Application of conditional independence to gapped quantum many-body systems

Isaac H. Kim

Institute of Quantum Information and Matter
California Institute of Technology
Pasadena, CA 91105, USA

January 17, 2013

Background

There are some surprising universal properties that arise in the ground state of gapped quantum many-body systems.

Background

There are some surprising universal properties that arise in the ground state of gapped quantum many-body systems.

- No fine-tuning of the hamiltonian is required.
- The predictions have been confirmed experimentally and numerically.
  - IQHE, FQHE, TI, etc...
  - Numerical calculation of topological entanglement entropy, entanglement spectrum, particle statistics, etc...

Background

Question : Why are these properties stable?

Background

Question : Why are these properties stable?
Answer 1: RG argument, low-energy effective field theory.

Background

Question : Why are these properties stable?
Answer 1: RG argument, low-energy effective field theory.
Answer 2: It can be mathematically proved! (Bravyi, Hastings, Michalakis 2010)

Background : Origin of the gap stability

According to Bravyi et al.'s work, there are properties of the ground state that protect the phase.

- Locality of the parent hamiltonian.
- Local indistinguishability : Different sectors of the ground state cannot be detected, nor be altered via local operation.
- Local ground state is equal to the global ground state.

Background : Origin of the gap stability

According to Bravyi et al.'s work, there are properties of the ground state that protect the phase.

- Locality of the parent hamiltonian.
- Local indistinguishability : Different sectors of the ground state cannot be detected, nor be altered via local operation.
- Local ground state is equal to the global ground state.

Background : Implication of the gap stability

If ground states of two different local hamiltonian can be adiabatically connected, there exists an *almost-local* "hamiltonian" that generates a unitary evolution between those two ground states.(Hastings, Wen 2005)

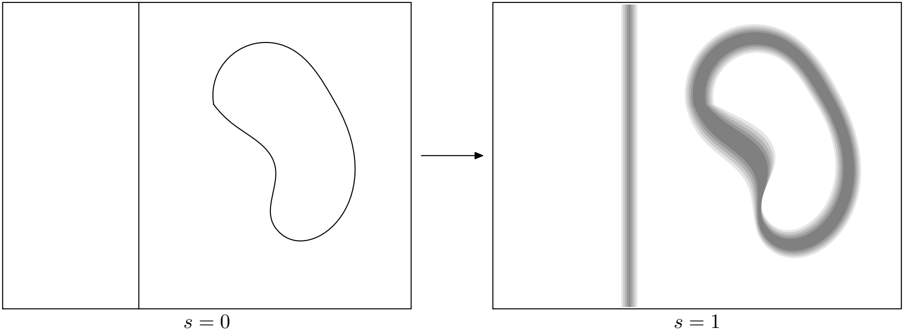
$$H = H_0 + sV, s \in (0, 1]$$

$$|\psi(s)\rangle = U(s)\,|\psi(0)\rangle$$

$$\frac{dU(s)}{ds} = i \sum_i h_i(s) U(s)$$

- $h_i(s)$ can be approximated by a strictly local operator with superpolynomially decaying tail.
- By using Trotter-Suzuki expansion, this unitary evolution can be approximated by a finite-depth local unitary circuit. (with small error)

Background : Implication of the gap stability



- Particle statistics is preserved.
- Logical operators are preserved.

Background : Unanswered questions

- Stability of topological entanglement entropy γ.
  - $S_A = a|\partial A| - \gamma$
  - γ is a universal constant : Levin and Wen(2006), Kitaev and Preskill (2006)
  - Passed number of numerical tests : Isakov et al. (2011), Jiang et al.(2012), Cincio and Vidal(2012), Selem et al.(2012),
- Locality of entanglement spectrum.
  - $\log \rho_A$ can be described by a local hamiltonian! : Li and Haldane(2008), Dubail et al. (2012), Cirac et al. (2011), Schuch et al. (2012)

Main message of this talk

There is another property of the ground state that protects the phase : **conditional independence**

Main message of this talk

There is another property of the ground state that protects the phase : **conditional independence**

1. Conditionally independent states naturally appear in the ground state of topologically ordered systems.

Main message of this talk

There is another property of the ground state that protects the phase : **conditional independence**

1. Conditionally independent states naturally appear in the ground state of topologically ordered systems.
2. Conditional independence can be exploited to produce some nontrivial statements about topological entropy, entanglement spectrum, etc.

What is conditional independence?

One of the most fundamental inequalities in quantum information theory is the strong subadditivity of entropy(Lieb, Ruskai 1972):

$$S_{AB} + S_{BC} - S_B - S_{ABC} \geq 0.$$

- This inequality holds for *any* tripartite quantum state!
- Many results in quantum information theory is based on this inequality.

What is conditional independence?

Alternatively,

$$I(A:C|B) \geq 0,$$

where $I(A:C|B)$ is *conditional mutual information*

$$I(A:C|B) = S_{AB} + S_{BC} - S_B - S_{ABC}$$

Definition : A tripartite state ρ_{ABC} is conditionally independent if the inequality is satisfied with an *equality.*

Why are conditionally independent states interesting? : Petz's theorem

Theorem 1. (Petz 2003) $I(A:C|B) = S_{AB} + S_{BC} - S_B - S_{ABC} = 0$ if and only if

$$\hat{H}_{A:C|B} := I_C \otimes \log \rho_{AB} + I_A \otimes \log \rho_{BC} - I_{AC} \otimes \log \rho_B - \log \rho_{ABC} = 0.$$

Corollary 1. Any first order perturbation of conditionally independent state vanishes.

$$\begin{aligned} \frac{dI(A:C|B)}{ds} &= \mathrm{Tr}\left(\frac{d\rho_{ABC}}{ds}\hat{H}_{A:C|B}\right) \\ &= 0 \end{aligned}$$

Why are conditionally independent states interesting?

Why are conditionally independent states interesting?

: Conditionally independent states naturally appear in all the known exactly solvable 2D topologically ordered systems.(Hastings, Poulin 2011)

Why are conditionally independent states interesting?

: Conditionally independent states naturally appear in all the known exactly solvable 2D topologically ordered systems.(Hastings, Poulin 2011)

- Note that $S_A = a|\partial A| - \gamma_A$, where γ_A only depends on the shape of A. One can find a set of subsystems for which the area terms as well as the topological terms cancel out.

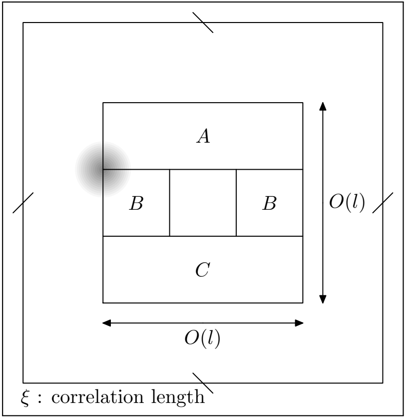
Application 1 : Topological entanglement entropy

Problem : Consider a unitary transformation generated by a sum of local hamiltonian $H = \sum_i h_i$. What is $\frac{d\gamma}{ds}|_{s=0}$?

$$S_A = a|\partial A| - \gamma$$

$$I(A:C|B) = S_{AB} + S_{BC} - S_B - S_{ABC} = 2\gamma$$

: Levin, Wen 2006



Application 1 : Topological entanglement entropy

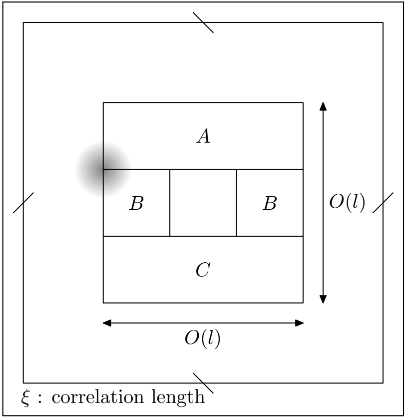
Simplifying the problem

- First order perturbation : effect of the perturbation can be decomposed into a sum of local contributions.
  - $\frac{dS_A}{ds} = i \sum_j \mathrm{Tr}([h_j, \rho] \log \rho_A)$.
- Local unitary transformation does not change entanglement entropy
  - $\mathrm{Tr}(-\rho_A \log(\rho_A)) = \mathrm{Tr}(-U_A^\dagger \rho_A U_A \log(U_A^\dagger \rho_A U_A))$.
  - It suffices to only consider the local terms on the boundary!

Application 1 : Topological entanglement entropy

Observation 1

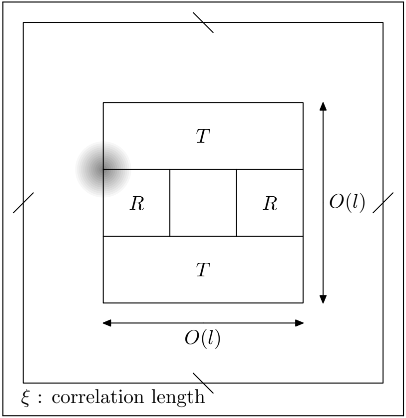
- $I(A:C|B) = I(C:A|B)$.
  - A, C : *target parties* ('T')
  - B : *reference party* ('R')



Application 1 : Topological entanglement entropy

Observation 1

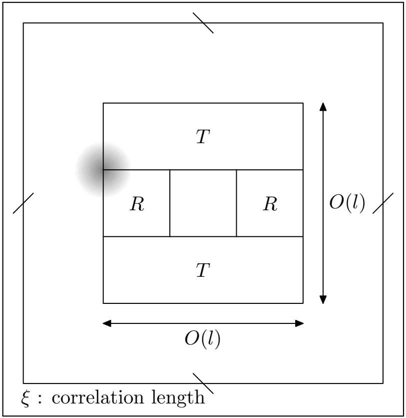
- $I(A : C|B) = I(C : A|B)$.
    - A, C : *target parties* ('T')
    - B : *reference party* ('R')



Application 1 : Topological entanglement entropy

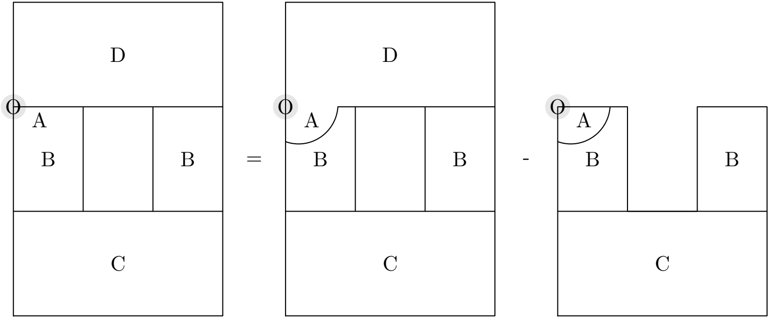
Observation 2

- **Chain rule of conditional mutual information**
  - $I(AD : C|B) = I(A : C|B) + I(D : C|AB)$.
- Main strategy : Given a unitary on the boundary, deform it away from the boundary.
  1. Isolation move : Isolate the unitary away from the reference party.
  2. Separation move : Separate the unitary away from the target(& reference party).



Application 1 : Topological entanglement entropy

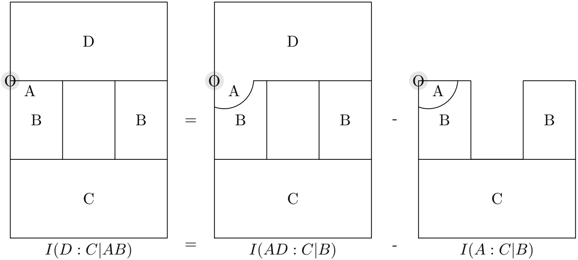
Isolation move : Isolate the unitary away from the reference party.



$$I(D:C|AB) = I(AD:C|B) - I(A:C|B)$$

Application 1 : Topological entanglement entropy

Isolation move : Isolate the unitary away from the reference party.



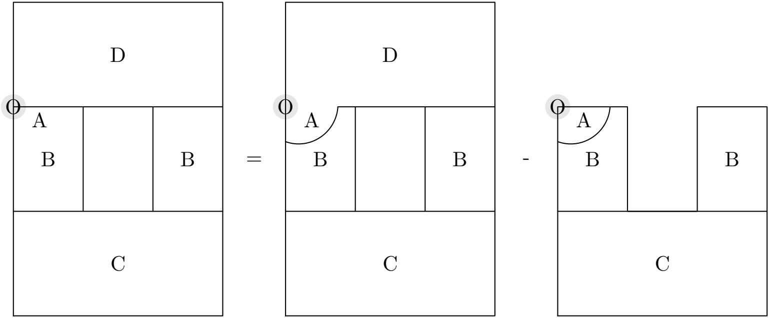
$$I(D:C|AB) = S_{ABD} + S_{ABC} - S_{AB} - S_{ABCD}$$

$$I(AD:C|B) = S_{ABD} + S_{BC} - S_{B} - S_{ABCD}$$

$$I(A:C|B) = S_{AB} + S_{BC} - S_{B} - S_{ABC}$$

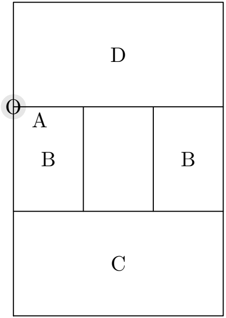
Application 1 : Topological entanglement entropy

Isolation move : Isolate the unitary away from the reference party.

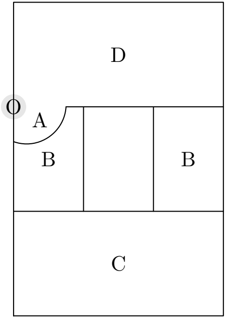


$$dI(D:C|AB)/ds = dI(AD:C|B)/ds - dI(A:C|B)/ds$$

Isolation move : Isolate the unitary away from the reference party.

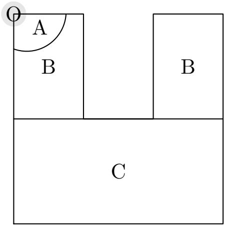


=



-

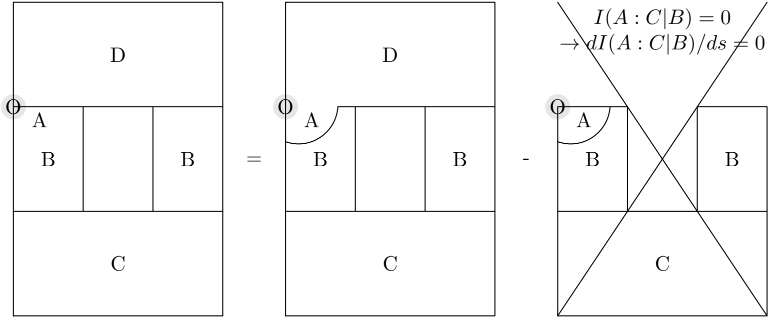
$$I(A:C|B)=0$$
$$\rightarrow dI(A:C|B)/ds=0$$
(Petz's-corollary)



$$dI(D:C|AB)/ds = dI(AD:C|B)/ds - dI(A:C|B)/ds$$

Application 1 : Topological entanglement entropy

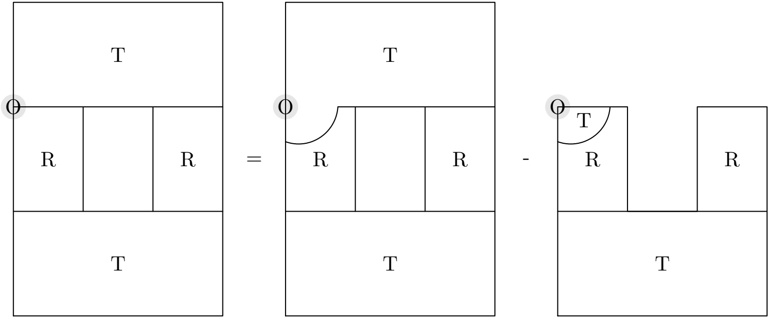
Isolation move : Isolate the unitary away from the reference party.



$dI(D:C|AB)/ds = dI(AD:C|B)/ds - dI(A:C|B)/ds$

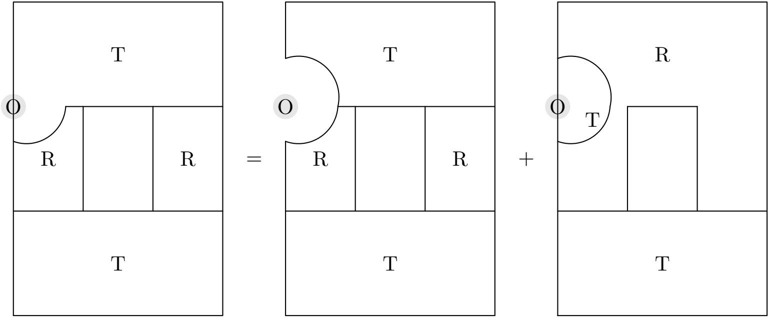
Application 1 : Topological entanglement entropy

Isolation move : Isolate the unitary away from the reference party.



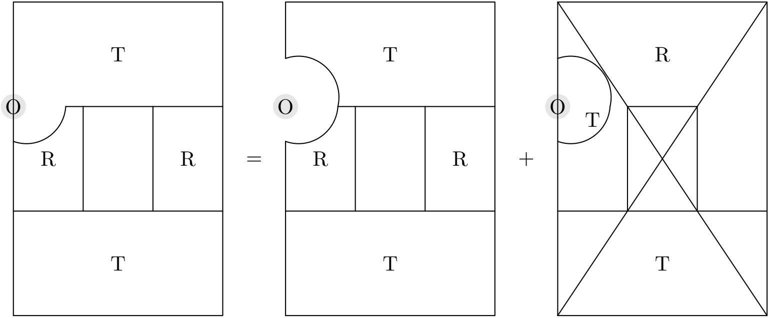
Application 1 : Topological entanglement entropy

Separation move : Separate the unitary away from the reference party.



Application 1 : Topological entanglement entropy

Separation move : Separate the unitary away from the reference party.



Application 1 : Topological entanglement entropy

Summary

1. Decompose the effect of the perturbation into a sum of quasilocal contributions.
2. Local approximation : Approximate the quasilocal operator by a strictly local operator.
3. Apply isolation and separation moves.
4. Bound the error from the local approximation : superpolynomially decaying tail is obtained by using Lieb-Robinson bound.

Application 1 : Topological entanglement entropy

Main Result 1:

$$\frac{d\gamma}{ds}|_{s=0} \leq cJ(\frac{\Gamma l}{v})^{10} l^4 u_{2/7}(c'\frac{\Gamma l}{v}).$$

- c, c' : some constants
- Γ : spectral gap
- v : Lieb-Robinson velocity
- l : size of the subsystem
- $u_{2/7}$: some superpolynomially decaying function

A segway to the next point...

When exact conditional independence is satisfied, entanglement spectrum can be "canceled out."

$$I(A:C|B) = 0 \longleftrightarrow \hat{H}_{A:C|B} = 0$$

What happens if conditional mutual information is approximately 0? There are many motivations for studying such scenario.

- Condensed matter theorists : This is a more realistic assumption.
- Quantum information theorist : Very few is known about the structure of states that are approximately conditionally independent. (except for a couple of notable results : Brandão et al. 2011, Li and Winter 2012)

A segway to the next point...

Main Result 2: Operator extension of strong subadditivity

$$\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B}) \geq 0.$$

A segway to the next point...

Main Result 2: Operator extension of strong subadditivity

$$\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B}) \geq 0.$$

Corollary 2:

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_C) \leq I(A:C|B)\|O_C\|.$$

A segway to the next point...

Main Result 2: Operator extension of strong subadditivity

$$\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B}) \geq 0.$$

Corollary 2:

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_C) \leq I(A:C|B)\|O_C\|.$$

Proof :

$$\begin{aligned}\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_C) &\leq \|O_C\||\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B})|_1 \\ &= \|O_C\|I(A:C|B).\end{aligned}$$

A segway to the next point...

Main Result 2: Operator extension of strong subadditivity

$$\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B}) \geq 0.$$

Corollary 2:

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_C) \leq I(A:C|B)\|O_C\|.$$

Proof :

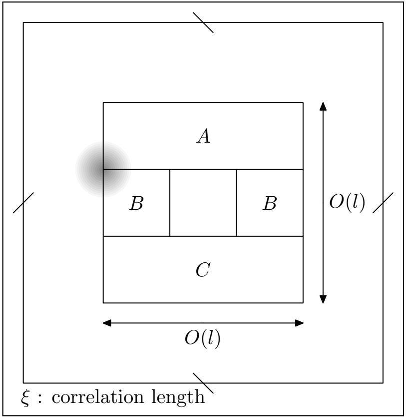
$$\begin{aligned}\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_C) &\leq \|O_C\||\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B})|_1 \\ &= \|O_C\|I(A:C|B).\end{aligned}$$

This isn't quite strong enough to prove the stability of topological entanglement entropy, but it has other applications.

Application 2 : Entanglement spectrum

Problem : If area law is satisfied approximately, do entanglement spectrum cancel out each other?

Answer : Yes!(with some caveats)



Application 2 : Entanglement spectrum

Easy case:
For an operator $O_C \in \mathcal{B}(\mathcal{H}_C)$,

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_C) \leq \|O_C\| I(A:C|B)$$

Application 2 : Entanglement spectrum

Less trivial case:
For an operator $O_B \in \mathcal{B}(\mathcal{H}_B)$,

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_B) \not\leq \|O_B\| I(A:C|B)$$

Application 2 : Entanglement spectrum

Less trivial case:
For an operator $O_B \in \mathcal{B}(\mathcal{H}_B)$,

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_B) \nleq \|O_B\| I(A:C|B)$$

Also, what if $I(A:C|B) = 2\gamma$ is not small?

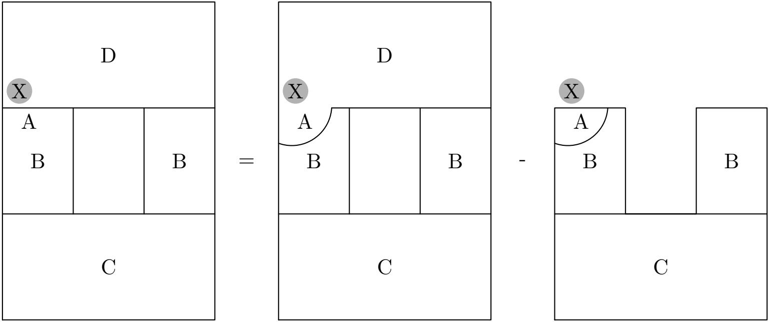
Application 2 : Entanglement spectrum

Less trivial case:
For an operator $O_B \in \mathcal{B}(\mathcal{H}_B)$,

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_B) \not\leq \|O_B\| I(A:C|B)$$

Also, what if $I(A:C|B) = 2\gamma$ is not small?
Apply deformation moves!

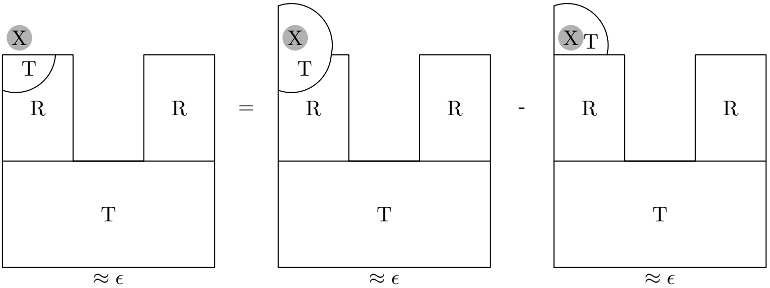
Isolation move revisited



$$Tr(\rho_{ABCD}\hat{H}_{D:C|AB}X) = Tr(\rho_{ABCD}\hat{H}_{AD:C|B}X) - Tr(\rho_{ABCD}\hat{H}_{A:C|B}X)$$

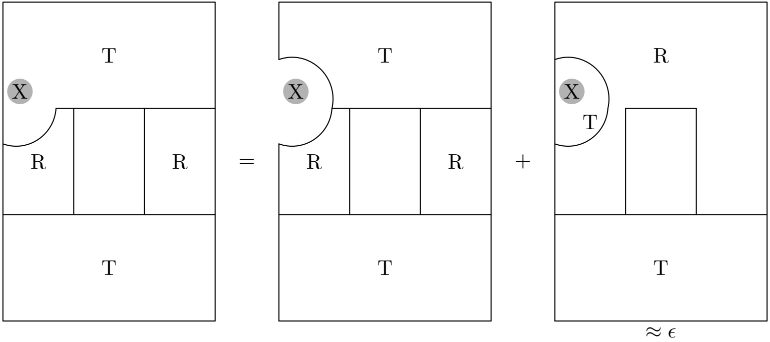
Application 2 : Entanglement spectrum

Absorption move



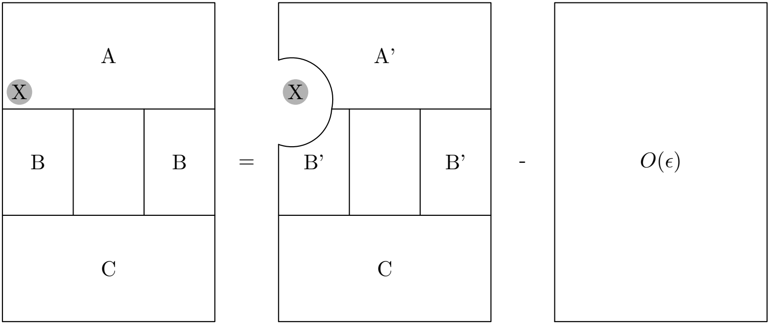
Application 2 : Entanglement spectrum

Separation move revisited



Application 2 : Entanglement spectrum

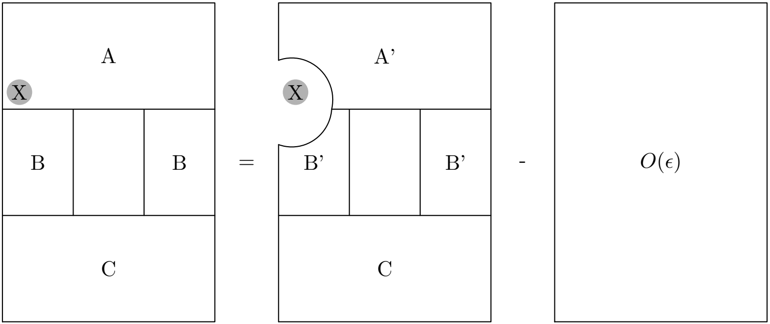
After all these moves... $(\langle\cdots\rangle = \mathrm{Tr}(\rho\cdots))$



$$\langle \hat{H}_{A:C|B} X \rangle \approx \langle \hat{H}_{A':C|B'} X \rangle$$

Application 2 : Entanglement spectrum

After all these moves... $(\langle \cdots \rangle = \mathrm{Tr}(\rho \cdots))$

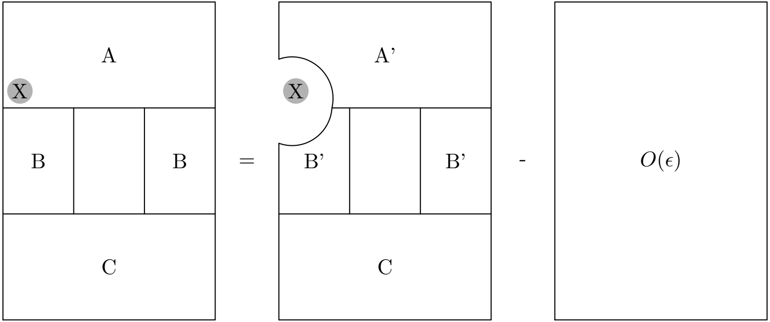


$$\langle \hat{H}_{A:C|B} X \rangle \approx \langle \hat{H}_{A':C|B'} X \rangle \approx \langle \hat{H}_{A':C|B'} \rangle \langle X \rangle$$

Exponential clustering theorem : Hastings and Koma(2006), Nachtergaele et al.(2006).

Application 2 : Entanglement spectrum

After all these moves... $(\langle \cdots \rangle = \mathrm{Tr}(\rho \cdots))$

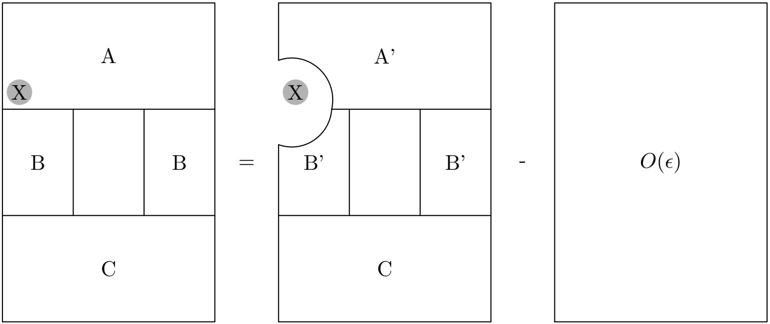


$$\langle \hat{H}_{A:C|B} X \rangle \approx \langle \hat{H}_{A':C|B'} X \rangle \approx \langle \hat{H}_{A':C|B'} \rangle \langle X \rangle = I(A':C|B') \langle X \rangle$$

Just using the definition...

Application 2 : Entanglement spectrum

After all these moves... $(\langle\cdots\rangle = \mathrm{Tr}(\rho\cdots))$



$$\langle \hat{H}_{A:C|B} X\rangle \approx \langle \hat{H}_{A':C|B'} X\rangle \approx \langle \hat{H}_{A':C|B'}\rangle\langle X\rangle = I(A':C|B')\langle X\rangle$$
$$\approx I(A:C|B)\langle X\rangle$$

Deformation moves, backward

Application 2 : Entanglement spectrum

Main Result 3: If $S_A = a|\partial A| - \gamma + \epsilon(|\partial A|)$,

$$|\langle \hat{H}_{A:C|B}, O\rangle| \leq \|O\|\mathcal{O}(|\partial A|^2 \epsilon(|\partial A|)).$$

$$\langle O_1, O_2\rangle = \langle O_1 O_2\rangle - \langle O_1\rangle\langle O_2\rangle.$$

for any O not overlapping with the boundary.

- We cannot prove that $\hat{H}_{A:C|B}$ is 0, but it is pretty close to being 0!
- We made no assumption about the property of the parent hamiltonian!

Conclusion

- Area law of entanglement entropy implies (approximate) conditional independence.
- Conditional independence implies i) first-order perturbative stability of topological entanglement entropy ii) local "cancelation" of entanglement spectrum.
- Structure of approximately conditionally independent state will have applications in quantum information theory for obvious reasons, but it will also benefit condensed matter theorists too.
  - Are there other extensions of strong subadditivity?

Another extension of SSA that can be useful

Conjecture :

$$\frac{1}{2}|\rho_{ABC} - \omega_{ABC}|_1^2 \leq I(A:C|B),$$

$$\omega_{ABC} = \rho_{AB}^{\frac{1}{2}} \rho_B^{-\frac{1}{2}} \rho_{BC} \rho_B^{-\frac{1}{2}} \rho_{AB}^{\frac{1}{2}}$$

This conjecture passed the following tests.

- Commuting case : Simply apply Pinsker's inequality for ρ_{ABC} and ω_{ABC}.
- This is even true for maximally antisymmetric states(Christiandl, Schuch, Winter 2009)!
- Numerical test : Tested $\geq 3 \times 10^6$ random mixed / pure states for $d_A = d_B = d_C = 2, 3, 4$, and no counterexamples were found.

Consequence

Conjecture :

$$\frac{1}{2}|\rho_{ABC} - \omega_{ABC}|_1^2 \leq I(A:C|B),$$

$$\omega_{ABC} = \rho_{AB}^{\frac{1}{2}} \rho_B^{-\frac{1}{2}} \rho_{BC} \rho_B^{-\frac{1}{2}} \rho_{AB}^{\frac{1}{2}}$$

If this is true

- It means that the performance of Petz's recovery map(transpose channel) for partial trace can be quite good, provided that $I(A:C|B)$ is small.
- There is a set of subsystems in the gapped quantum many-body systems for which one can argue that $I(A:C|B)$ is small.
- By choosing B and BC to be some localized region, one can recover the global density matrix from the *local density matrix* with good accuracy.
    - Note : If conditional independence holds exactly, the recovery map works perfectly.(Hastings and Poulin 2011)

Consequence

Theorem(Unpublished) : If the aforementioned conjecture is true, for systems satisfying area law, i.e. $S_A = a|\partial A| - \gamma + \epsilon(|\partial A|)$, trace distance between two locally indistinguishable states ψ_1, ψ_2 is bounded by the topological entanglement entropy.

$$|\psi_1 - \psi_2|_1 \leq 4g\sqrt{\gamma} + O(\epsilon(l/4)).$$

l : size of the system, g : genus.

- Existence of locally indistinguishable ground states implies a universal lower bound for topological entanglement entropy : $\gamma \geq \frac{1}{16}$.
- Topological degeneracy implies nonzero topological entanglement entropy, but not the other way around.
  - $g = 0$ for a sphere, but still topological entanglement entropy is nonzero.
- Topological entanglement entropy is an obstruction that prohibits a reconstruction of the global density matrix from the local density matrix.

Some remarks

Monotonicity of quantum relative entropy : Recall

$$D(\rho\|\sigma) \geq D(T\rho\|T\sigma),$$

where $D(\rho\|\sigma) = Tr(\rho(\log\rho - \log\sigma))$, quantum operation T.
The conjecture is a strengthening for the case $\rho = \rho_{ABC}$, $\sigma = \rho_{AB} \otimes \frac{I_C}{d_C}$, $T = Tr_A$. In particular,

$$\omega_{ABC} = R_\sigma T \rho_{ABC},$$

where $R_\sigma(\cdot) = \sigma^{\frac{1}{2}} T^*[(T\sigma)^{-\frac{1}{2}}(\cdot)(T\sigma)^{-\frac{1}{2}}]\sigma^{\frac{1}{2}}$ is nothing but Petz's recovery channel(transpose channel). Further, $D(\rho\|\sigma) = D(T\rho\|T\sigma)$ iff $\rho = R_\sigma T\rho$.(Petz 2003)