Two dimensional quantum memories

Commuting projector codes

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Joint work with Sergey Bravyi and Barbara Terhal

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Outline

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Definitions

- Λ is a 2D lattice.
- Each vertex occupied by d-level quantum particle.
- Hamiltonian $H = -\sum_{X \subset \Lambda} P_X$ with
  - $P_X = 0$ if radius$(X) \geq w$.
  - $[P_X, P_Y] = 0$.
  - P_X are projectors (optional).
- Code $\mathcal{C} = \{\psi : P_X|\psi\rangle = |\psi\rangle\}$
  = ground space of H
  = image of code projector $\Pi = \prod_X P_X$
- With proper coarse graining, we can assume that
  - Λ is a regular square lattice.
  - Each P_X acts on 2×2 cell.

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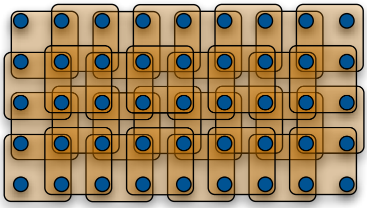

Well known examples

- Kitaev's toric code
- Bombin's topological color codes
- Levin & Wen's string-net models
- Turaev-Viro models
- Kitaev's quantum double models
- Most known models with topological quantum order

Remark

The first two example are simple because they are stabilizer codes. Most things I will say are trivial to prove in this case.

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Subsystem codes do not belong to this family.

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Standard definitions

Correctable region

A region $M \subset \Lambda$ is *correctable* if there exists a recovery operation $\mathcal{R}$ such that $\mathcal{R}(\mathrm{Tr}_M \rho) = \rho$ for all code states ρ.

Minimum distance

The minimum distance d is the size of the smallest non-correctable region.

Logical operator

Operator L such that $L|\psi\rangle$ is a code state for any code state $|\psi\rangle$.

Rate (capacity)

The rate of a code is $R = \frac{k}{n}$ where $k = \log \dim(\mathcal{C})$ and $n = |\Lambda|$ in the number of particles.

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Statement of the lemma

Holographic disentangling lemma

Let $M \subset \Lambda$ be a correctable region and suppose that its boundary ∂M is also correctable. Then, there exists a unitary operator $U_{\partial M}$ acting only on the boundary of M such that, for any code state $|\psi\rangle$,

$$U_{\partial M}|\psi\rangle = |\phi_M\rangle \otimes |\psi'_{\overline{M}}\rangle$$

for some *fixed* state $|\phi_M\rangle$ on M.

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For a trivial code $k = 0$, every region is correctable, so we recover the area law $S(M) \leq |\partial M|$ for commuting Hamiltonians of Wolf, Verstraete, Hastings, and Cirac.

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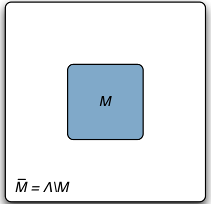
Proof

- Let M be correctable.
- Assume ∂M is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{AB} P_{B\overline{M}}$ with $[P_{AB}, P_{B\overline{M}}] = 0$.
- $\mathcal{H}_B = \bigoplus_J \mathcal{H}_{B_L^J} \otimes \mathcal{H}_{B_R^J}$ and $\Pi = \bigoplus_J P_{AB_L^J} \otimes P_{B_R^J \overline{M}}$
- This last sum over J contains only one non-zero factor since $B \subset M$ is correctable.
- We can divide B into two subsystems B^1 and B^2 such that $\Pi = V_B P_{AB^1} \otimes P_{B^2\overline{M}} V_B^\dagger$. $(\star)$



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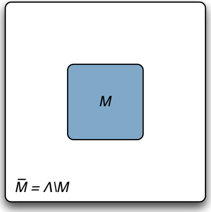

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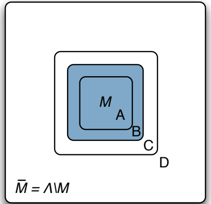

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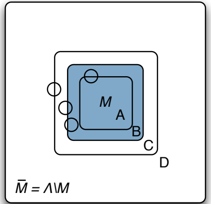

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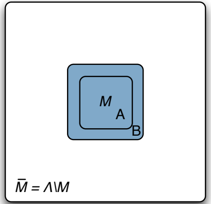

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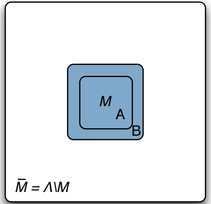

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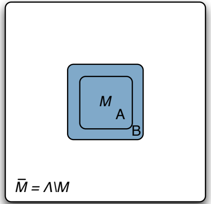

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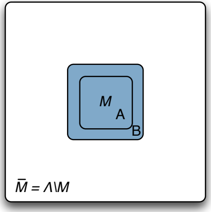

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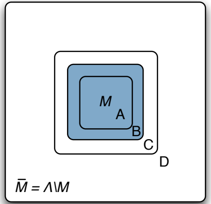

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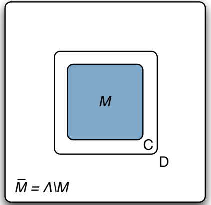

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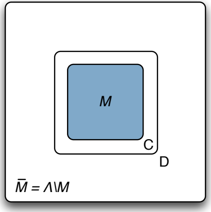

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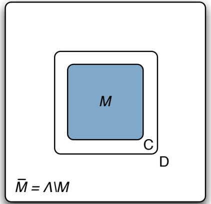

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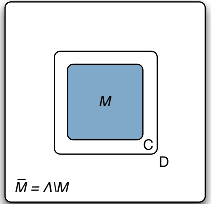

- $\mathcal{H}_C = \bigoplus_J \mathcal{H}_{C_L^J} \otimes \mathcal{H}_{C_R^J}$ and $\Pi = \bigoplus_J P_{MC_L^J} \otimes P_{C_R^J D}$
- This last sum over J contains only one non-zero factor since $C \subset \partial M$ is correctable.
- We can divide C into two subsystems C^1 and C^2 such that $\Pi = V_C P_{MC^1} \otimes P_{C^2 D} V_C^\dagger$. $(\star\star)$

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- Let M be correctable.
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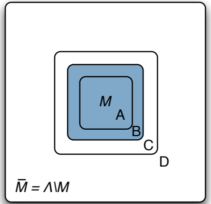

- Combining $(\star)$ with $(\star\star)$, $\Pi' = V_B^\dagger V_C^\dagger \Pi V_B V_C = P_{AB^1} P_{B^2C^1} P_{C^2D}$
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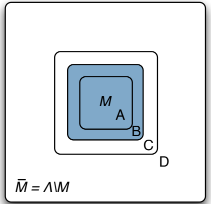

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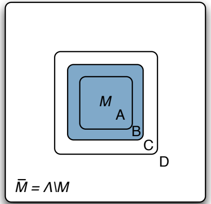

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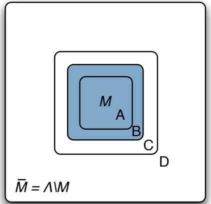

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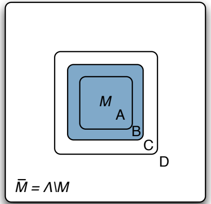

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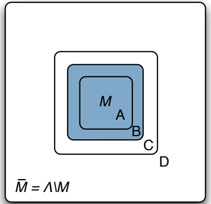

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Holographic minimum distance

Region $M \subset \Lambda$ is correctable if its boundary is smaller than the minimum distance $|\partial M| \leq cd$.

- Bulky errors are not problematic: it's the skinny ones we need to worry about.
- This hints at our next result: string-like logical operators.

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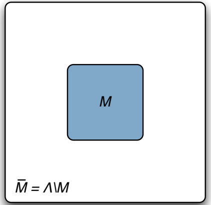

- Let $M \subset \Lambda$ be a correctable region.
- If $|\partial M| \leq d$, then ∂M is also correctable.
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- But from the Holographic disentangling lemma, $\rho_{AD} = \eta_A \otimes \rho_D$ with η_A independent of the encoded state ρ.
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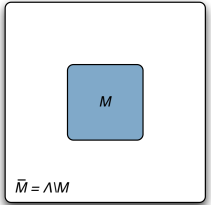

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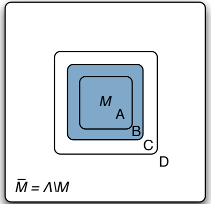

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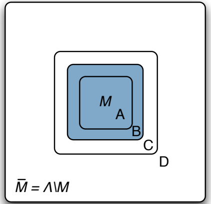

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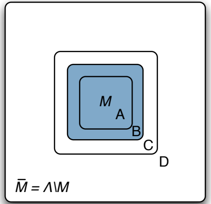

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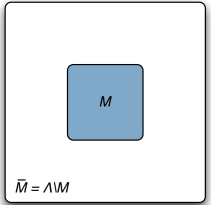

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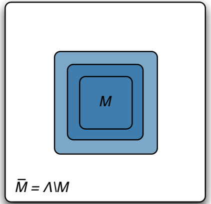

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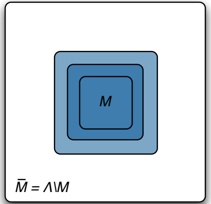

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Statement of the result

Capacity-Stability Tradeoff

$$k \leq c\frac{n}{d^2}$$

- Singleton's bound: $k \leq n - 2(d-1)$.
- Hamming bound: $k \leq n\left[1 - \frac{d}{2n}\log 3 - H(\frac{d}{2n})\right]$.
- Kitaev's codes (with punctures) saturate this bound, so it is tight.
  - Holes of linear size ℓ separated by distance ℓ.
  - Minimum distance $d \propto \ell$.
  - Number of logical qubits $k \propto$ number of holes $\propto n/\ell^2 \propto n/d^2$.
- No "good codes" in 2D, i.e. $k \propto n$ and $d \propto n$.
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Tool 1

Information-theoretic condition for error correction

M is correctable iff $S(M\overline{M}) = S(\overline{M}) - S(M)$ for any code state ρ.

- Obvious for pure states.
- Let $\rho_{M\overline{M}}$ be a code state and $\rho_{M\overline{M}R}$ its purification.
- By assumption, there exists $\mathcal{R}$ on Λ such that $\mathcal{R}(\mathrm{Tr}_M \rho_{M\overline{M}R}) = \rho_{M\overline{M}R}$ and $\mathcal{R}(\mathrm{Tr}_M \rho_{M\overline{M}} \otimes \rho_R) = \rho_{M\overline{M}} \otimes \rho_R$.
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Information-theoretic condition for error correction

M is correctable iff $S(M\overline{M}) = S(\overline{M}) - S(M)$ for any code state ρ.

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Union of correctable regions

Let M_1 and M_2 be correctable distant regions and suppose that ∂M_1 is also correctable. Then, $M_1 \cup M_2$ is correctable.

- Trivial for syndrome-based error correction (e.g. stabilizer codes).
- We will prove the Knill-Laflamme condition $\Pi O_{M1} \otimes O_{M_2} \Pi \propto \Pi$.
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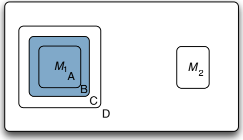

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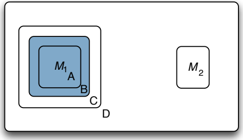

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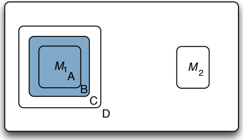
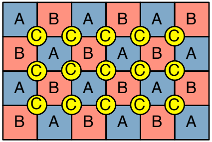


Proof of Capacity-Stability tradeoff



- Squares have perimeter $\approx d$.
- Region A (union of blue squares) is correctable.
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- Let's apply the information theoretic conditions to maximally mixed code state $\rho = \Pi/\mathrm{Tr}(\Pi)$ in two different ways:
- Using $S(BC) \leq S(B) + S(C)$ and $S(AC) \leq S(A) + S(C)$

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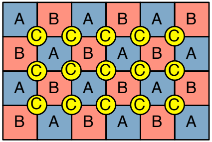

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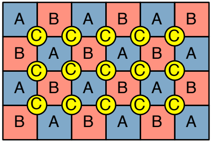

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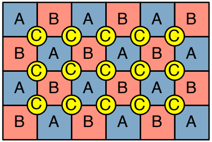

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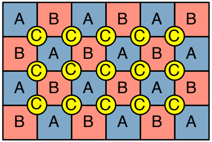

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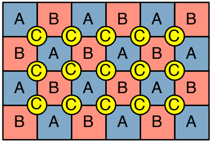

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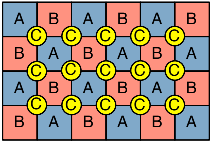

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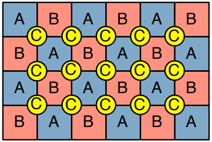

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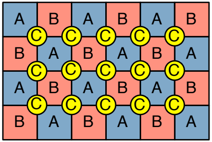

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Outline

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There exists a non-trivial logical operator supported on a string-like region.

- Well known for Kitaev's toric code.
- Intuitive for known models that support anyons:
  - The ground state can be changed by dragging an anyon around a topologically non-trivial loop.
  - This process is realized on a string, and generated a logical operation.
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There exists a string-like region that is not-correctable.

- Let M be a string-like region.
- Suppose M is correctable.
- Consider its boundary $\partial M = \partial M_L \cup \partial M_R$.
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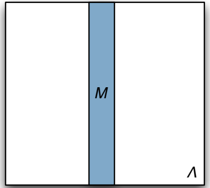

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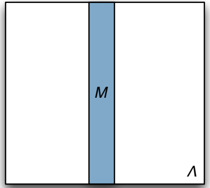

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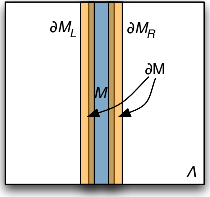
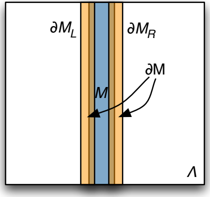


Proof, part 1

There exists a string-like region that is not-correctable.

- Let M be a string-like region.
- Suppose M is correctable.
- Consider its boundary $\partial M = \partial M_L \cup \partial M_R$.
- If either ∂M_L or ∂M_R are not correctable, we are done.



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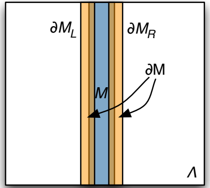

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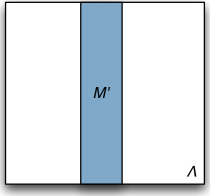

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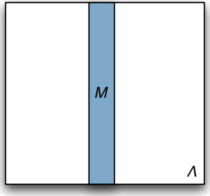
Proof, part 2

- Let M be a non-correctable string-like region.
- There exists O_M such that $\Pi O_M \Pi \not\propto \Pi$.
- Let $\Pi_M = \prod_{X \cap M \neq \emptyset} P_X$
- Then $X = \Pi_M O_M \Pi_M$ is a non-trivial logical operator supported on $M \cup \partial M$.
- Any function of X, e.g. $\exp(-iX\theta)$, is also a logical operator with the same support.



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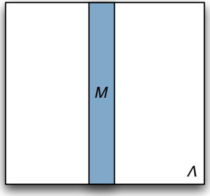

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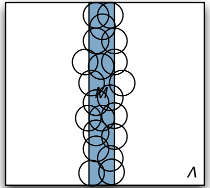

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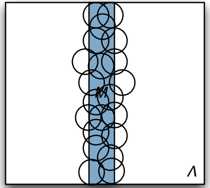

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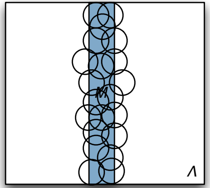

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Outline

- All our results extend to D-dimensional lattices, e.g. $k \leq cn/d^{\frac{2}{D-1}}$
  - How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)
- String-like logical operators $\Rightarrow$ constant energy barrier.
  - This is not directly related to thermal instability.
  - 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
  - What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).
  - Can we characterize all string-like logical operators?
  - Relation between commuting projector codes and anyon models.
- Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).
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