

# Chern-Simons-Maxwell from fermion lattice model

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**EPSRC**

Pioneering research  
and skills



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# Motivation

Topological systems fascinate:

- New *physics* (material science)
- New *reliable* technologies (quantum computation)



They can support **anyons**

Anyonic particles are interesting on their own right:

- Experimental detection (Abelian/Non-Abelian anyons)
- Kinematics, interactions, transport physics

**Main problems:**

**realisation  
detection**

# Commercial Break:

Combining physics, mathematics and computer science, topological quantum computation is a rapidly expanding research area focused on the exploration of quantum evolutions that are immune to errors. In this book, the author presents a variety of different topics developed together for the first time, forming an excellent introduction to topological quantum computation.

The makings of anyonic systems, their properties and their computational power are presented in a pedagogical way. Relevant calculations are fully explained, and numerous worked examples and exercises support and aid understanding. Special emphasis is given to the motivation and physical intuition behind every mathematical concept.

Demystifying difficult topics by using accessible language, this book has broad appeal and is ideal for graduate students and researchers from various disciplines who want to get into this new and exciting research field.

**Jiannis K. Pachos** is a Reader in the School of Physics and Astronomy at the University of Leeds, UK. He works on a variety of research topics, ranging from quantum field theory to quantum optics. Dr Pachos is a University Research Fellow of the Royal Society.

Pachos Introduction to Topological Quantum Computation

## Introduction to Topological Quantum Computation

Jiannis K. Pachos

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# Introduction

## Abelian Chern-Simons-Maxwell:

Gauge theory,  $U(1)$

Chern-Simons: *topological*

Maxwell is non-topo., *confining in 2+1 dimensions*

## Massive Thirring model:

Relativistic Dirac fermions with mass, *interacting*

## Lattice models:

"Easy" to implement in the laboratory (atoms in optical lattices, engineered materials etc.)

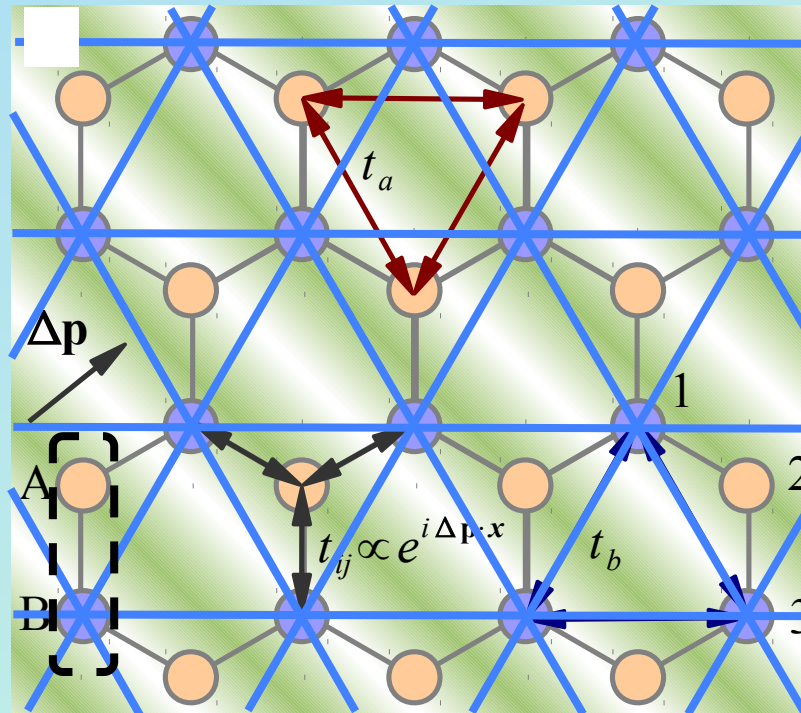
From Haldane (IQHE) + interactions (FQHE) **analytically**

# Extended Haldane's model

Two triangular sublattices:  $A$ ,  $B$ , each loaded with **fermionic atoms** in internal states  $b$ ,  $w$

Hamiltonian:

$$H = -t \sum_{\langle j,k \rangle} (b_j^\dagger w_k + w_k^\dagger b_j) - \sum_{\langle\langle j,k \rangle\rangle} (-t_w w_j^\dagger w_k + e^{i\phi_{jk}} t_b b_j^\dagger b_k) + U \sum_j w_j^\dagger w_j b_j^\dagger b_j$$



# Extended Haldane's model

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Two Fermi points  
Two massless Dirac particles

Interactions between  
Dirac fermions

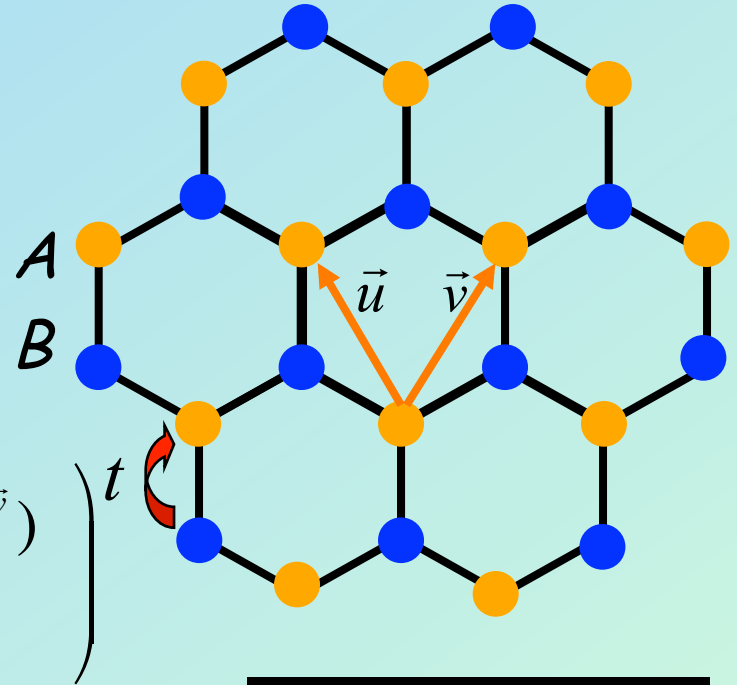
Gives mass to  
Dirac fermions and *maybe more...*

# Graphene-like structure

$$H = -t \sum_{\langle i,j \rangle} (w_i^+ b_j + b_j^+ w_i)$$

Fourier transformation:

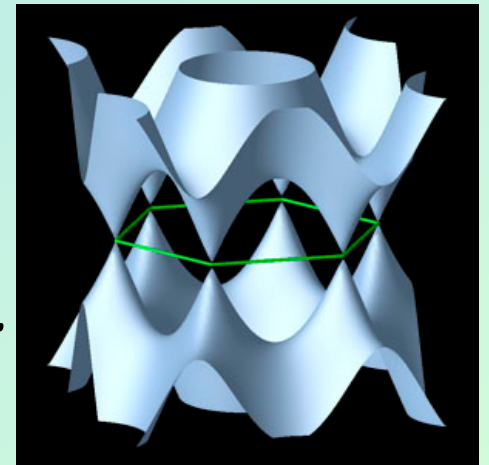
$$H_{\vec{p}} = \begin{pmatrix} 0 & -t(1 + e^{-i\vec{p}\cdot\vec{u}} + e^{-i\vec{p}\cdot\vec{v}}) \\ -t(1 + e^{i\vec{p}\cdot\vec{u}} + e^{i\vec{p}\cdot\vec{v}}) & 0 \end{pmatrix} t$$



$$E(\vec{p}) = \pm t \sqrt{3 + 2 \cos \vec{p} \cdot \vec{u} + 2 \cos \vec{p} \cdot \vec{v} + 2 \cos \vec{p} \cdot (\vec{u} - \vec{v})}$$

Fermi points:  $E(p)=0$

$E(p_x, p_y):$



# Graphene-like structure

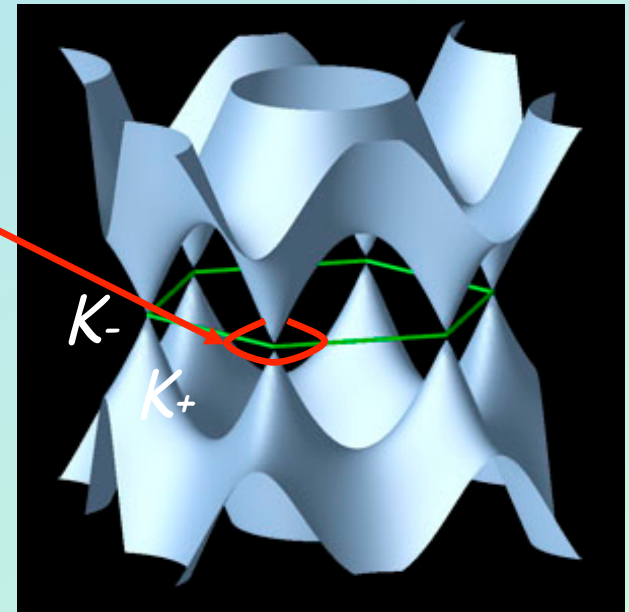
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Linearise energy  $E(\vec{k})$  around a conical point:

$$\vec{p} = \vec{K} + \vec{k}$$

$$H_{\vec{k}} \approx \pm \frac{3t}{2} \begin{pmatrix} 0 & k_x + ik_y \\ k_x - ik_y & 0 \end{pmatrix} = \pm \frac{3t}{2} \vec{\sigma} \cdot \vec{k}$$

Relativistic Dirac equation!  
-> Relativistic QFT.





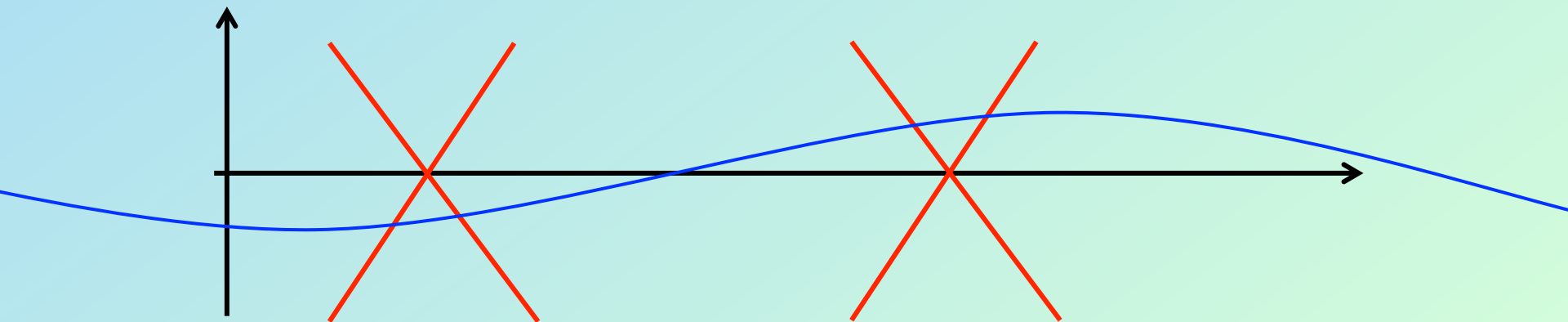
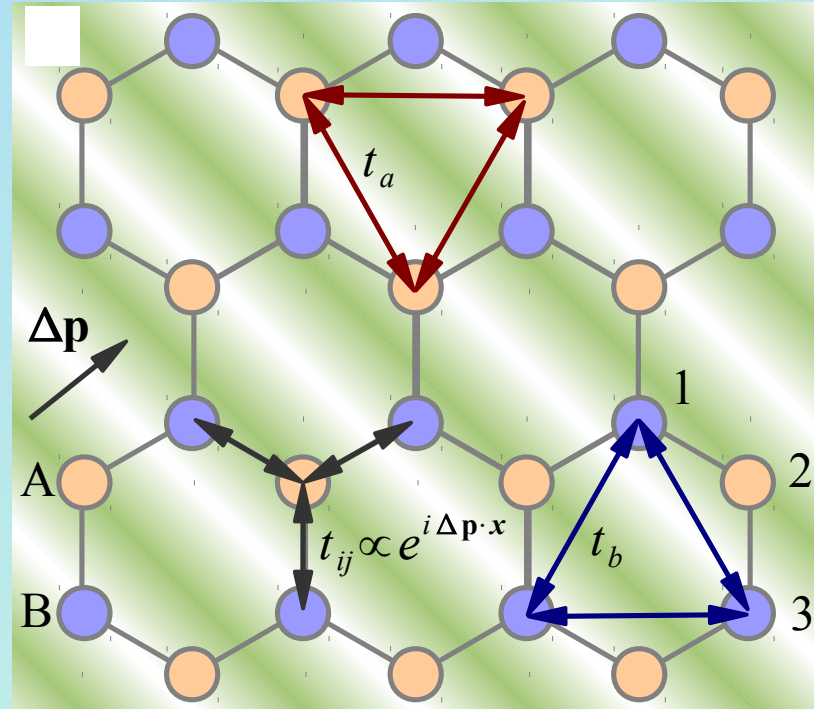
# Extended Haldane's model

Linearisation of Hamiltonian around Fermi points

$$H = -E \sum_k (w_k^+ \ b_k^+) \mathbf{S}(k) \cdot \boldsymbol{\sigma} \begin{pmatrix} w_k \\ b_k \end{pmatrix}$$

$$\mathbf{S}(k) \cdot \boldsymbol{\sigma} \approx \underbrace{k_x \sigma_x + k_y \sigma_y}_{t\text{-term}} + \underbrace{m(k) \sigma_z}_{t_w, t_b \text{-terms}}$$

"mass"  $m(k)$  can be +ve or -ve



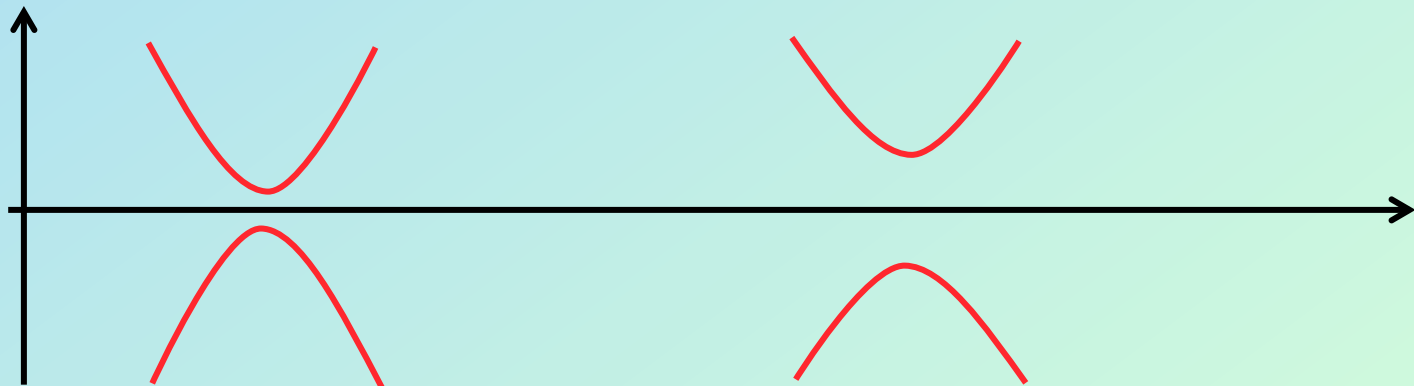
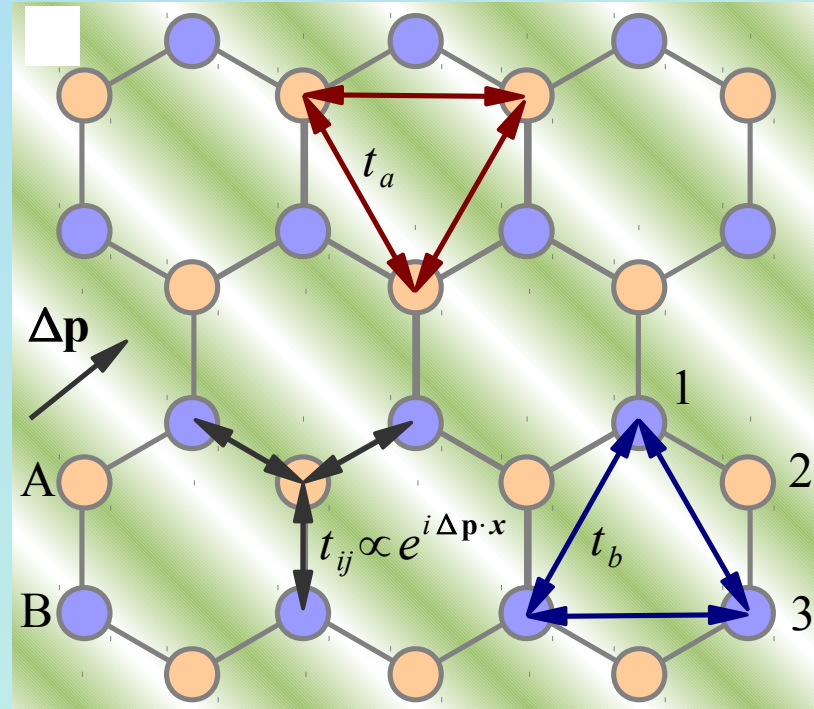
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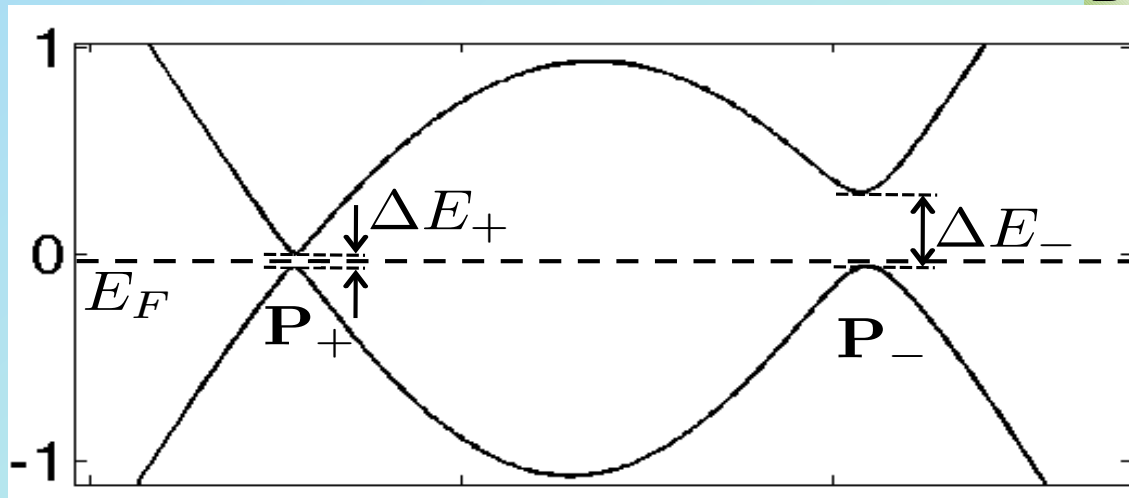
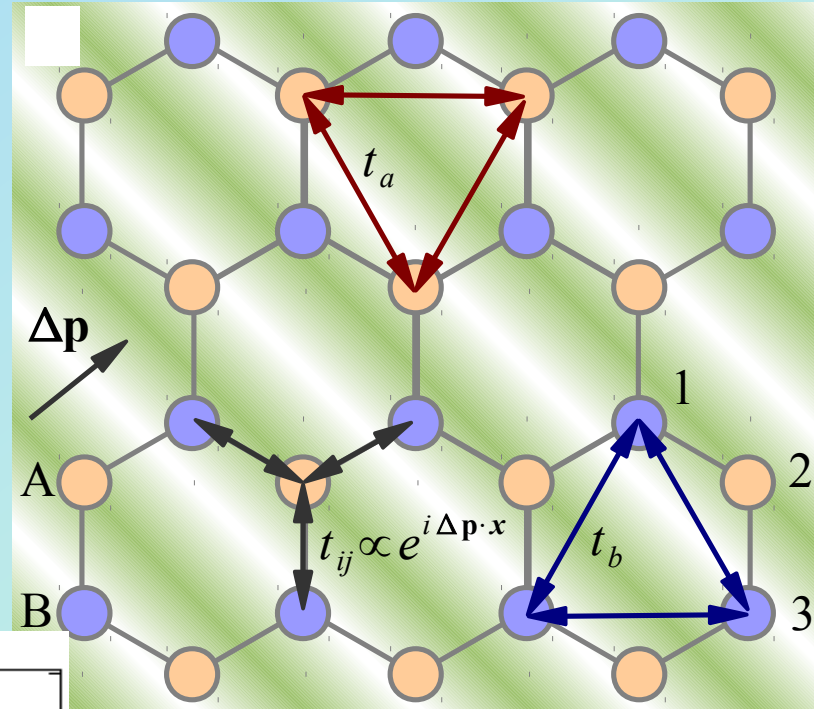


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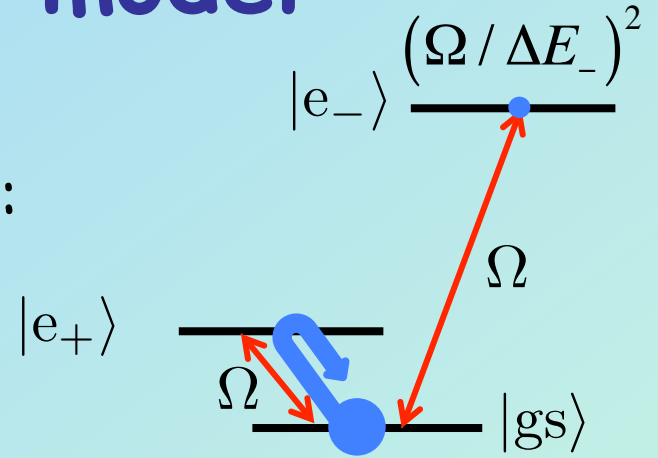


Adiabatic elimination of  $P_-$

# 2+1 dim Thirring model

Eliminate one Fermi point so only one Dirac fermion out of the lattice model:

Introduce interactions:



$$U \sum_j w_j^+ w_j b_j^+ b_j \rightarrow U \int w^+(r) w(r) b^+(r) b(r)$$

Altogether:

$$H \approx \int d^2 r \left[ \psi^\dagger (c \boldsymbol{\sigma} \cdot \mathbf{p} + \sigma_z M c^2) \psi + \frac{g^2}{2} j^\mu j_\mu \right]$$

$$g^2 = \frac{U}{3}, \quad c = \frac{3}{2} t \quad \text{and} \quad M = \frac{2}{3} \frac{t_b}{t^2}$$

# Chern-Simons-Maxwell

2+1 dim Thirring  $\rightarrow$  Chern-Simons-Maxwell theory

$$Z_{\text{Th}} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left\{ - \int d^3x \left[ \bar{\psi} (c \not{\partial} - M c^2) \psi - \frac{g^2}{2} j^\mu j_\mu \right] \right\}$$

$$\exp \left( \int d^3x \frac{g^2}{2} j^\mu j_\mu \right) = \int \mathcal{D}a_\mu \exp \left[ - \int d^3x \left( \frac{1}{2} a^\mu a_\mu + g j^\mu a_\mu \right) \right]$$

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[ -c \int d^3x \bar{\psi} \left( \not{\partial} + \frac{g}{c} \not{a} - M c \right) \psi \right] =$$

$$= \exp \left\{ c \log \left[ \det \left( \not{\partial} + \frac{g}{c} \not{a} - M c \right) \right] \right\} = \exp(-S_{\text{eff}}[a])$$

[Fradkin & Schaposnik 1994]

# Chern-Simons-Maxwell

2+1 dim Thirring  $\rightarrow$  Chern-Simons-Maxwell theory

$$S_{\text{eff}}[a] = \frac{ig^2}{8\pi c} \frac{Mc}{|Mc|} \int d^3x \epsilon^{\lambda\mu\nu} a_\lambda \partial_\mu a_\nu + \mathcal{O}\left(\frac{\partial}{Mc}\right)$$

$$S_{\text{I}}[a, A] = \int d^3x \left( \frac{1}{2} a^\mu a_\mu - i\epsilon^{\lambda\mu\nu} a_\lambda \partial_\mu A_\nu + \frac{2\pi ic}{g^2} \epsilon^{\lambda\mu\nu} A_\lambda \partial_\mu A_\nu \right)$$

$$Z_{\text{I}} = \int \mathcal{D}a_\mu \mathcal{D}A_\mu e^{-S_{\text{I}}[a, A]} =$$

$$\int \mathcal{D}a_\mu \exp \left[ - \int d^3x \left( \frac{ig^2}{8\pi c} \epsilon^{\lambda\mu\nu} a_\lambda \partial_\mu a_\nu + \frac{1}{2} a^\mu a_\mu \right) \right] =$$

$$\int \mathcal{D}A_\mu \exp \left[ - \int d^3x \left( \frac{2\pi i c}{g^2} \epsilon^{\lambda\mu\nu} A_\lambda \partial_\mu A_\nu + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \right] = Z_{\text{CSM}}$$

[Deser & Jackiw 1984]

# Measuring protocol

Observables of Chern-Simons theory: Wilson loop ops

$$W(L) = \exp\left(\frac{i}{g} \oint_L A_\mu dx^\mu\right)$$

$$\langle W(L) \rangle_{\text{CSM}} = \exp(\pm i\Phi_L/8\pi)$$

Take  $L$  on the space plane,  $L_0$ , and a single loop:  $\Phi_{L_0} = 0$

$$\langle \Psi_{\text{CSM}} | W(L_0) | \Psi_{\text{CSM}} \rangle = 1.$$

Ground state is stabilised by **all** Wilson loop operators

-> topologically ordered

- Topo entanglement entropy (Levin and Wen)
- Topo degeneracy (Freedman et al.)

# Measuring protocol

Bosonisation procedure: [Fradkin & Schaposnik 1994]

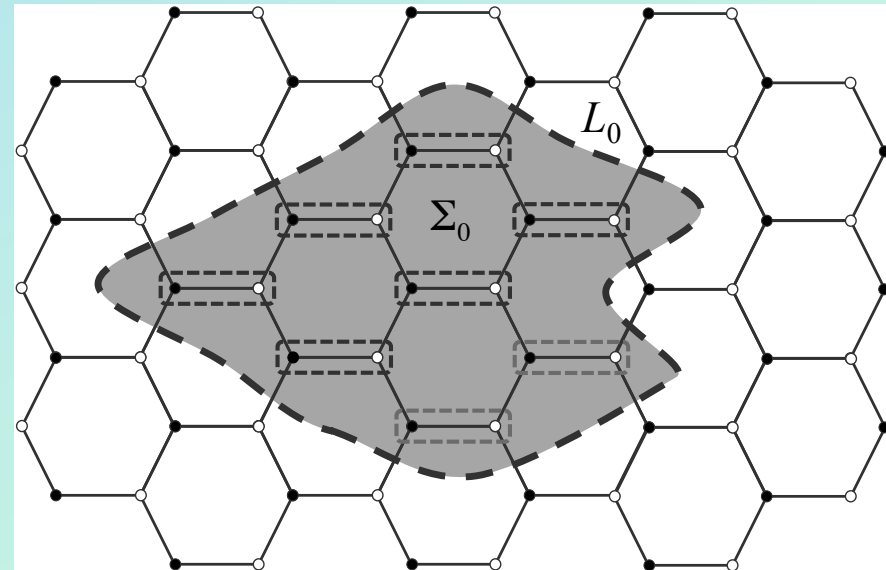
$$\langle W(L) \rangle_{\text{CSM}} = \langle \exp \left( i \int_{\Sigma} dS_{\mu} \bar{\psi} \gamma^{\mu} \psi \right) \rangle_{\text{Th}}$$

$$\int_{\Sigma_0} dS_{\mu} \bar{\psi} \gamma^{\mu} \psi = \int_{\Sigma_0} dS [b(\mathbf{r})^{\dagger} b(\mathbf{r}) + w(\mathbf{r})^{\dagger} w(\mathbf{r})]$$

$$\langle \Psi_{\text{TB}} | \exp \left[ i \sum_{i \in \Sigma_0} (b_i^{\dagger} b_i + w_i^{\dagger} w_i) \right] | \Psi_{\text{TB}} \rangle$$

$\sim 1$  Chern-Simons

$\sim \exp(-k|\Sigma_0|)$  Maxwell





# Conclusions

- Method to **eliminate the “doubling”** of Fermi points in lattice models -> possible to realise chiral models.
- **Analytically tractable** model for FQHE.
- **No need for magnetic field.**
- Wilson loop behaviour by **local density** measurements.
  
- **Future:**
  - **Non-Abelian anyons**
  - **SU(N) Yang-Mills theories 2+1 dims**
  - **Extension to 3+1 dims**