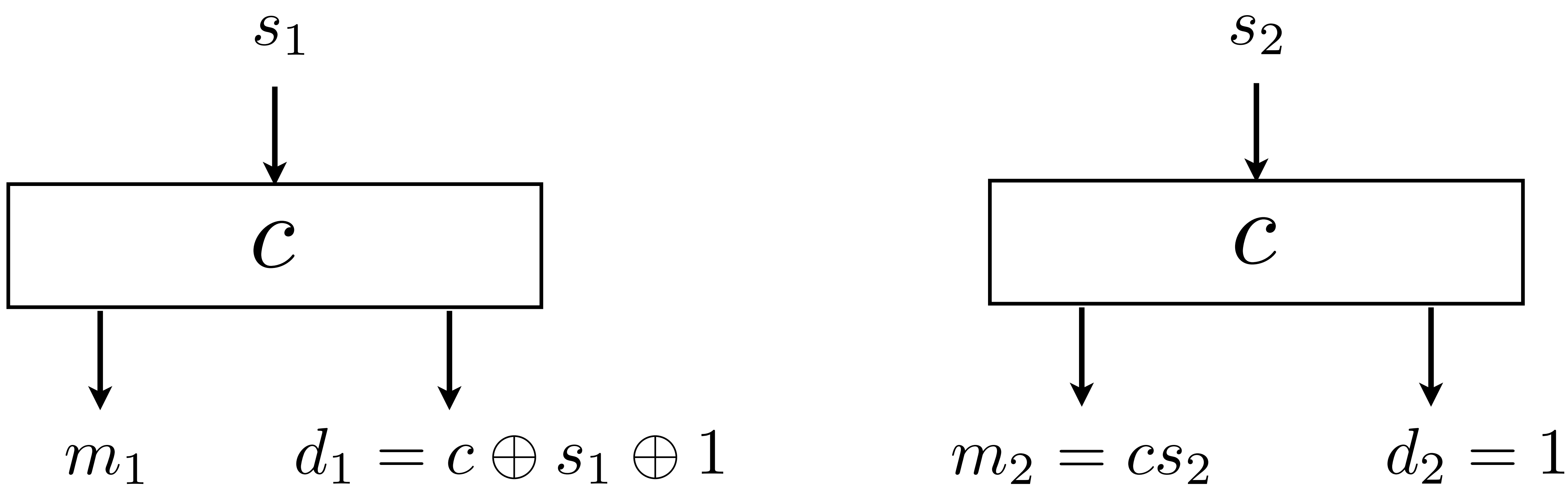
- Post-selection allows the hidden variable **c** to "learn" the value of ***s₁***.
- It is only the **lack of knowledge** of other inputs which restricted us to **linear functions** before.
- Post-selection can correlate inputs ***s*** with LHVs and the **LHVs** (shared by all parties) act as a **broadcast channel**.

The detector loophole

- The detector loophole can be understood via a similar model.
- We model an imperfect detector as a box with **2** outputs.
- The **second** output d_j will now determine whether the detector fires (**1**) or not (**0**).
- The **first** output m_j represents the output of the detector in the event that it fires.

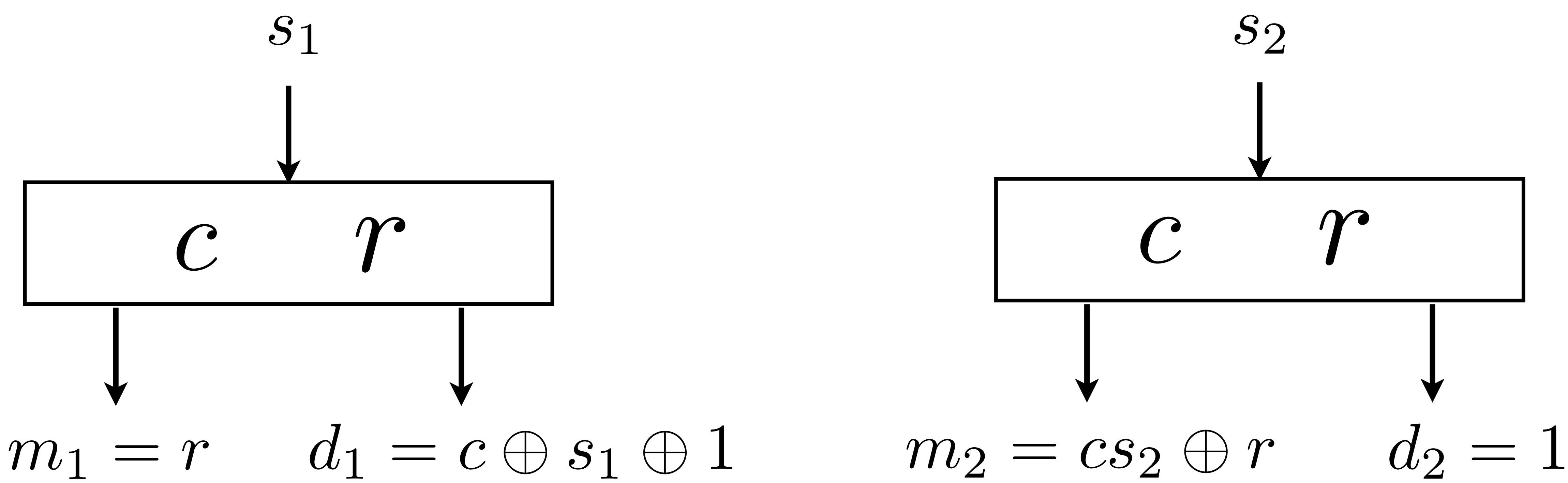


The detector loophole



- We now post-select on $d_1 = 1$.
- Assuming **c** is unbiased, we get a "click" half of the time.
- The output of detector 2 (which always clicks) equals $s_1 s_2$.

The detector loophole



- Adding shared unbiased bit **r**, we recover the statistics of the Popescu-Rohrlich **non-local box**.
- Via a further shared unbiased bit, we can **symmetrise**.
  - Half the time: **Above** strategy
    Half the time: **Mirror-flipped** strategy

The detector loophole

- We need one final step to “fake” inefficient quantum detectors.
- In symmetrised strategy:

$$p(\text{click}) = 3/4$$

$$p(\text{click},\text{click}) = 1/2$$

- Quantum detectors fail **independently**. i.e. we need:

$$p(\text{click},\text{click}) = p(\text{click})^2$$

- Solution: Add **correlated fail outcomes**.
  - Can then fake **independent detectors** with **efficiency 2/3** and perfectly simulate a non-local box.

The detector loophole

- **Garg and Mermin**

$$E_{0,0} + E_{0,1} + E_{1,0} - E_{1,1} \leq \frac{4}{\eta} - 2$$

- The model saturates Garg and Mermin's inequality for **η = 2/3**.
- By modifying the strategy, we can **boost** the faked efficiency at the cost of **lower Bell inequality violation**.
- That model then saturates G & M's inequality for all **η**.

Avoiding post-selection loopholes

- Can we post-select **without** creating loopholes?
- Post-selection can enable non-linear maps in only two ways
  - The post-selection itself induces an explict **non-linear** relationship between input bits and output.
  - Post-selection **correlates** input bits and LHVs.

Post-selection is universal

- We post-select in **every** Bell inequality experiment!

s_1	m_2	m_1	m_2
0	1	1	0
1	1	0	1
1	1	1	1
0	0	0	0
0	1	0	0

- Let $\mathbf{x}$ label the particular conditional probability we want to calculate. Then we post-select on data satisfying $\mathbf{s} = \mathbf{x}$.

Post-selection is universal

- E.g. Setting **x = 01**

s_1	m_2	m_1	m_2
0	1	1	0
1	1	0	1
1	1	1	1
0	0	0	0
0	1	0	0

- To compute $p(\bigoplus_j m_j = 1 | s = 01)$ we post-select on data where **s = 01**.

Post-selection is universal

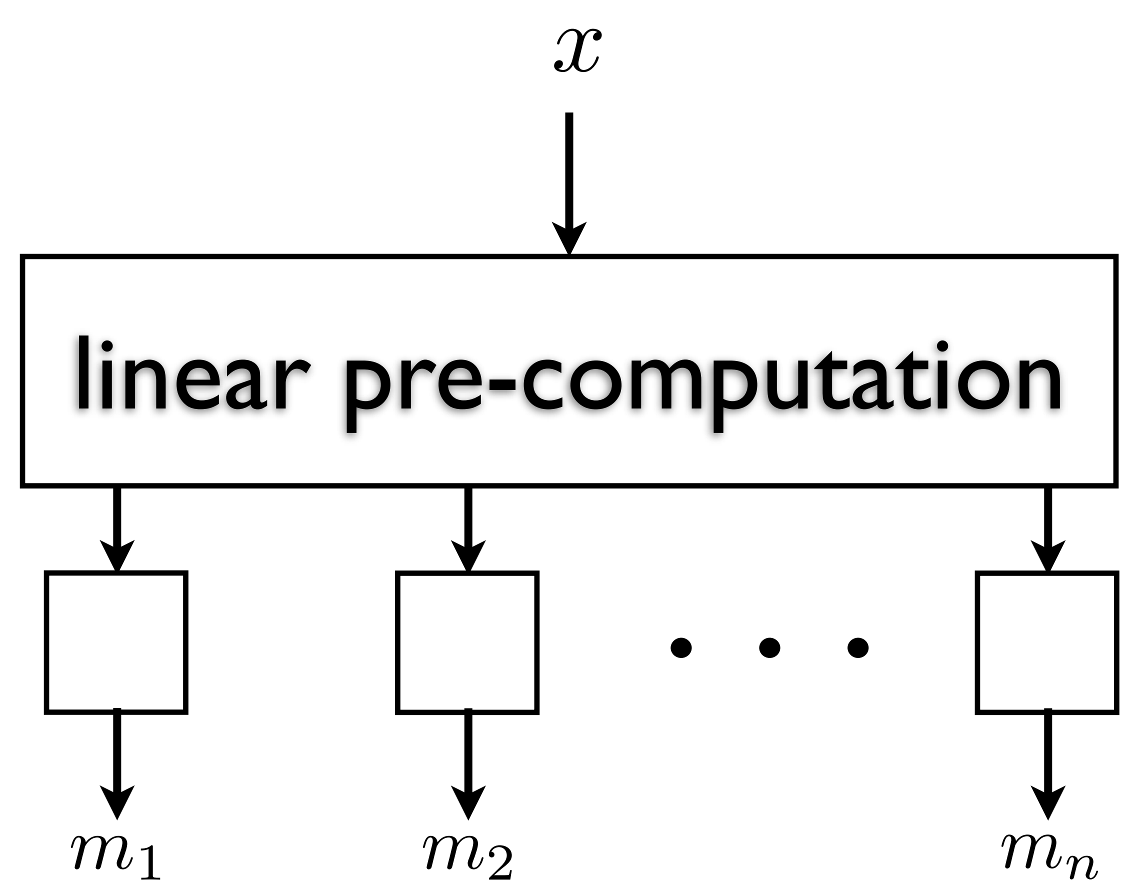
- We make this distinction since
  - **s** is an **unbiased random string**
  - **x** is **not**
- We can use this observation to **post-select** in a non-trivial way without introducing loopholes.

Loophole-free post-selection

- For example, we can post-select such that each setting bit $\mathbf{s_j}$ depends **linearly** on the bits of $\mathbf{x}$.

$$s_j = f_j(x)$$

where $\mathbf{f_j}$ is linear in $\mathbf{x}$.

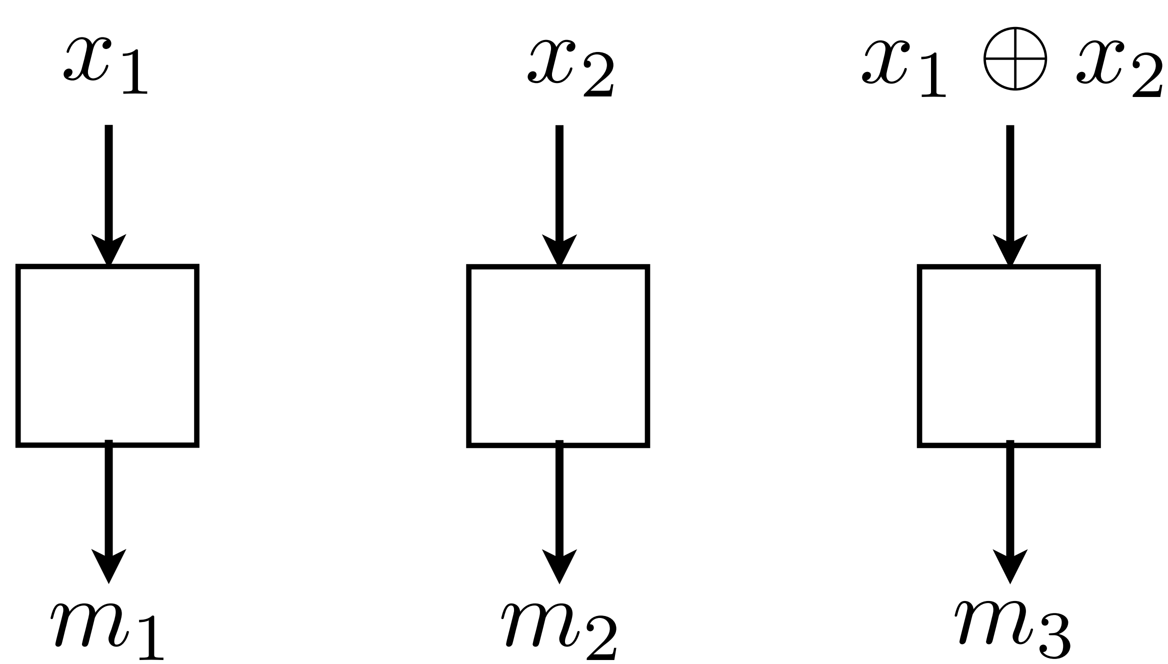


- This is equivalent linear pre-computation on $\mathbf{x}$.

- Via our earlier argument, the parity of outputs inhabits convex hull of functions linear in $\mathbf{x}$.

Loophole-free post-selection

- This isn't really new. In fact, this is the type of post-selection you'd do in a **GHZ** experiment.



- Note also, such post-selection reduces the dimension of the linear polytope from 2^**|s|**-bits to 2^**|x|**.

Loophole-free post-selection

- More interestingly, we can introduce post-selection on settings and **outputs**.

$$s_j = f_j(x) \oplus g_j(m)$$

  where $\mathbf{f}_j$ and $\mathbf{g}_j$ are linear functions.

- This looks dangerous. We know that measurement bits can act as a **conduit** to map information onto the shared LHVs.

- Surprisingly, after such post-selection, the parity of output bits remains **linear**. No loophole is induced.

Loophole-free post-selection

- The intuition of why this post-selection induces no loopholes is the following:

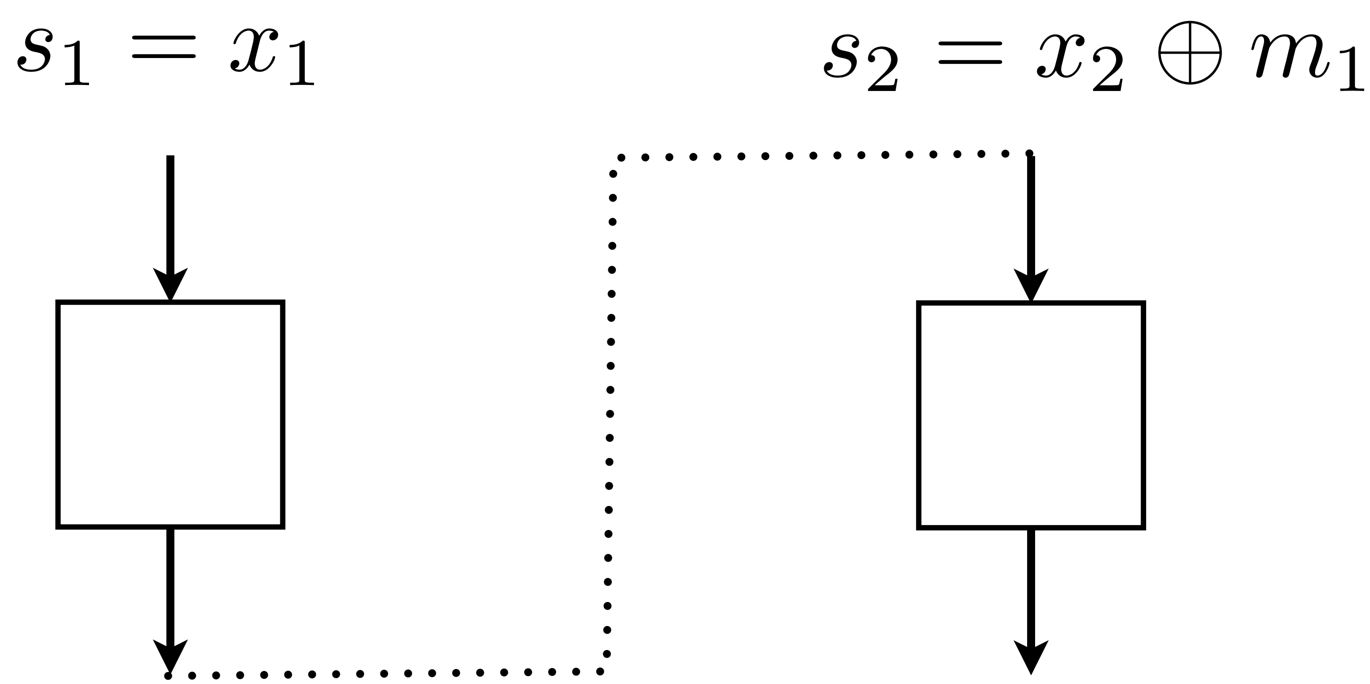
$$s_j = f_j(x) \oplus g_j(m)$$

- $\mathbf{s_j}$ is an **unbiased bit**. It thus acts as a "pad" preventing the measurement bits from "learning" any information about **x**.

- It doesn't matter whether the $\mathbf{s_j}$'s are correlated, only that their marginals are unbiased.

Loophole-free post-selection

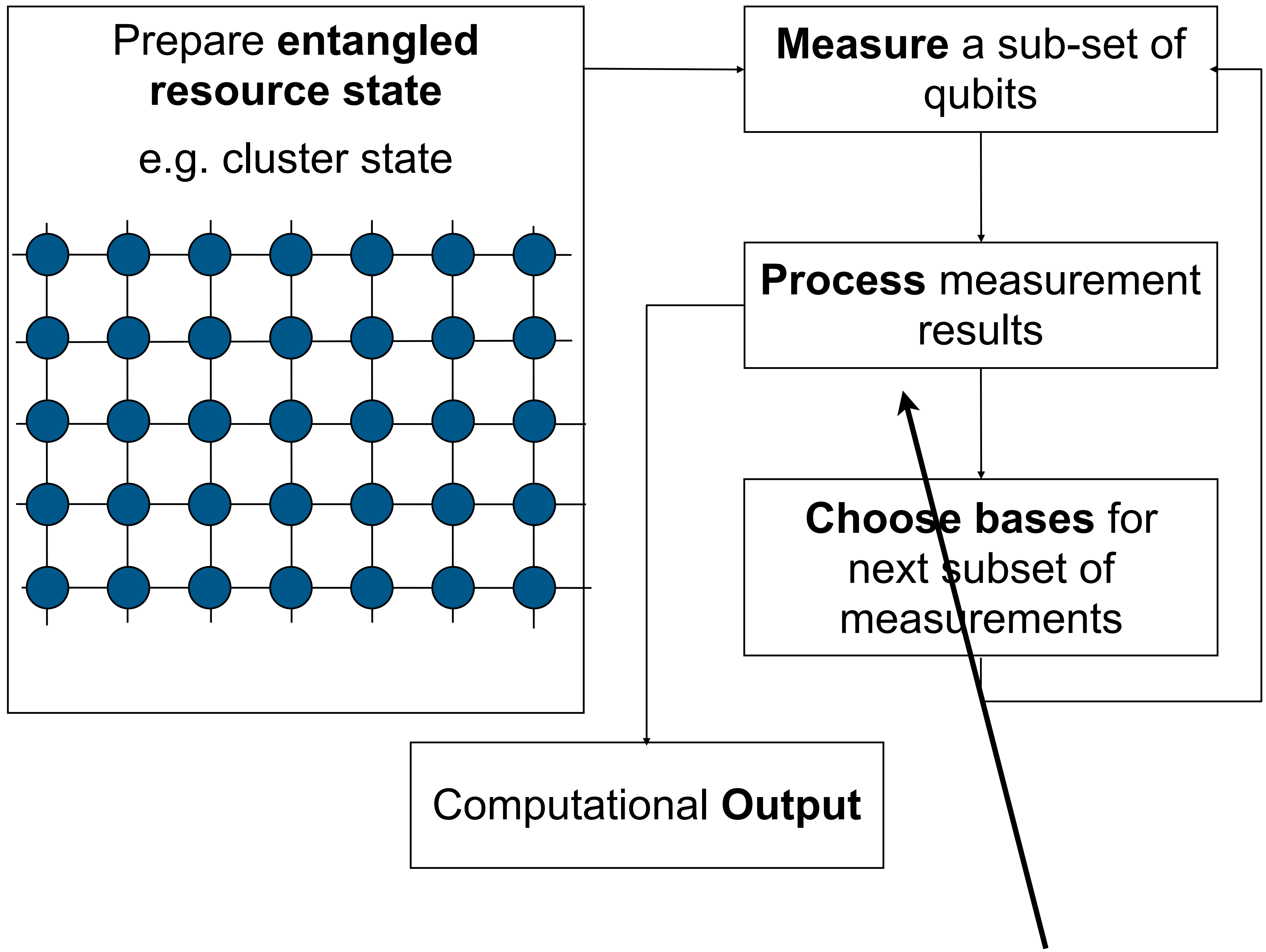
- This type of post-selection can "simulate" an **adaptive measurement.**
- E.g



- Provided that adaptivity is linear, e.g. settings depend only **linearly** on other measurement outcomes.

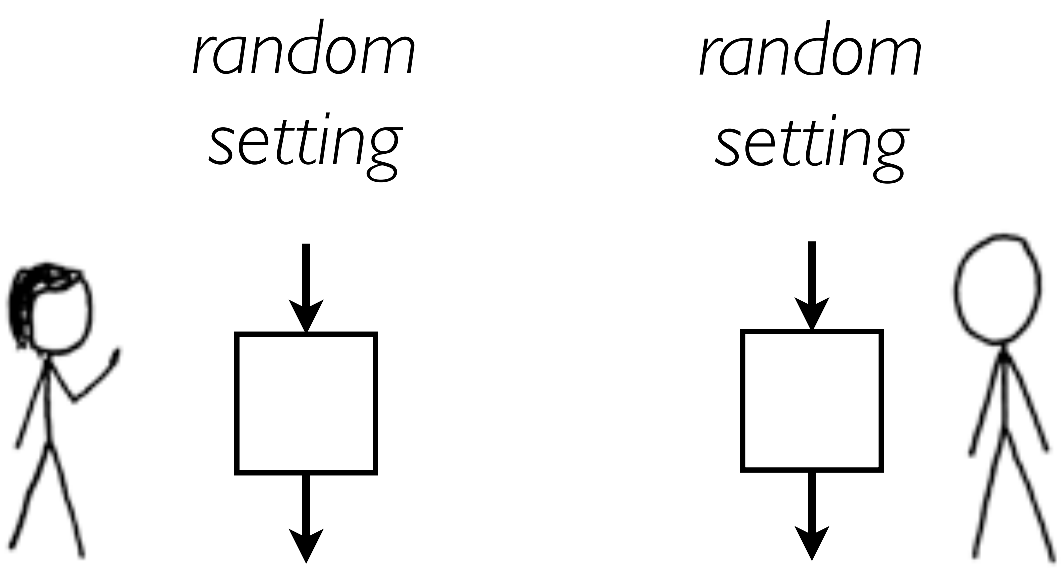
Bell tests vs MQBC

Measurement-based quantum computation



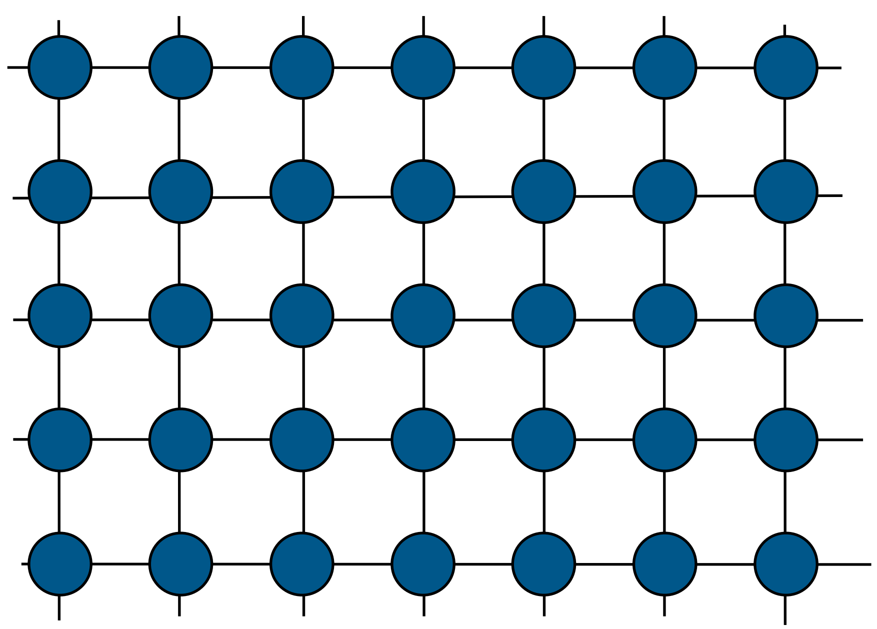
Measurements are adaptive

Bell Tests vs MBQC



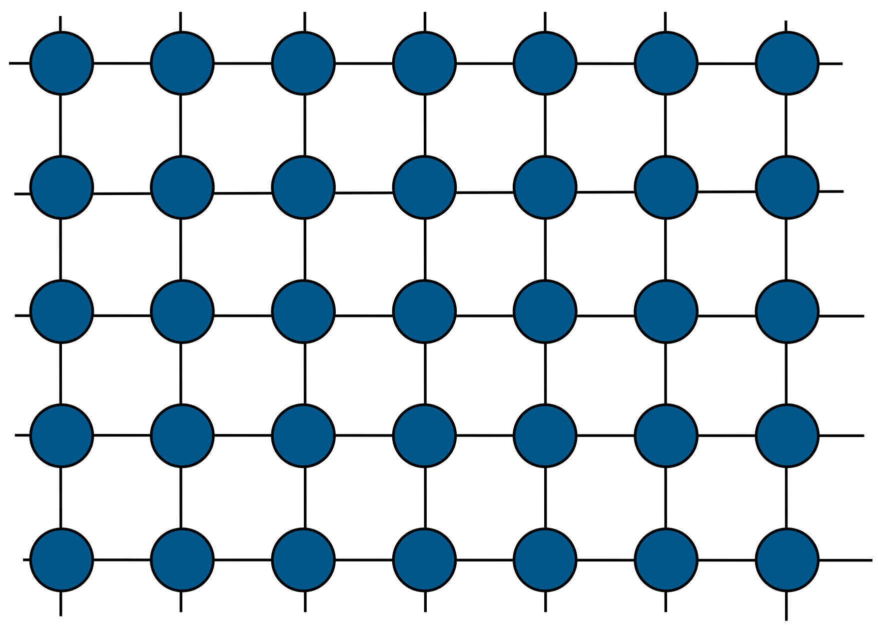
- **Bell test**
  - **Single-site** measurements
    - *Random settings, space-like separated*
  - on an **Entangled** State
  - to achieve a **Non-classical Correlation**
    - and hence refute Local Hidden Variable (LHV) Theories

Bell Tests vs MBQC

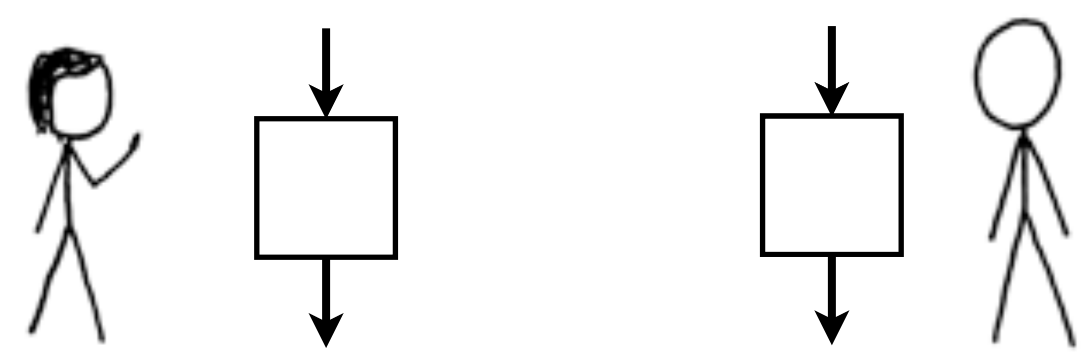


- **Measurement-based Quantum Computing**
  - **Single-site** measurements
    - *Adaptive*
  - on an **Entangled** State
  - to achieve a **Non-Classical Computation**

Bell Tests vs MBQC



vs



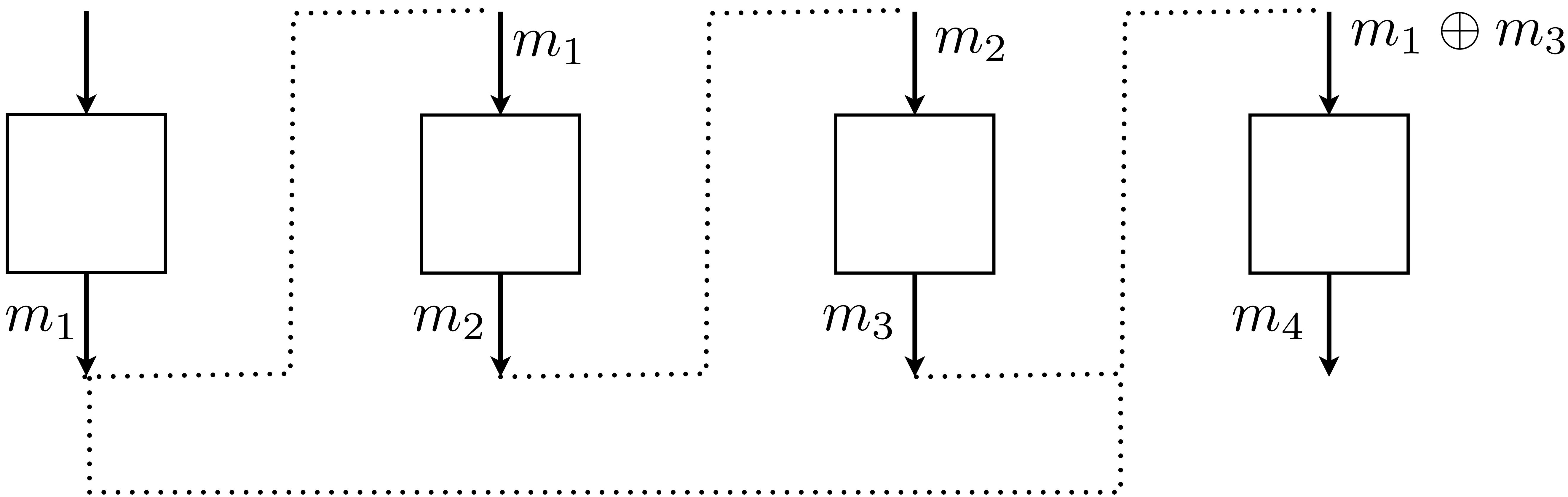
- *Adaptive*

vs

- *Random settings, space-like separated*

Measurement-based quantum computation

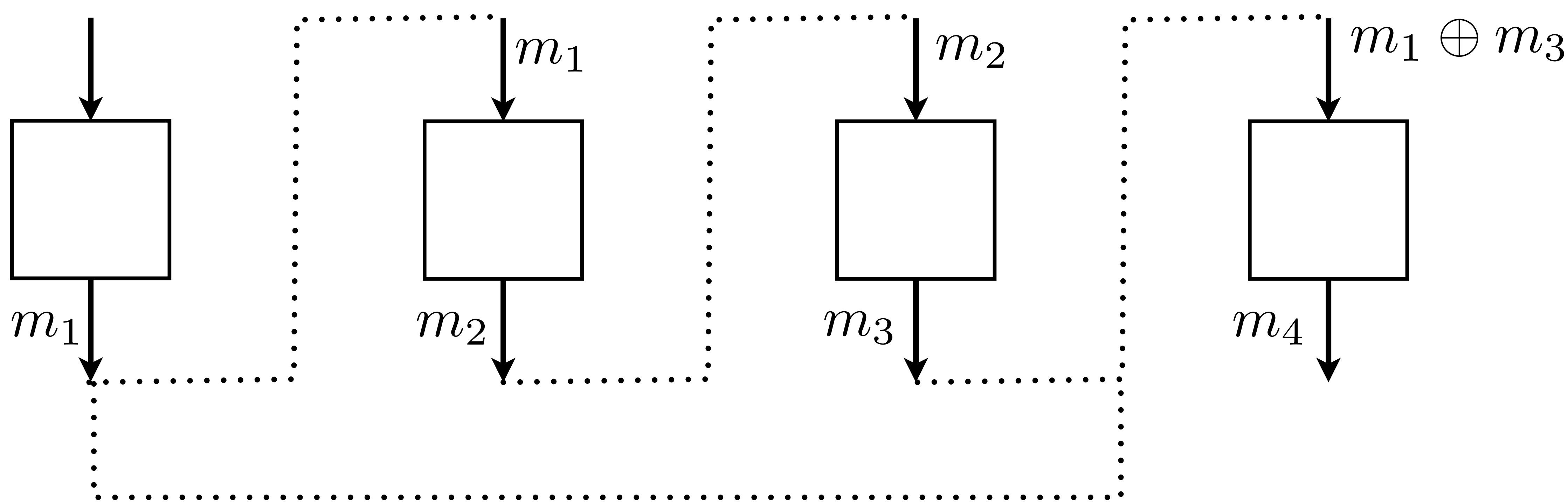
In Raussendorf and Briegel's cluster state MBQC, adaptivity is **linear**!



Every measurement setting is a **linear** function of **previous** measurement outcomes.

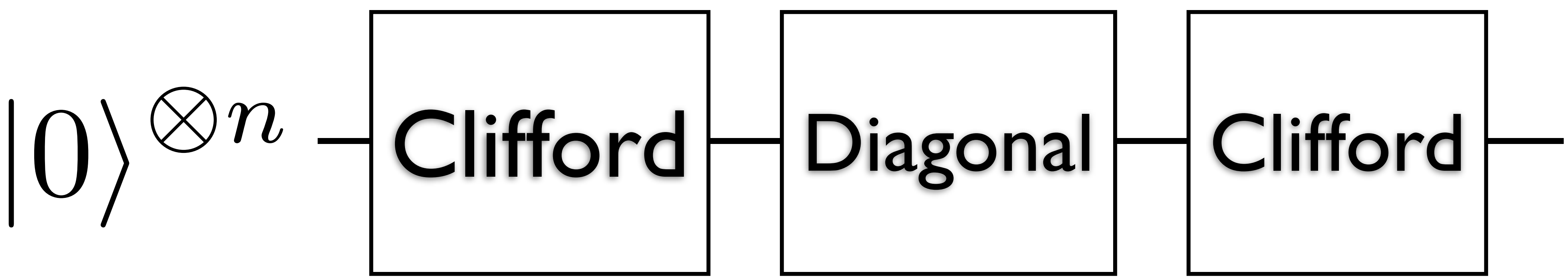
Bell inequalities for MBQC?

- This means that with **loophole-free post-selection**, we can **simulate** the MBQC-type correlations in a Bell-type experiment.
- MBQC and BI violations have a similar foundation.



Adaptivity in MBQC

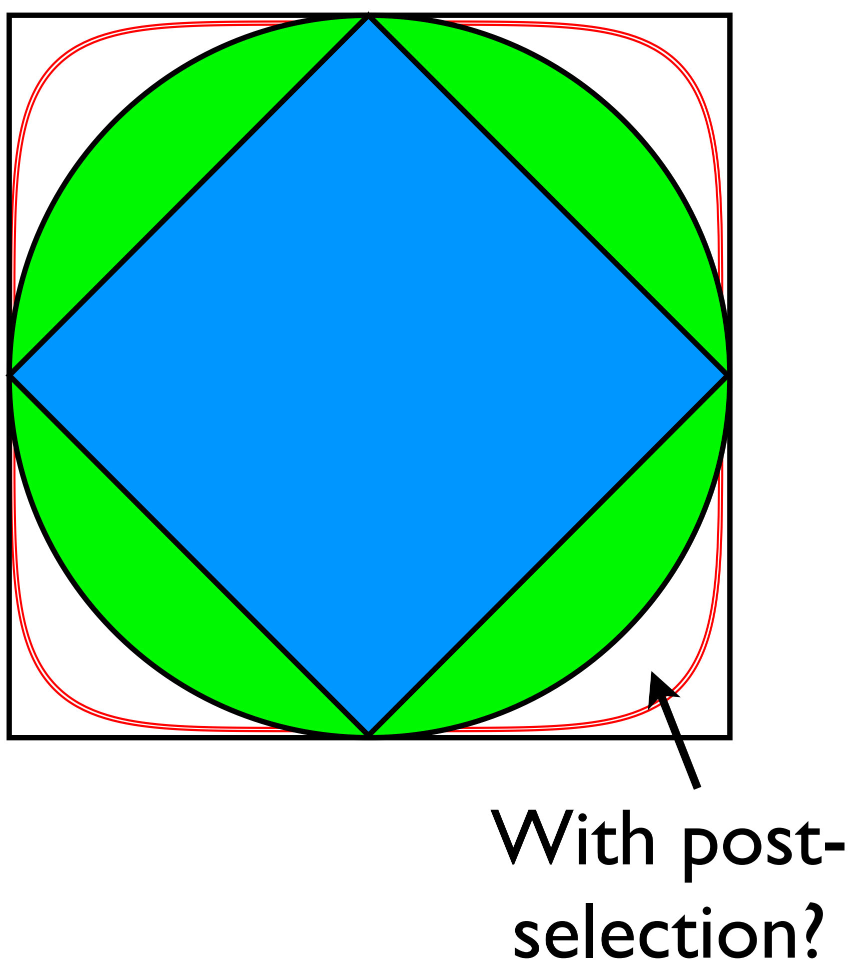
- We believe that **adaptive measurement** is required in MBQC to achieve **universality**.
- With simultaneous measurements we can only achieve circuits of the form:



- This is closely related to Bremner and Shepherd's **IQP** model.
- This model is **not universal**.

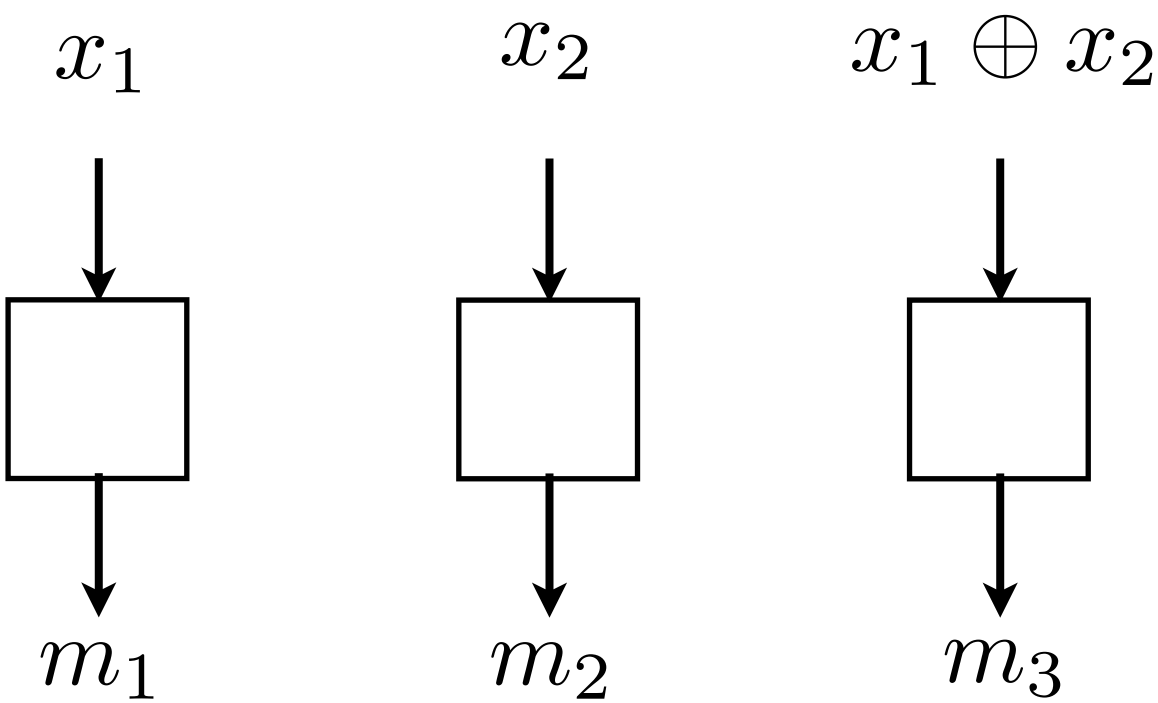
A larger quantum region?

- We'd thus expect to achieve correlations with linear adaptivity **impossible** without it.
- This implies that the post-selection, which left the LHV region invariant, might increase the quantum region.
- Can we show this?
- Yes.



A larger quantum region

- Consider the function: $f(x) = x_1 x_2 x_3$

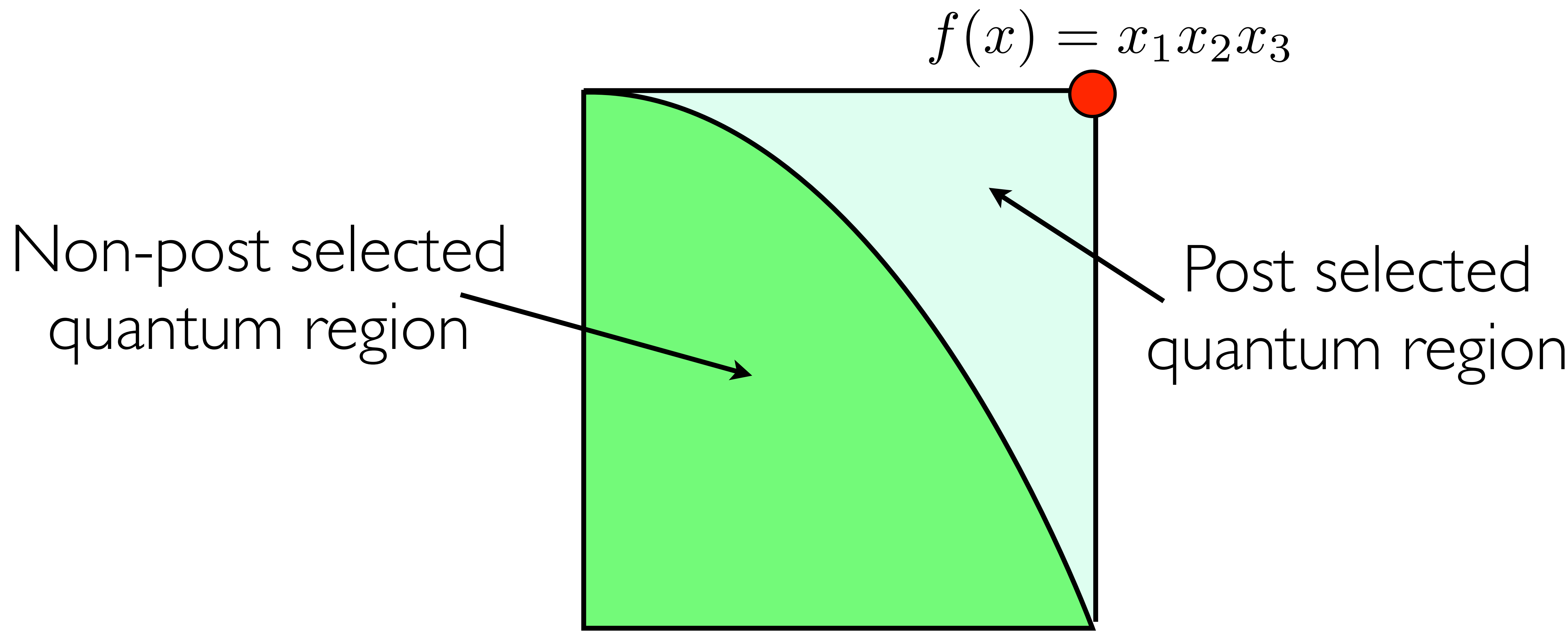


$$m_1 \oplus m_2 \oplus m_3 = x_1 x_2$$

- We can compute f(x) with two AND gates - using the GHZ correlation twice using linear adaptivity.

A larger quantum region

CHSH correlation space



Using methods adapted from Werner and Wolf we can show that this lies **outside** the standard quantum region.

Summary

- In CHSH experiments, LHV region is characterised by the set of **linear functions** on the input settings.
  - Hence, **loopholes** = source of **non-linearity**.
  - We can **post-select** in a non-trivial way without introducing a **loophole**.
- Post-selection simulates the **adaptivity** structure of Raussendorf and Briegel **MBQC**.
- We see a **concrete connection** between **Bell inequality violation** and (**quantum) computation**.
- **Loophole-free post-selection** can **enlarge** the region of quantum correlations.

Outlook and Open Questions

- **Better characterisation** of linearly adaptive quantum region?
- Consider **more general correlations** (i.e. than just CHSH-parity)? Other "quantum games"?
- Study **other detection loopholes** (E.g. Eberhard's analysis).
- Our methods generalise to **higher dimensions**, though the post-selection result fails. Is there a "safe" form of post-selection in higher **d**? Consider high-d cluster state computation?
- Are there implications for attempts to **axiomatise** quantum correlations (currently good for bi-partite case only). **Which region** should one axiomatise?
- Use MBQC correspondence for **quantum circuit** bounds? E.g. Heuristics for IQP vs BQP?

Acknowledgements

- These results follow on from **earlier work** with::
  - Janet Anders, Earl Campbell and Klearchos Loukopoulos.

- **References**
  - R. Werner and M. Wolf, Phys. Rev. A 64 32112 (2001)
  - J. Anders and D. E. Browne, Phys. Rev. Lett. (2009)
  - M. J. Hoban, E.T. Campbell, K. Loukopoulos, D.E. Browne, "Non-adaptive measurement-based quantum computation and multi-party Bell inequalities", New Journal of Physics, in press.
  - **These results:** M. J. Hoban and D. E. Browne, arxiv soon.

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