Many-body models based on quantum double algebras

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ARC CENTRE OF EXCELLENCE FOR ENGINEERED QUANTUM SYSTEMS

Toric code

Topologically ordered spin system

Toric code → Quantum double model

Cluster state → ???

Color code → ???

Many-body model → ???

Outline

- $\mathbb{Z}_2$ & Qubits
- Group algebras & Qudits
- Toric code and quantum double models
- Examples
  - Quantum double cluster state
  - Non-Abelian color code
- More general algebras

$\mathbb{Z}_2$

Elements	$g \in \{0,1\}$
Multiplication	$\oplus g$ $0 \oplus 0 = 1 \oplus 1 = 0$ $0 \oplus 1 = 1 \oplus 0 = 1$
Unirreps	$\Gamma \in \{+,-\}$ $\Gamma_+(0) = \Gamma_+(1) = +1$ $\Gamma_-(0) = +1$ $\Gamma_-(1) = -1$

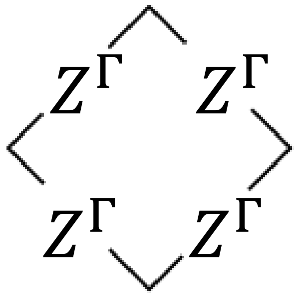
$\mathbb{Z}_2$	**Qubit**
Elements $\{0,1\}$	$\|0\rangle, \|1\rangle$
Multiplication $\oplus g$	$X^g\|h\rangle = \|g \oplus h\rangle$
Reps $\Gamma \in \{+,-\}$	$\|\Gamma\rangle = \Gamma(0)\|0\rangle + \Gamma(1)\|1\rangle$ $\|+\rangle, \|-\rangle$
Rep Action $\Gamma(g)$	$Z^\Gamma\|h\rangle = \Gamma(h)\|h\rangle$
$\mathbb{Z}_2 \cong Rep(\mathbb{Z}_2)$	Hadamard
$g, h \rightarrow g \oplus h$	CNOT$\|g\rangle\|h\rangle = \|g\rangle\|g \oplus h\rangle$
$g, \Gamma \rightarrow \Gamma(g)$	CNOT$\|g\rangle\|\Gamma\rangle = \Gamma(g)\|g\rangle\|\Gamma\rangle$

G	**Qudit**
Elements $g \in G$	$\vert g\rangle \quad (d = \vert G\vert)$
Multiplication	$X_+^g \vert h\rangle = \vert gh\rangle$, $X_-^g \vert h\rangle = \vert hg^{-1}\rangle$
Reps $\Gamma \in Rep(G)$	$\vert \Gamma_{i,j}\rangle = \sum_g \Gamma_{i,j}(g) \vert g\rangle$
Rep Action $\Gamma(g)$	$Z^{\Gamma_{i,j}} \vert h\rangle = \Gamma_{i,j}(h) \vert h\rangle$
	Hadamard? Y? CPhase?
$g, h \to gh$ or hg^{-1}	CMULT$\vert g\rangle \vert h\rangle = \vert g\rangle \vert gh\rangle$
$g, \Gamma_{i,j} \to \Gamma_{i,j}(g)$	CRepEl (nonunitary)

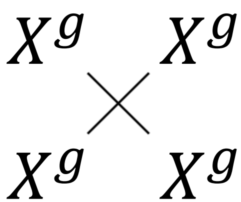
Qubit	Qudit
$\|0\rangle, \|1\rangle$	$\|g\rangle \quad (d = \|G\|)$
$X^g\|h\rangle = \|g \oplus h\rangle$	$X_+^g\|h\rangle = \|gh\rangle$, $X_-^g\|h\rangle = \|hg^{-1}\rangle$
$\|\Gamma\rangle = \Gamma(0)\|0\rangle + \Gamma(1)\|1\rangle$ $\|+\rangle, \|-\rangle$	$\|\Gamma_{i,j}\rangle = \sum_g \Gamma_{i,j}(g)\|g\rangle$
$Z^\Gamma\|h\rangle = \Gamma(h)\|h\rangle$	$Z^{\Gamma_{i,j}}\|h\rangle = \Gamma_{i,j}(h)\|h\rangle$
Hadamard, Y, CPhase	N/A
CNOT$\|g\rangle\|h\rangle = \|g \oplus h\rangle$	CMULT$\|g\rangle\|h\rangle = \|g\rangle\|gh\rangle$
CNOT$\|g\rangle\|\Gamma\rangle = \Gamma(g)\|g\rangle\|\Gamma\rangle$	CRepEl (nonunitary)

Toric Code from $\mathbb{Z}_2$

$B_p^\Gamma =$

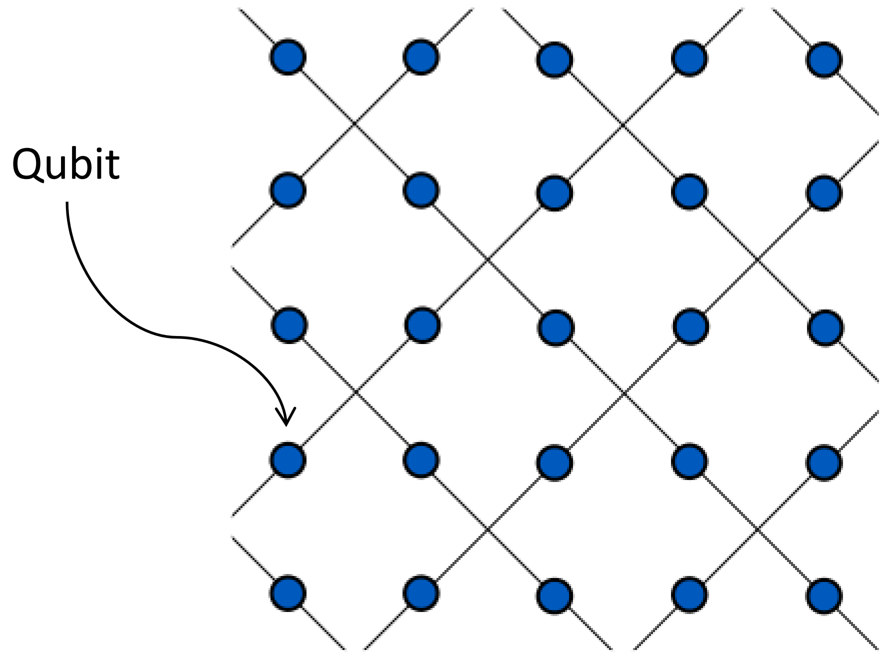


$A_v^g =$



$$B_p = \sum_\Gamma B_p^\Gamma$$

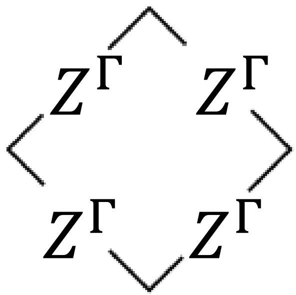
$$A_v = \sum_{\mathrm{g}} A_v^g$$



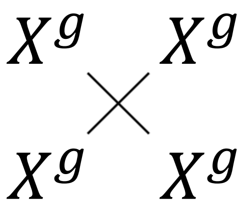
$$H = -\sum_v A_v - \sum_p B_p$$

Toric Code from $\mathbb{Z}_2$

$B_p^{\Gamma} =$

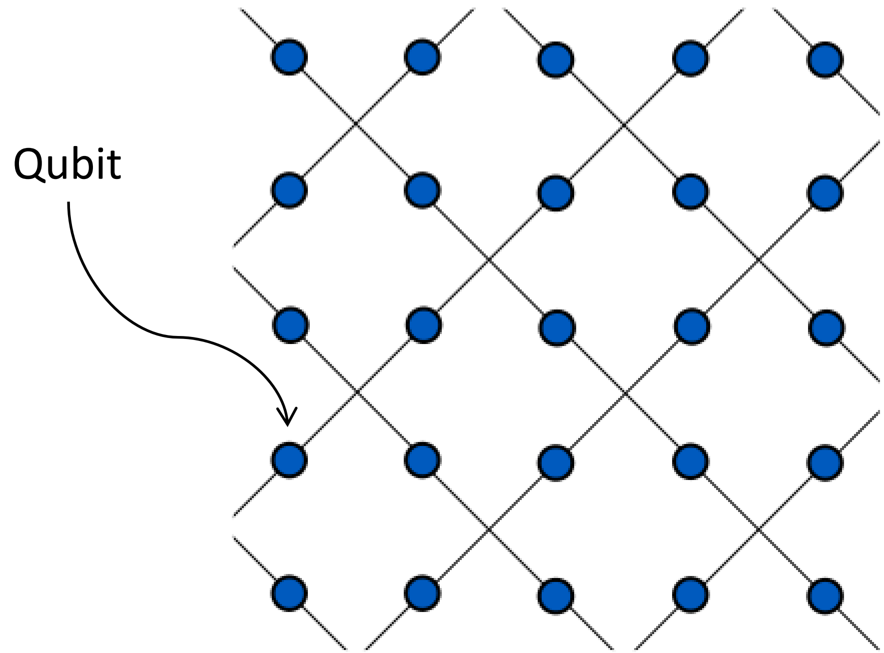


$A_v^{g} =$



$$B_p = \sum_{\Gamma} B_p^{\Gamma}$$

$$A_v = \sum_{g} A_v^{g}$$

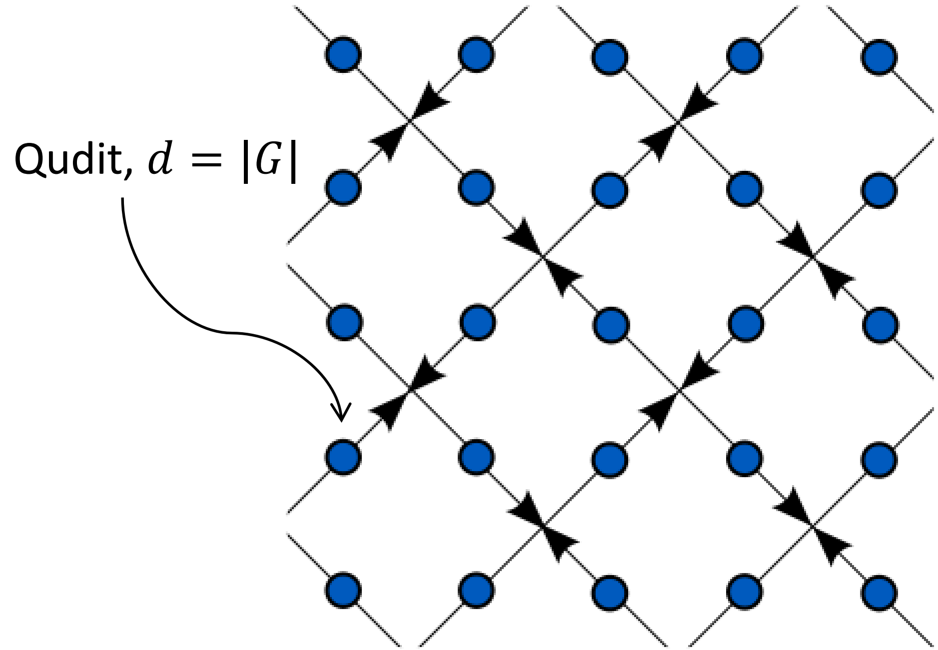


$B_p = 1$: Identity Element

$A_v = 1$: Trivial Representation

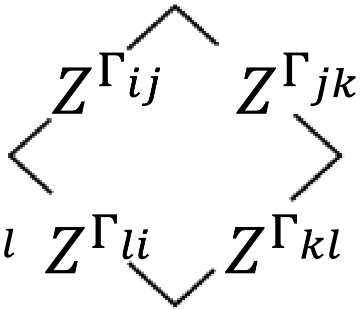
$B_p, A_v \neq 1$: Anyonic Charges

Quantum Double Models from G

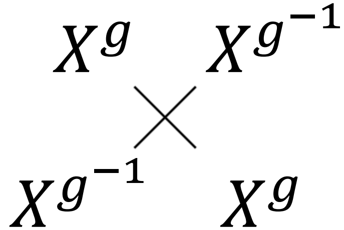


Quantum Double Models from G

$B_p^\Gamma = \sum_{i,j,k,l}$

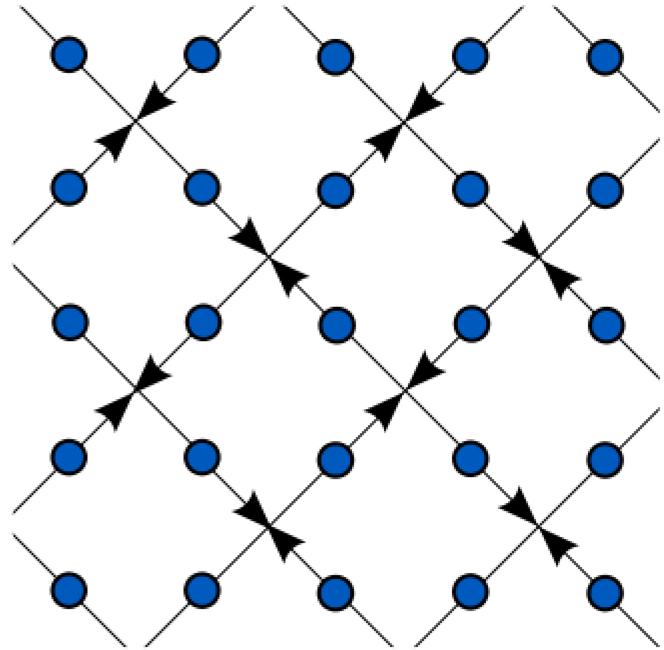


$A_v^g =$



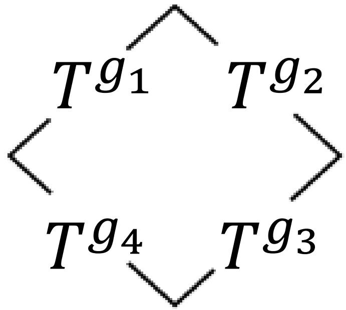
$$B_p = \sum_\Gamma d_\Gamma B_p^\Gamma$$

$$A_v = \sum_g A_v^g$$

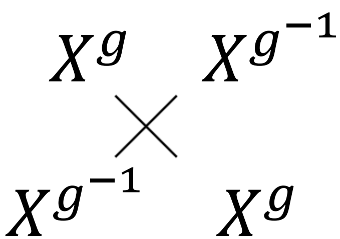


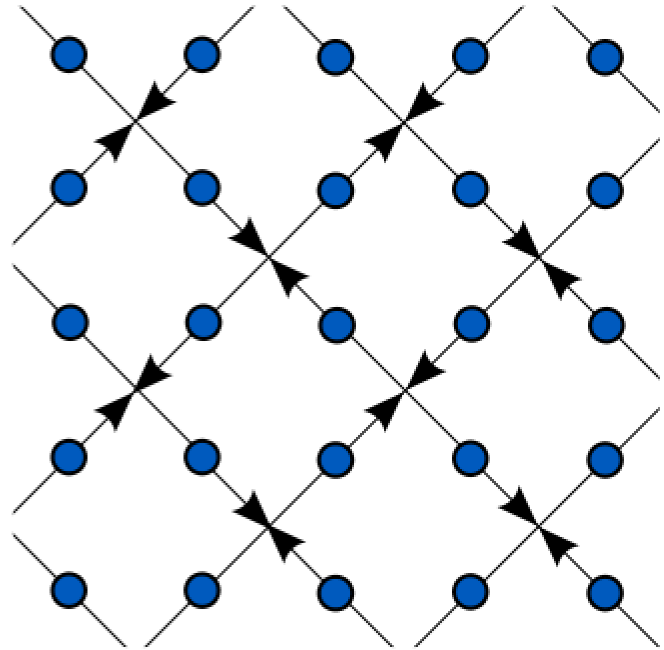
$$H = -\sum_v A_v - \sum_p B_p$$

Quantum Double Models from G

$B_p^g = \sum_{g_1 g_2 g_3 g_4 = g}$ 

$T^g = |g\rangle\langle g|$

$A_v^g =$ 



$B_p = B_p^e$

$A_v = \sum_{\mathrm{g}} A_v^g$

$$H = -\sum_v A_v - \sum_p B_p$$

Cluster State

- MBQC resource
- Output of constant depth circuit
- GS of commuting local Hamiltonian
- Related to toric code, color code, etc

Cluster State

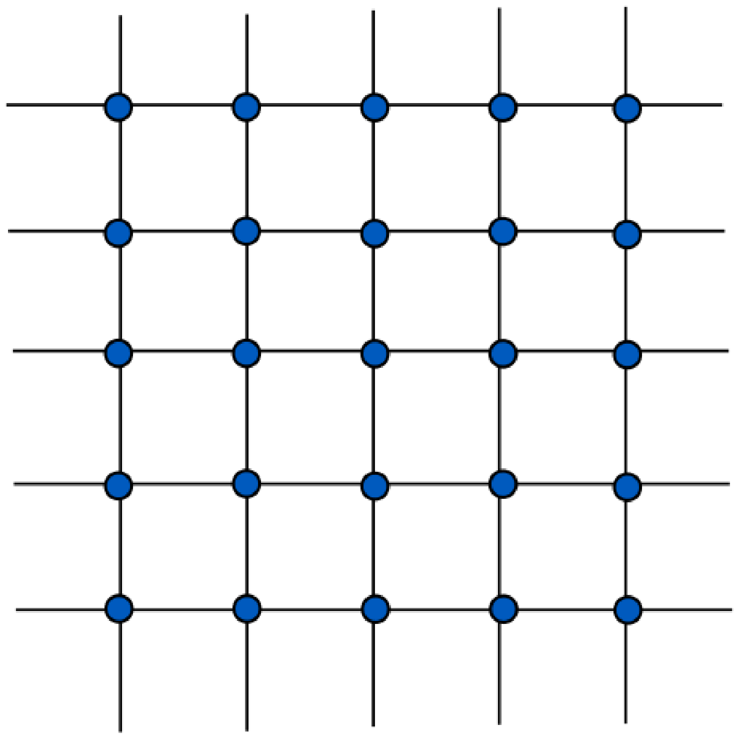
- Constructive

  Prepare $|+\rangle$

  Perform CPhase

- Stabilizer

$$S_v = \begin{matrix} & Z & \\ Z & X & Z \\ & Z & \end{matrix}$$



Problem

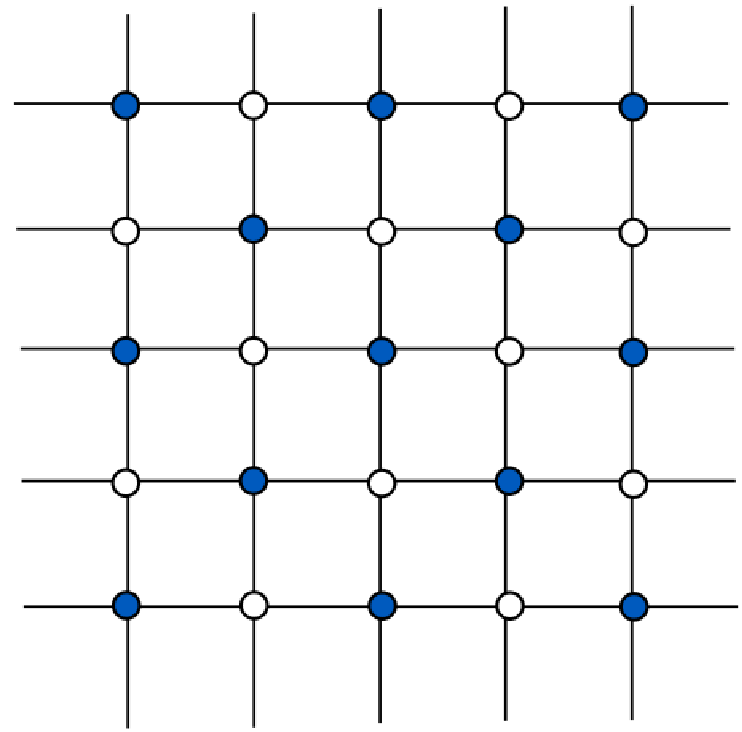
- Construction circuit for cluster state involves
  - $|+\rangle$
  - CPhase

- No obvious generalizaton of CPhase

- Solution: modify the cluster state

Fourier Variant Cluster State

- Locally equivalent on bipartite graph
- Constructive
  - Prepare even $|+\rangle$
  - Prepare odd $|0\rangle$
  - CNOT with even as control

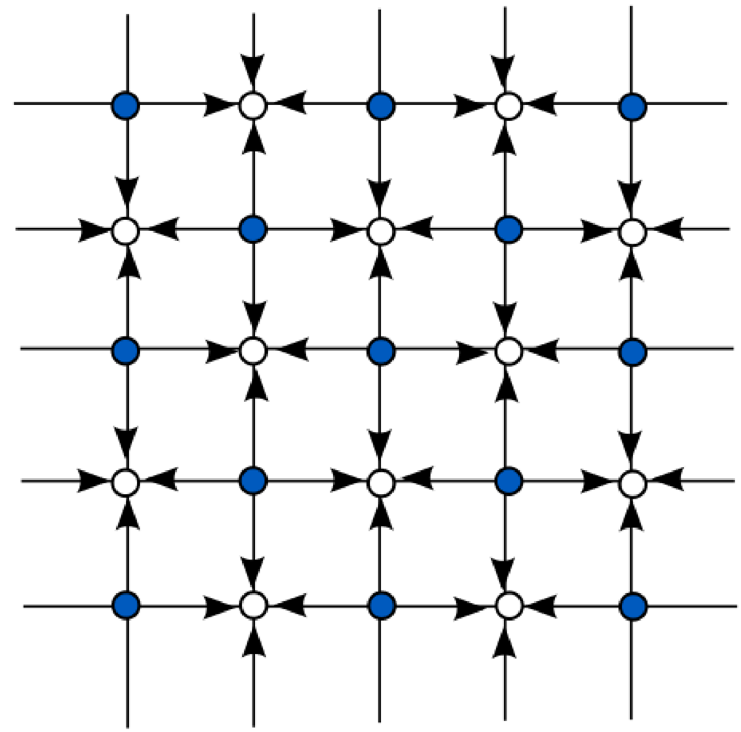


- Stabilizer
  - Even: $\begin{matrix} & X & \\ X & X & X \\ & X & \end{matrix}$ Odd: $\begin{matrix} & Z & \\ Z & Z & Z \\ & Z & \end{matrix}$

Quantum Double Cluster State

Ingredients:

- Directed bipartite graph
- Vertex ordering



- Constructive
  - Prepare even $|I\rangle$ ●
  - Prepare odd $|e\rangle$ ○
  - CMULT with even as control

Quantum Double Cluster State

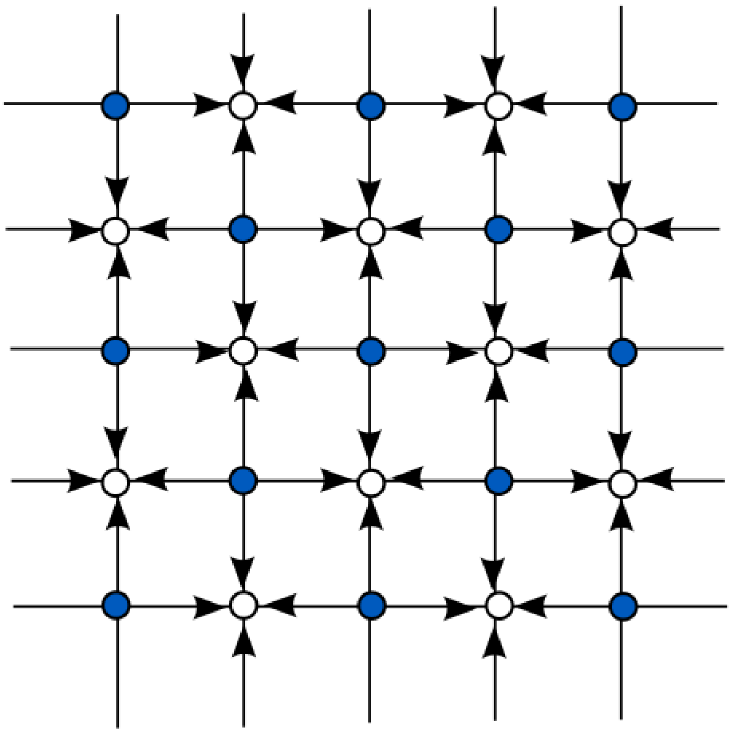
- Stabilizer

Even:

$$\sum_{g_1 g_2 g_3 g_4 = h} \begin{matrix} & T^{g_1} & \\ T^{g_4} & T^{h} & T^{g_2} \\ & T^{g_3} & \end{matrix}$$

Odd:

$$\sum_{k, g_i} \begin{matrix} T^{g_5} & & & \\ & T^{g_1} & X_+^{m_1} & \\ & X_+^{m_4} & X_+^{k} & X_+^{m_2} \\ & T^{g_4} & X_+^{m_3} & T^{g_2} \\ & & T^{g_3} & \end{matrix}$$



$m_1 = (g_1)\ \mathrm{k}\ (g_1)^{-1}$

$m_2 = \mathrm{k}$

$m_3 = (g_4 g_3 g_2)\ \mathrm{k}\ (g_4 g_3 g_2)^{-1}$

$m_4 = (g_5 g_4)\ \mathrm{k}\ (g_5 g_4)^{-1}$

Color Code

- Topological code
- Large transversal gate set
- Anyonic statistics
- Related to topological subsystem code
- Higher dimensional generalizations
- Related to toric code

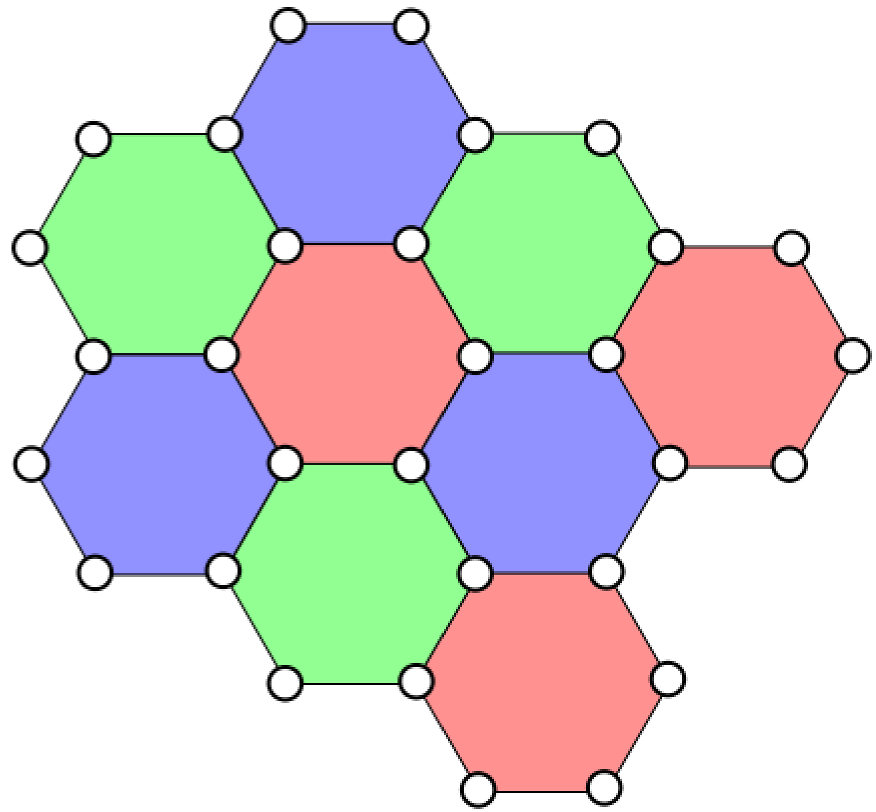
Color Code

$$A_p^g = \begin{array}{ccc} & X^g \;—\; X^g & \\ / & & \backslash \\ X^g & & X^g \\ \backslash & & / \\ & X^g \;—\; X^g & \end{array}$$

$$B_p^\Gamma = \begin{array}{ccc} & Z^\Gamma \;—\; Z^\Gamma & \\ / & & \backslash \\ Z^\Gamma & & Z^\Gamma \\ \backslash & & / \\ & Z^\Gamma \;—\; Z^\Gamma & \end{array}$$

$$B_p = \sum_\Gamma B_p^\Gamma$$

$$A_v = \sum_g A_v^g$$



$B_p = 1$: Identity Element

$A_v = 1$: Trivial Representation

$B_p, A_v \neq 1$: Anyonic Charges

Problem

- Naively converting all A_p and B_p to general groups:
  - Non-commuting.

- Solution:
  - Choose a preferred color (red)
  - Use abelianization of G for red plaquettes
  - $G^{ab} = G/[G,G] \qquad [G,G] := g^{-1}h^{-1}gh$

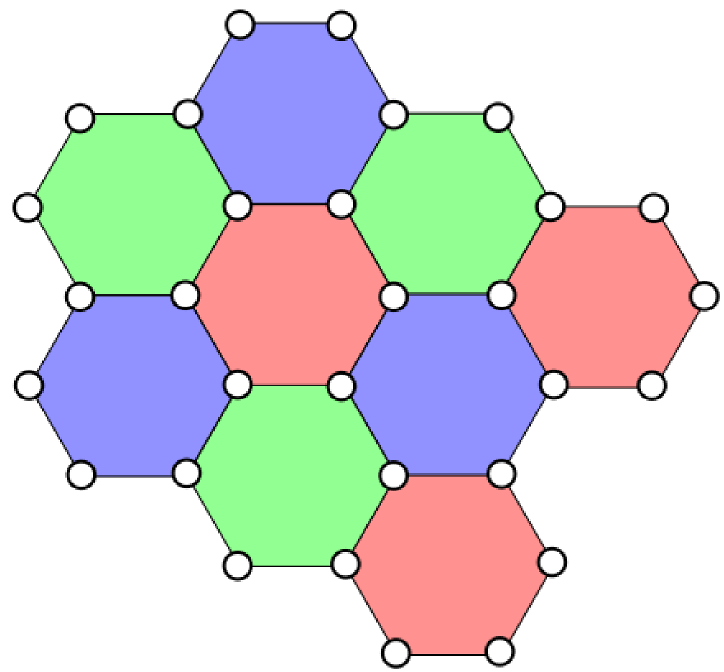
Non-Abelian Color Code

$$B_p^g \sim B_p^g = \sum_{\prod g_i = g} \begin{array}{ccccc} & T^{g_1} & - & T^{g_2} & \\ / & & & & \backslash \\ T^{g_6} & & & & T^{g_3} \\ \backslash & & & & / \\ & T^{g_5} & - & T^{g_4} & \end{array}$$

$$B_p^g = \sum_{\prod g_i \in g[G,G]} \begin{array}{ccccc} & T^{g_1} & - & T^{g_2} & \\ / & & & & \backslash \\ T^{g_6} & & & & T^{g_3} \\ \backslash & & & & / \\ & T^{g_5} & - & T^{g_4} & \end{array}$$

$$A_p^g = \begin{array}{ccccc} & X^g & - & X^g & \\ / & & & & \backslash \\ X^g & & & & X^g \\ \backslash & & & & / \\ & X^g & - & X^g & \end{array}$$

$$C_l^g = \begin{array}{c} X^g \\ | \\ X^g \end{array}$$



$$B_p = B_p^e$$

$$A_v = \sum_{\mathrm{g}} A_v^g$$

Further extensions

- Other models
- New algebras:
  - Hopf Algebras
    - Non-co-commutative
  - Weak Hopf Algebras
  - Fusion Categories
    - Toric code $\rightarrow$ String net

Conclusions

- General program of using group algebras to build interesting families of models from simple ones
- Examples include
  - Quantum double cluster states
  - Non-abelian color codes
- Further generalization to more complicated algebras should be possible

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