Sydney Quantum Information Theory Workshop
Coogee'13

Towards a complete characterization of *Emergent Topological Order* *from a microscopic Hamiltonian*

Guifre Vidal

Perimeter Institute

Based on ***Lukasz Cincio, G. V., arXiv:1208.2623 (accepted in PRL)***

Coogee, Australia 15th-18th January 2013

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Towards a complete characterization of Emergent Topological Order from a microscopic Hamiltonian

Collaboration with
Lukasz Cincio
(Perimeter Institute)

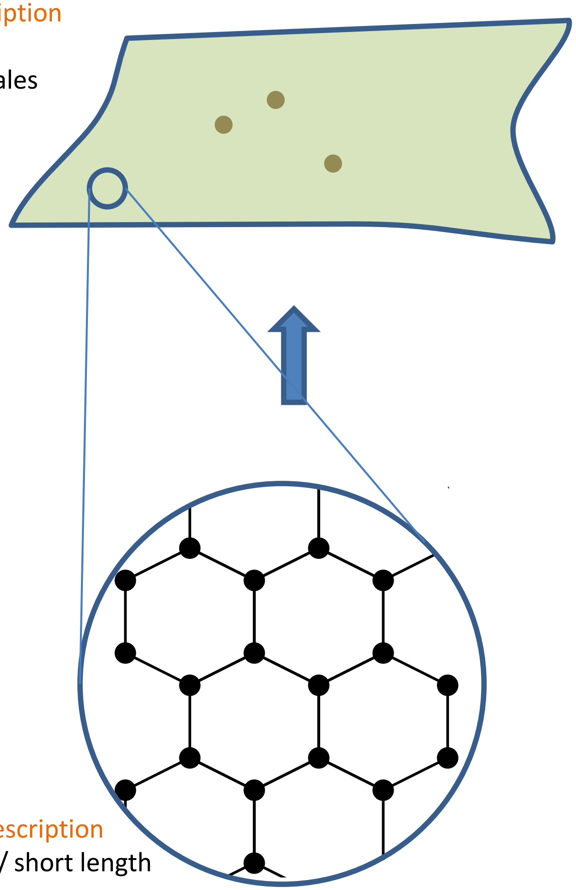
Based on ***Lukasz Cincio, G. V., arXiv:1208.2623 (accepted in PRL)***

Introduction

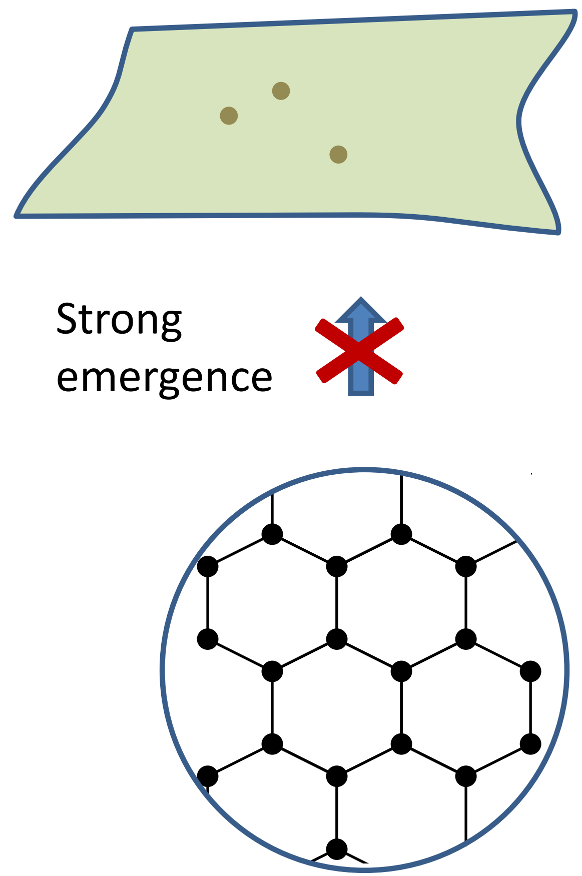
Reductionism versus Emergence

(bottom up)

effective description
at low energy/
large length scales



microscopic description
at high energy/ short length
scales



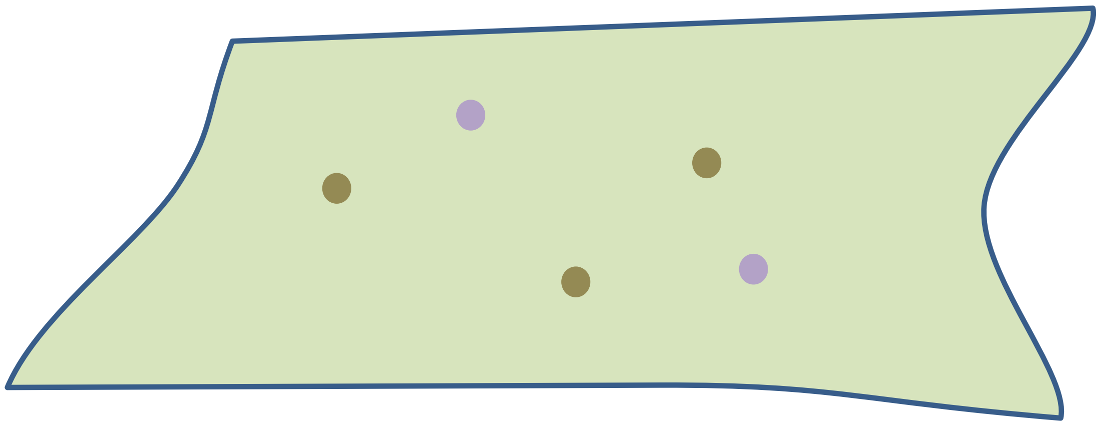
Strong
emergence

P. W. Anderson

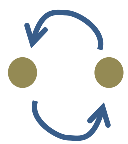
R. B. Laughlin

Introduction

Emergent topological order



- exotic quasi-particle excitations

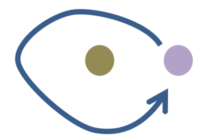


self statistics

$$|\Psi\rangle \to e^{i\varphi}|\Psi\rangle$$ anyon

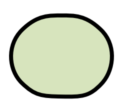
$|\Psi\rangle \to |\Psi\rangle$ boson

$|\Psi\rangle \to -|\Psi\rangle$ fermion

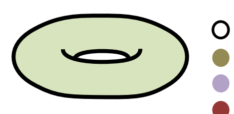


mutual statistics

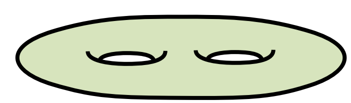
- ground state degeneracy depends on topology



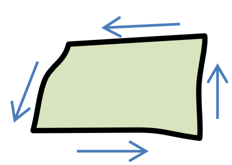
sphere

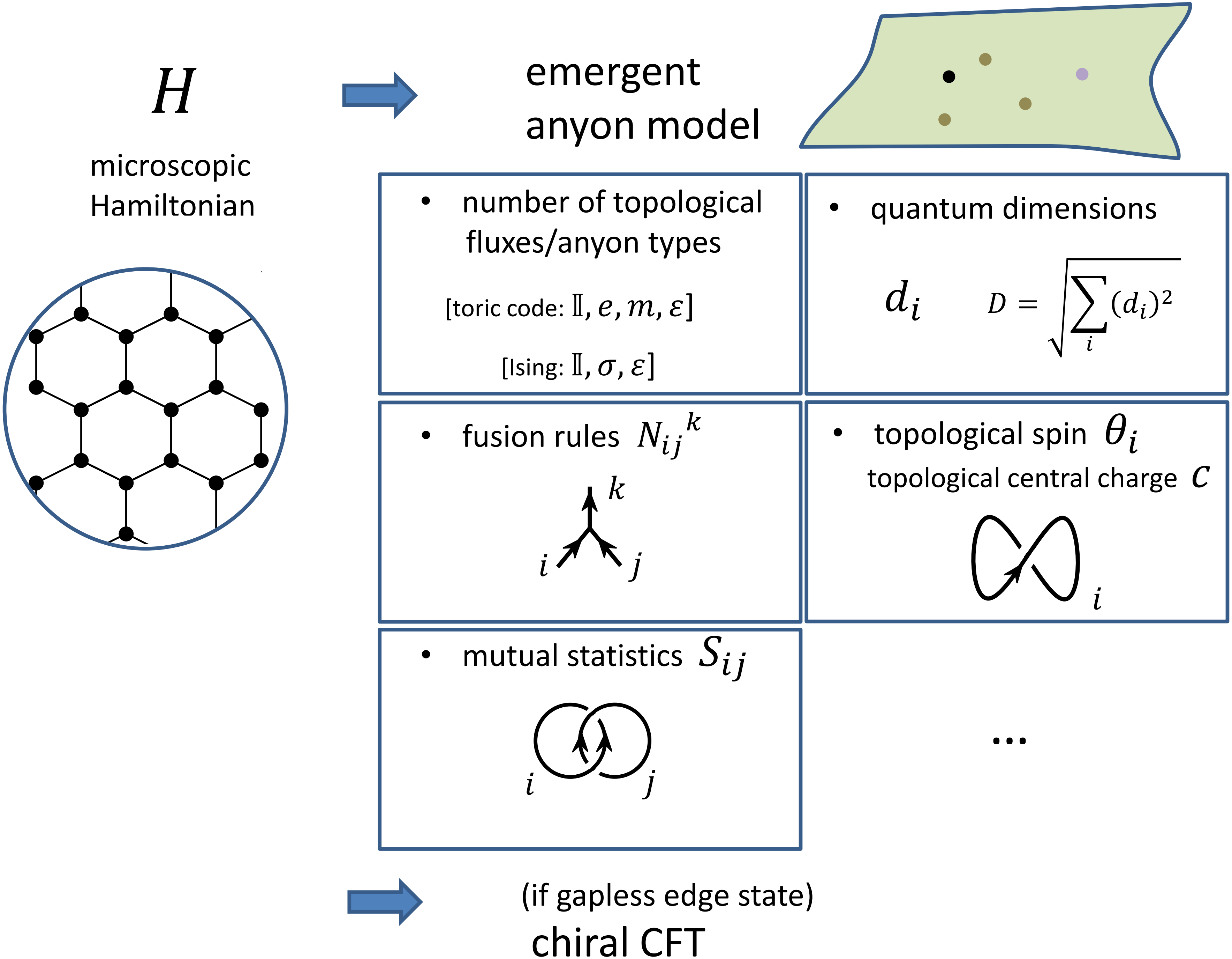


torus



- (if chiral) protected gapless edge modes at boundary





H
microscopic
Hamiltonian
emergent
anyon model
• number of topological fluxes/anyon types
[toric code: $\mathbb{I}, e, m, \varepsilon$]
[Ising: $\mathbb{I}, \sigma, \varepsilon$]
• quantum dimensions
d_i
$D = \sqrt{\sum_i (d_i)^2}$
• fusion rules $N_{ij}{}^{k}$
k
i
j
• topological spin θ_i
topological central charge c
i
• mutual statistics S_{ij}
i
j
...
(if gapless edge state)
chiral CFT

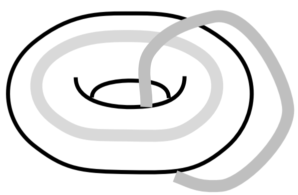
Clarification (1)

Kitaev's toric code (quantum double of Z_2)

Ground subspace on the torus

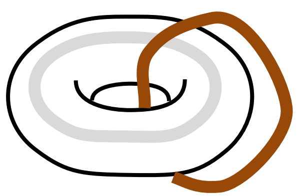
$$|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$$

One possible basis:



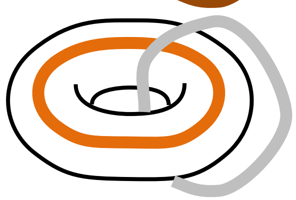
$\mathbb{I} \times \mathbb{I}$

$|00\rangle$



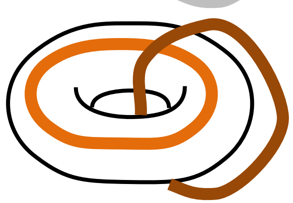
$\mathbb{I} \times m$

$|01\rangle$



$m \times \mathbb{I}$

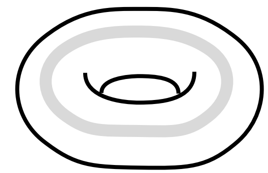
$|10\rangle$



$m \times m$

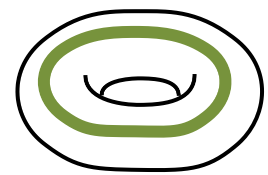
$|11\rangle$

Another basis:



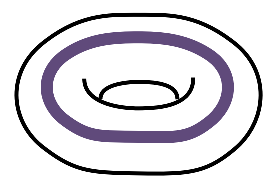
$\mathbb{I}$

$|0\,+\rangle$



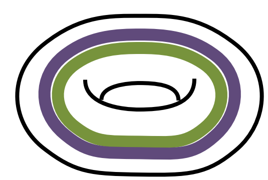
e

$|0\,-\rangle$



m

$|1\,+\rangle$



$\varepsilon = em$

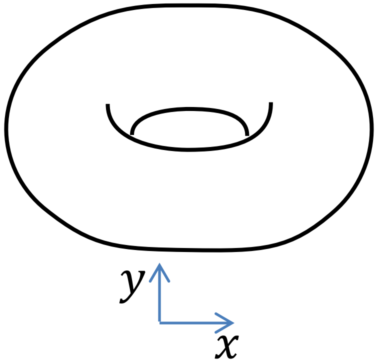
$|1\,-\rangle$

Clarification (2)

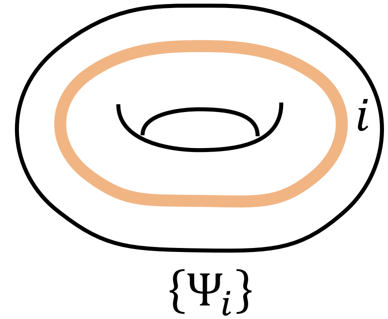
- complete set of ground states of a lattice Hamiltonian H

X.-G. Wen, 1989

A) on a torus



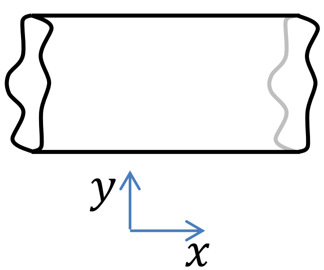
$$L_x, L_y \gg \xi$$



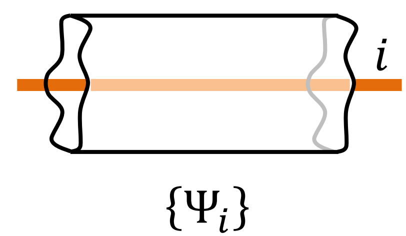
fact: each ground state has a well-defined anyon flux in x-direction

example: toric code $i = \mathbb{I}, e, m, \varepsilon$

B) on an infinite cylinder (*with no boundaries*)



$$L_x = \infty; \;\; L_y \gg \xi$$



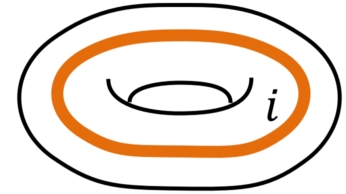
fact: each ‘ground state’ has a well-defined anyon flux in x-direction

example: toric code $i = \mathbb{I}, e, m, \varepsilon$

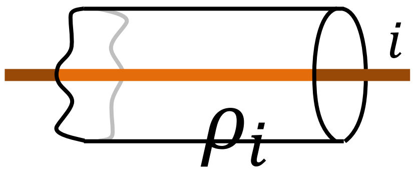
Background

Infinite cylinder

Finite torus



Entanglement spectrum



spectrum of gapless edge state (chiral CFT)

$$H_i^{(boundary)}$$

H. Li, F. D. M. Haldane, PRL 2008

X.-L. Qi, H. Katsura, A. W. W. Ludwig, PRL 2012

Topological entanglement entropy

$$S_L = \mathrm{a}L - \gamma$$

$$\gamma = \log\left(\frac{D}{d_i}\right)$$

quantum dimensions

$$\frac{d_i}{D} \qquad D = \sqrt{\sum_i (d_i)^2}$$

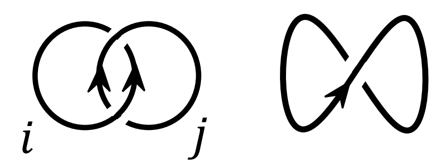
A. Kitaev, J. Preskill, PRL 2006, M. Levin, X.-G. Wen, PRL 2006

S. Dong, E. Fradkin, R. Leigh, S. Nowling, JHEP 2008

Modular transformations

topological S, U matrices

$$V_{ij} = \langle \Psi_i{}^{tor} | R\pi_{/3} | \Psi_j{}^{tor} \rangle \qquad V = DUS^{-1}D^{\dagger}$$

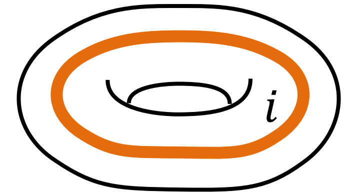


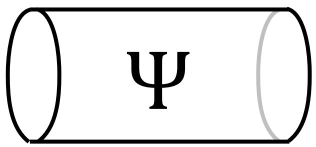
Y. Zhang, T. Grover, A. Turner, M. Oshikawa, A. Vishwanath, PRB 2012

Background

Infinite cylinder

Finite torus





ground state
on finite cylinder

2D DMRG

S. Yan, D. A. Huse, S. R. White, Science 2011

$$D = \sqrt{\sum_i (d_i)^2}$$

H.-C. Jiang, H. Yao, L. Balents, PRB 2012

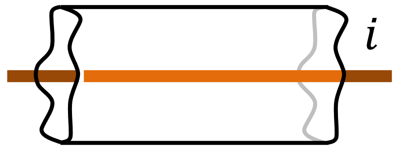
H.-C. Jiang, Z. Wang, L. Balents, arXiv:1205.4289

S. Depenbrock, I. P. McCulloch, U. Schollwoeck, PRL 2012

OUTLINE

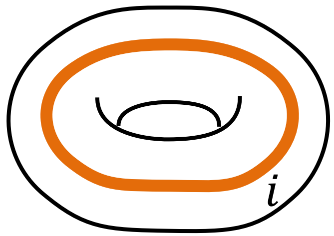
1) GROUND STATES

Infinite cylinder



- edge spectrum
  - quantum dimensions
  - chiral CFT

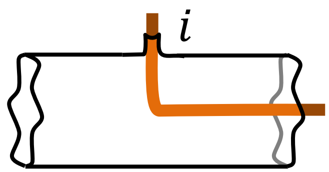
Finite torus



- S matrix
  - mutual statistics
  - quantum dimensions
  - fusion rules
- U matrix
  - central charge
  - topological spins

2) QUASIPARTICLE EXCITATIONS

Infinite cylinder

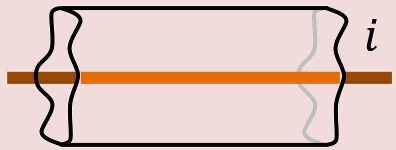


- integer excitations
- fractionalized excitations

OUTLINE

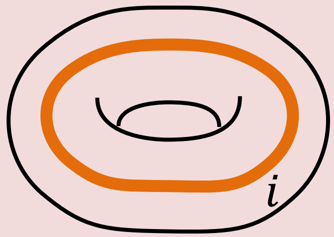
1) GROUND STATES

Infinite cylinder



- edge spectrum
  - quantum dimensions
  - chiral CFT

Finite torus



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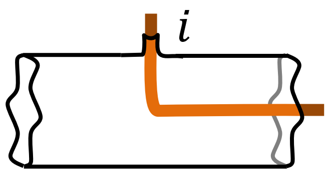
2) QUASIPARTICLE EXCITATIONS

Infinite cylinder



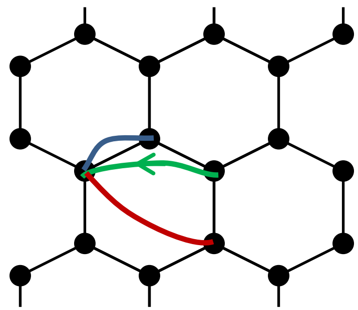
- integer excitations
- fractionalized excitations

LATTICE MODELS

Haldane

(hardcore bosons on honeycomb)

F.D.M. Haldane, PRL 1988



$t = 1$

$t' = 0.6$

$\phi = 0.4\pi$

$t'' = -0.58$

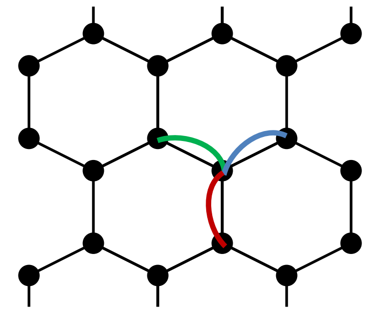
$$H = -t \sum_{\langle rr' \rangle} b_r{}^{\dagger} b_{r'} - t' \sum_{\langle\langle rr' \rangle\rangle} b_r{}^{\dagger} b_{r'} e^{i\phi_{rr'}} - t'' \sum_{\langle\langle\langle rr' \rangle\rangle\rangle} b_r{}^{\dagger} b_{r'}$$

Y.-F. Wang, Z.-C. Gu, C.-D. Gong, D.N. Sheng, PRL 2011

Kitaev Honeycomb

(non-Abelian phase with magnetic field)

A. Kitaev , Annals of Physics 2006

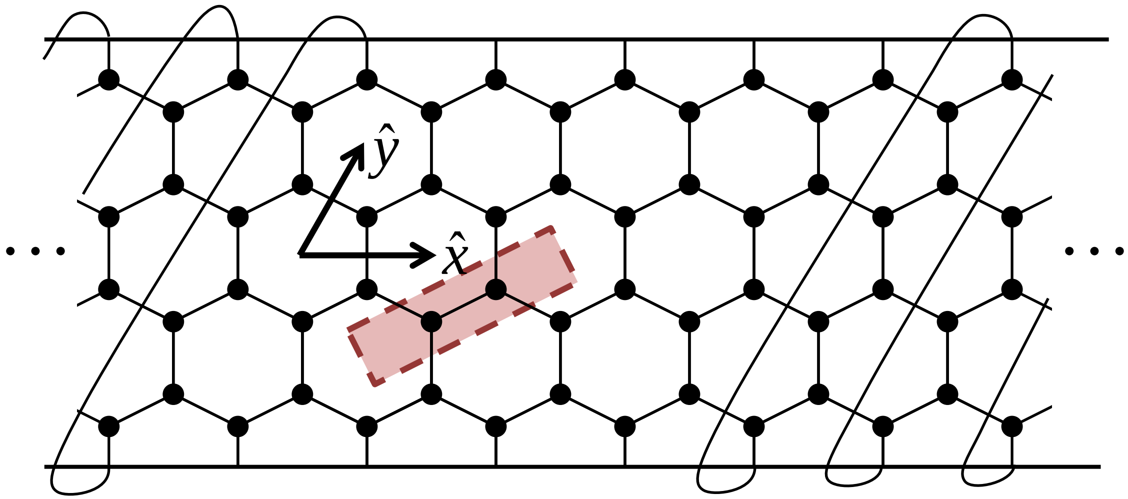


h=0.01

$$H = \sum_{\langle rr' \rangle_x} \sigma^x{}_r \sigma^x{}_{r'} + \sum_{\langle rr' \rangle_y} \sigma^y{}_r \sigma^y{}_{r'} + \sum_{\langle rr' \rangle_z} \sigma^z{}_r \sigma^z{}_{r'}$$

$$+h \sum_r (\sigma^X{}_r + \sigma^Y{}_r + \sigma^z{}_r)$$

VARIATIONAL WAVEFUNCTION



$L_y = 4$

$(XC8-4)?$

$L_x = \infty$

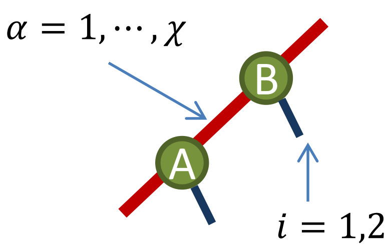
MPS / 2D DMRG

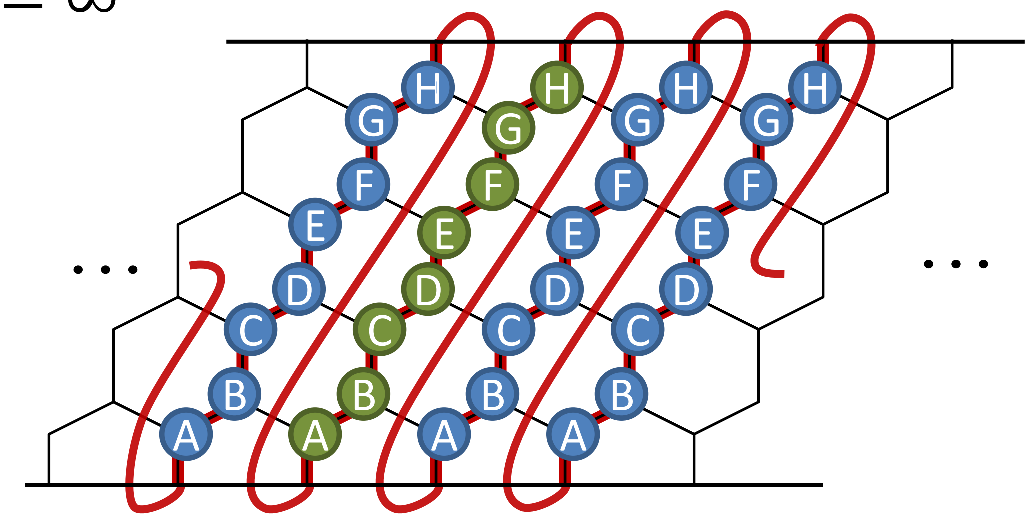
(Matrix Product State)

S. White, PRL 1992

S. Yan, D. A. Huse, S. R. White, Science 2011

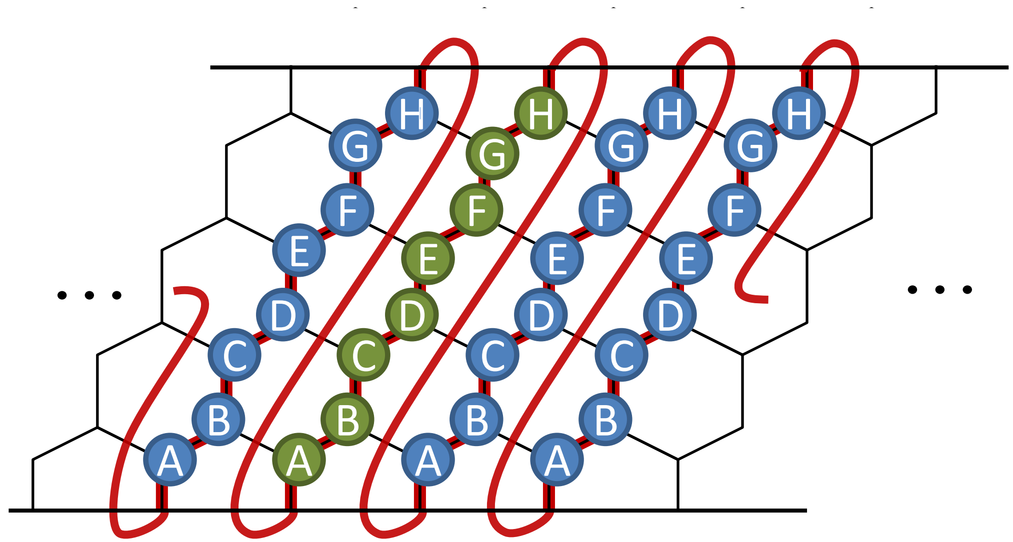
H.-C. Jiang, H. Yao, L. Balents, PRB 2012



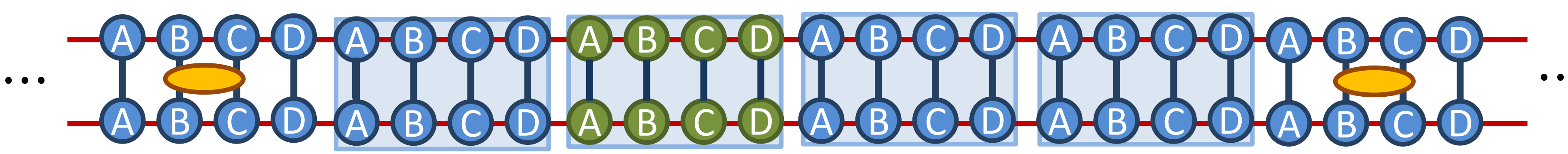


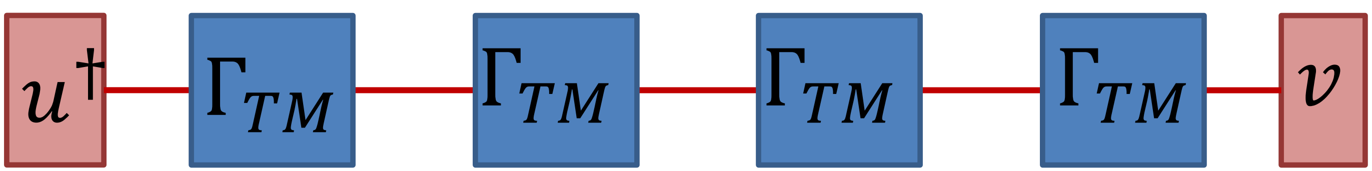
Computational cost $O(\chi^3) \sim e^{L_y}$ $L_y \gg \xi$

CORRELATION LENGTH



$$\langle\Psi|o(0,0)o(x,y)|\Psi\rangle =$$



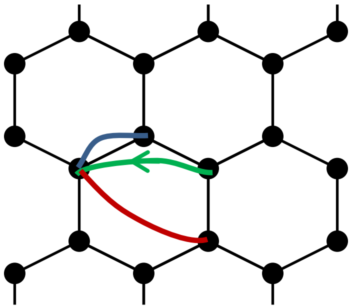


$$\approx \lambda^x = e^{-x/\xi_{TM}}$$

$$\xi_{TM} \stackrel{\text{def}}{=} -\frac{1}{\log(\lambda)}$$

Haldane model (hardcore bosons)

F.D.M. Haldane, PRL 1988



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Y.-F. Wang, Z.-C. Gu, C.-D. Gong, D.N. Sheng, PRL 2011

(flat band!)

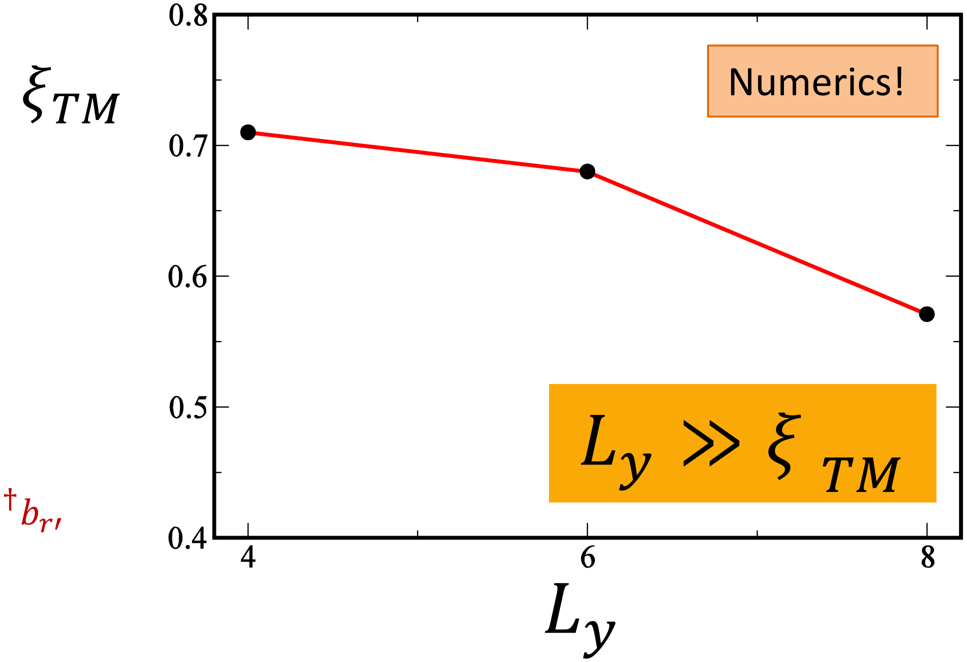
We find 2 ‘ground states’:

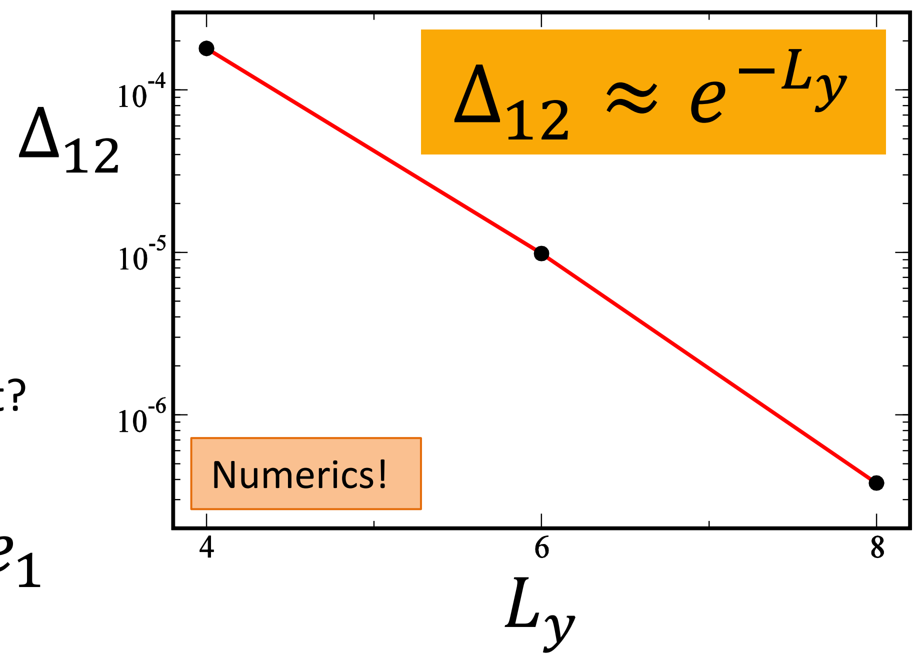
H

Numerics!

Ψ_1 Ψ_2 complete set?

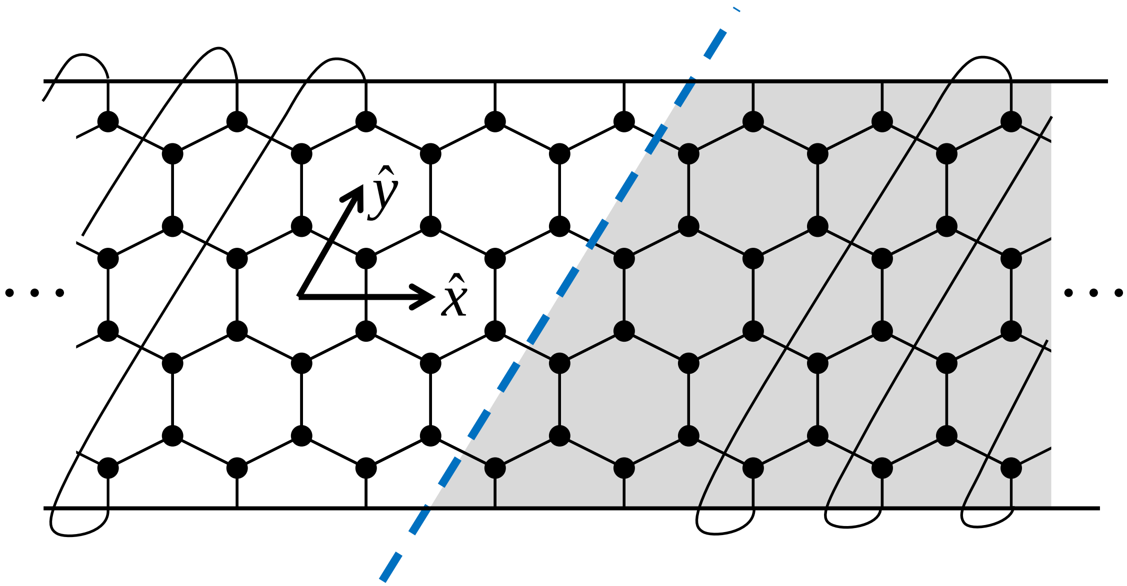
$$e_i \stackrel{\text{def}}{=} \frac{E_i}{L_x L_y} \qquad \Delta_{12} \stackrel{\text{def}}{=} e_2 - e_1$$

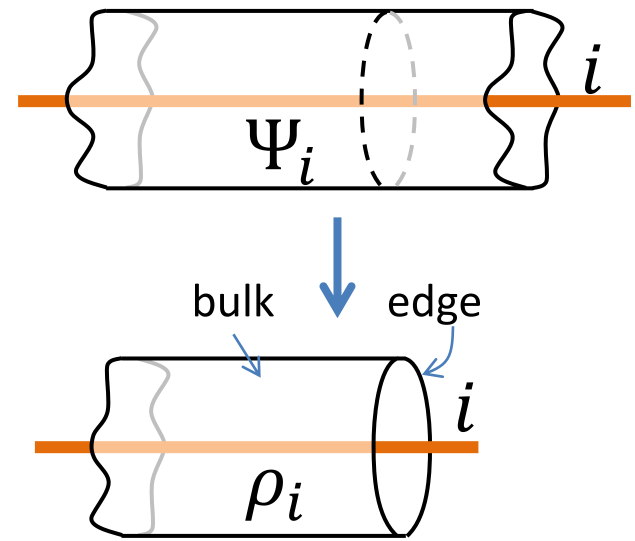




Haldane model
(hardcore bosons)

ENTANGLEMENT SPECTRUM (I)





$$H \to \begin{cases} \Psi_1 \to \rho_1 \to \{p_{1,\alpha}\} \\ \Psi_2 \to \rho_2 \to \{p_{2,\alpha}\} \end{cases}$$

'ground states' infinite cylinder	density matrices semi-infinite cylinder	spectra

$$\rho_i |p_{i,\alpha}\rangle = p_{i,\alpha} |p_{i,\alpha}\rangle$$

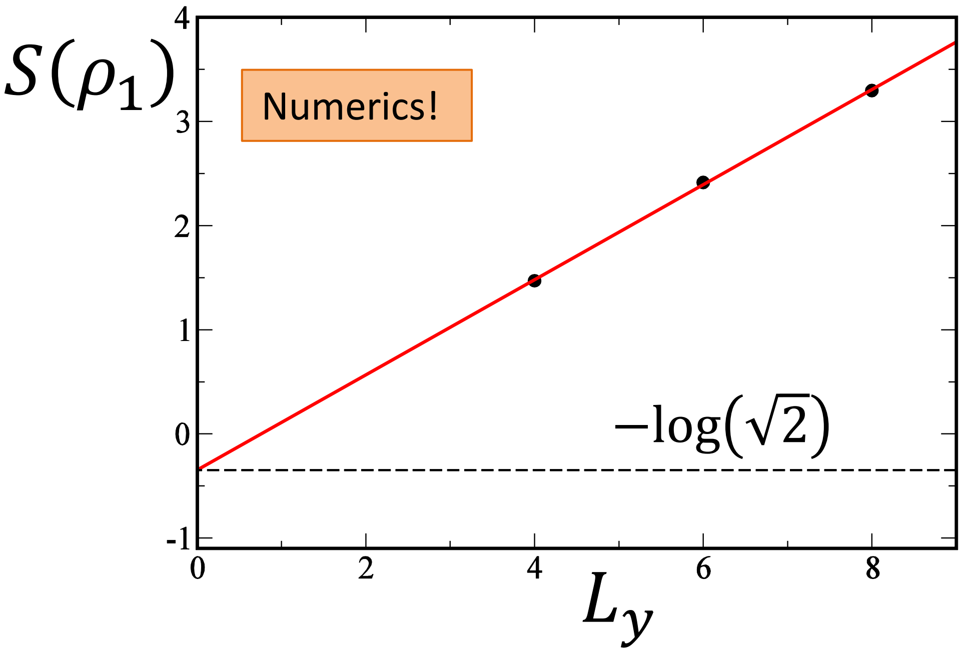
Haldane model
(hardcore bosons)

$$\{p_{1,\alpha}\},\{p_{2,\alpha}\} \longrightarrow S(\rho_1),S(\rho_2)$$

spectrum

Scaling of entanglement entropy

A. Kitaev, J. Preskill, PRL 2006
M. Levin, X.-G. Wen, PRL 2006
S. Dong, E. Fradkin, R. Leigh, S. Nowling, JHEP 2008



Region with flux i

$$S_L = \mathrm{a}L - \log\left(\frac{D}{d_i}\right)$$

*For one ground state in large finite cylinder, H.-C. Jiang, H. Yao, L. Balents, PRB 2012,
H.-C. Jiang, Z. Wang, L. Balents, arXiv:1205.4289

Numerics!

$$\frac{d_1}{D} = 0.7079 \approx \frac{1}{\sqrt{2}} \ (0.1\%)$$

$$S(\rho_1) - S(\rho_2) = \log\left(\frac{d_1}{d_2}\right)$$

Numerics!

$$d_1/d_2 = 1.005$$

We found 2 'ground states':

$$H$$

Numerics!

$$\Psi_1 \qquad \Psi_2$$

complete set?

$$D \stackrel{\text{def}}{=} \sqrt{\sum_i (d_i)^2}$$

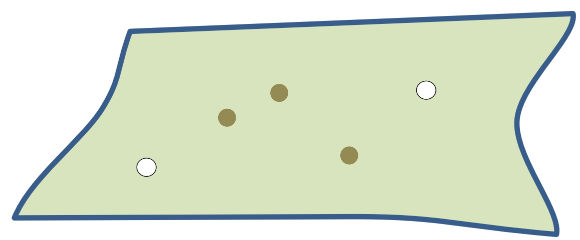
$$\Downarrow$$

$$\sum_i \left(\frac{d_i}{D}\right)^2 = 1$$

Numerics!

$$\left(\frac{d_1}{D}\right)^2 + \left(\frac{d_2}{D}\right)^2 = 1.007$$

$\Longrightarrow$ complete set



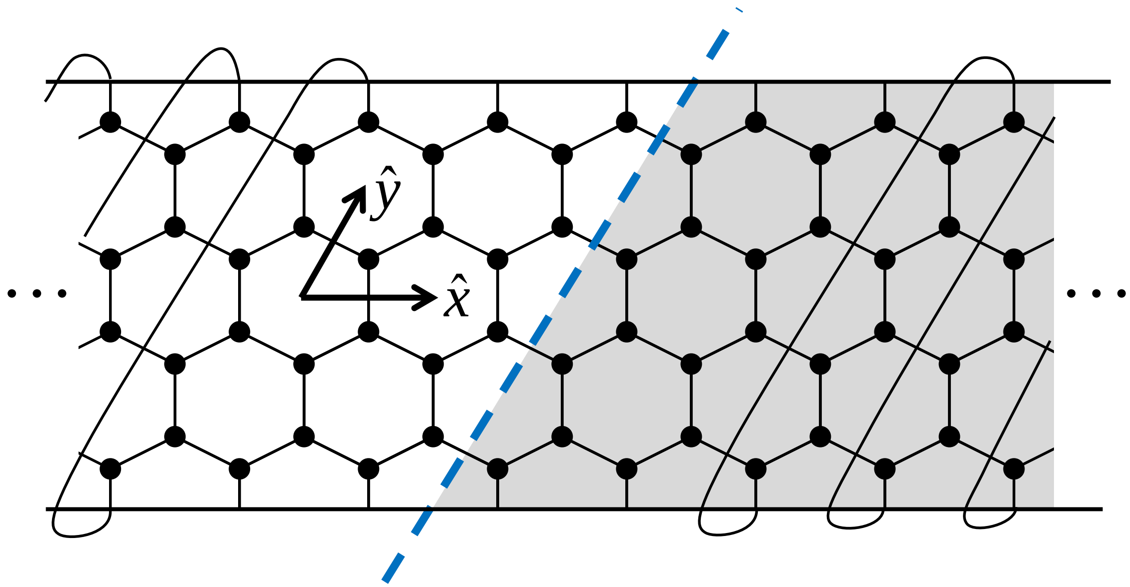
Any anyon model has identity $i = \mathbb{I}$, with quantum dimension $d_{\mathbb{I}} = 1$

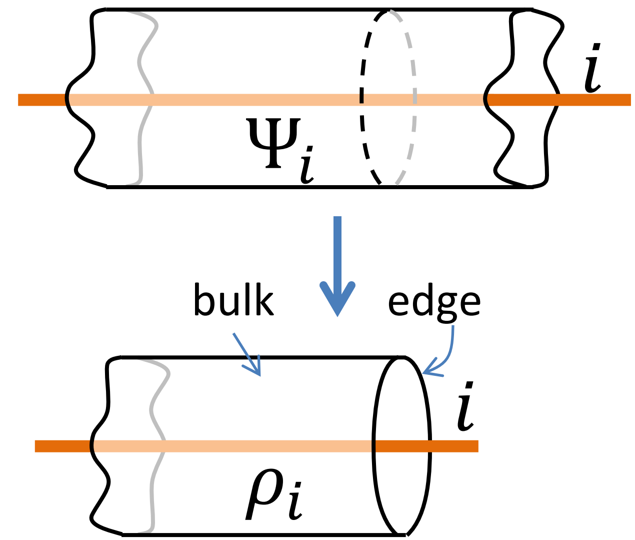
$$d_1 = 1, \quad \Longrightarrow \quad d_2 = 1.005 \approx 1,\ D = 1.413 \approx \sqrt{2}\ (0.1\%),$$

Numerics!

Haldane model
(hardcore bosons)

ENTANGLEMENT SPECTRUM (II)





$$H \nearrow \Psi_1 \rightarrow \rho_1 \rightarrow \{p_{1,\alpha}; k_{1,\alpha}\}$$

$$H \searrow \Psi_2 \rightarrow \rho_2 \rightarrow \{p_{2,\alpha}; k_{2,\alpha}\}$$

'ground states'
infinite cylinder

density matrices
semi-infinite cylinder

spectra

entanglement energies

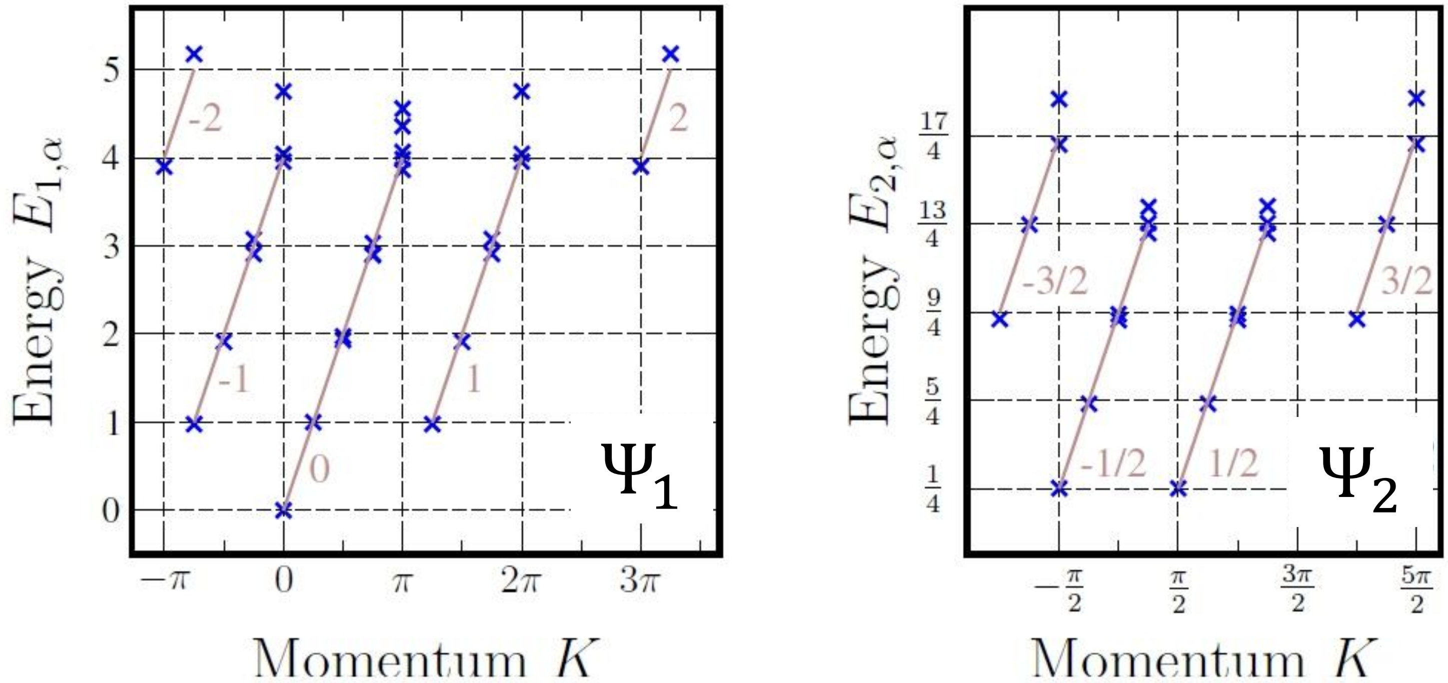
$$\rho_i |p_{i,\alpha}; k_{i,\alpha}\rangle = p_{i,\alpha} |p_{i,\alpha}; k_{i,\alpha}\rangle$$

$$E_i \stackrel{\text{def}}{=} -\log(p_{i,\alpha})$$

$$T_{y1} |p_{i,\alpha}; k_{i,\alpha}\rangle = e^{-i\frac{2\pi}{L_y} k_{i,\alpha}} |p_{i,\alpha}; k_{i,\alpha}\rangle$$

momentum in y-direction

H. Li, F. D. M. Haldane, PRL 2008
X.-L. Qi, H. Katsura, A. W. W. Ludwig, PRL 2012

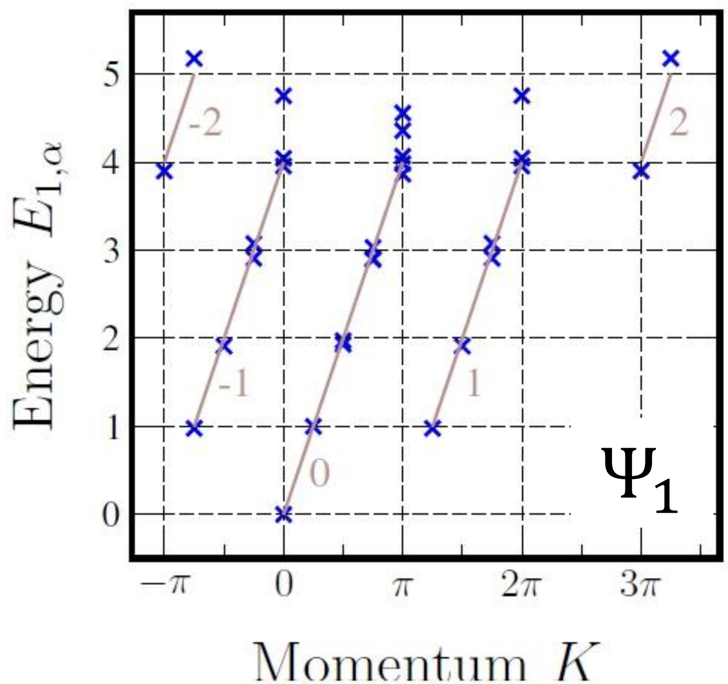


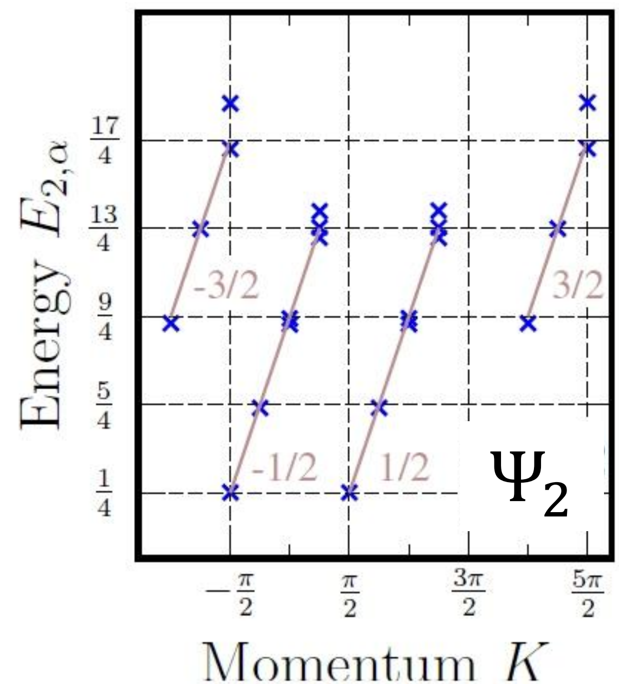
- Spectrum organized as multiplets of emergent SU(2) [lattice model is only U(1) symmetric]

Ψ_1 $\quad m_z = \cdots -2, -1, 0, 1, 2 \cdots$ $\qquad$ Ψ_2 $\quad m_z = \cdots \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}, \cdots$

integer irreps $\quad s = 0,1,2,\cdots$ $\qquad$ integer irreps $\quad s = 0,1,2,\cdots$

- Degeneracy pattern: $\{1,1,2,3,5,\cdots\}$ $\qquad$ Xiao-Gang: “bosonic Gaussian theory”

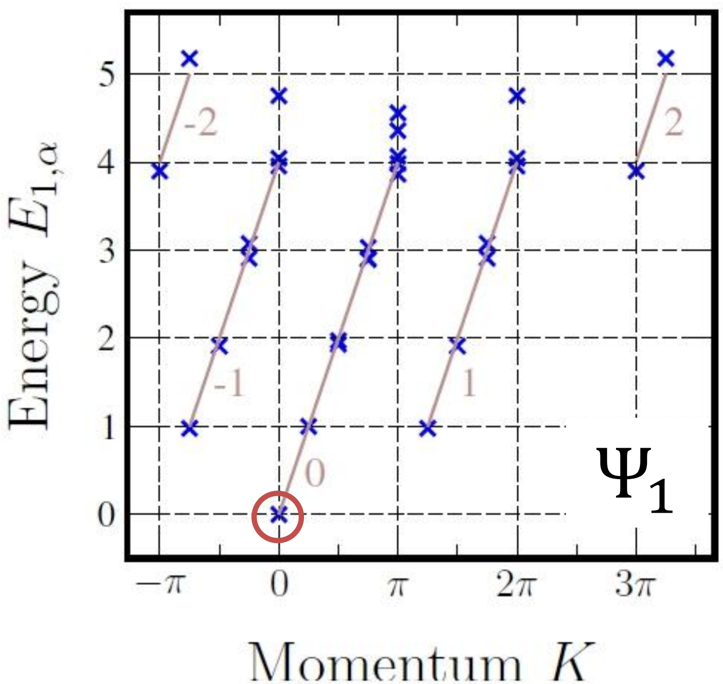


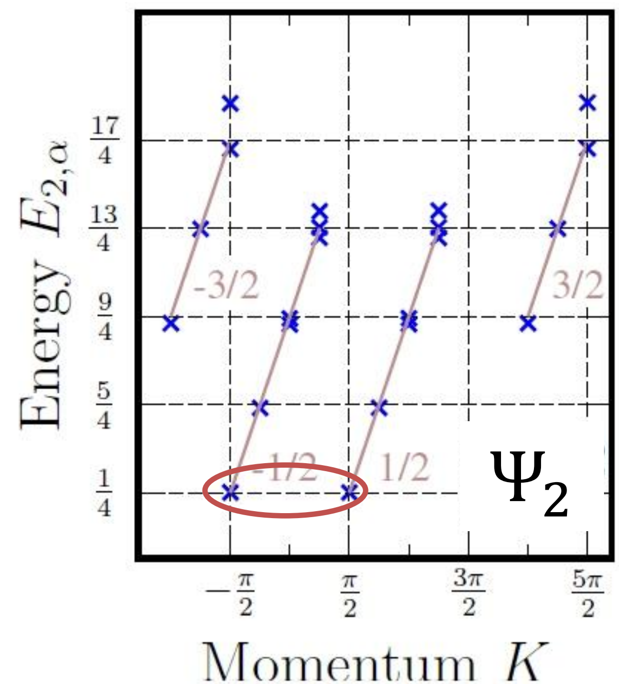


L_0	m: −2	−1	0	1	2	$su(2)$ decomposition
0			1			(0)
1		1	1	1		(2)
2		1	2	1		(2)+(0)
3		2	3	2		2(2)+(0)
4	1	3	5	3	1	(4)+2(2)+2(0)
5	1	5	7	5	1	(4)+4(2)+2(0)
6	2	7	11	7	2	2(4)+5(2)+4(0)

L_0	m: −2	−1	0	1	2	3	$su(2)$ decomposition
$\frac{1}{4}$			1	1			(1)
$\frac{5}{4}$			1	1			(1)
$\frac{9}{4}$		1	2	2	1		(3)+(1)
$\frac{13}{4}$		1	3	3	1		(3)+2(1)
$\frac{17}{4}$		2	5	5	2		2(3)+3(1)
$\frac{21}{4}$		3	7	7	3		3(3)+4(1)
$\frac{25}{4}$	1	5	11	11	5	1	(5)+4(3)+6(1)

P. Di Francesco, P. Mathieu, D. Senechal, *Conformal Field Theory*, 1997





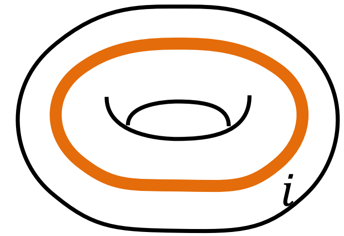
chiral $SU(2)_1$ Wess-Zumino-Witten CFT

Ψ_i primary field + tower of (Virasoro and Kac-Moody) descendants

Ψ_1 identity I, $SU(2)$ singlet $\longrightarrow$ $\Psi_{\mathbb{I}}$ identity

Ψ_2 chiral vertex operator $e^{i\varphi/\sqrt{2}}$, $SU(2)$ doublet $\longrightarrow$ $\Psi_{\mathbb{S}}$ semion

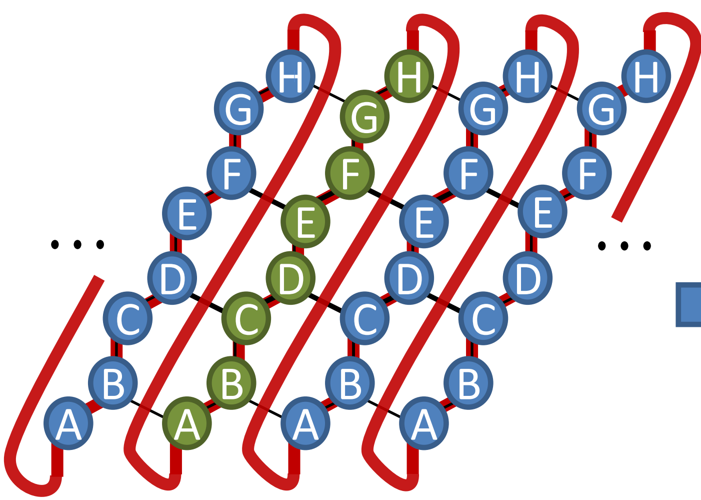
infinite cylinder

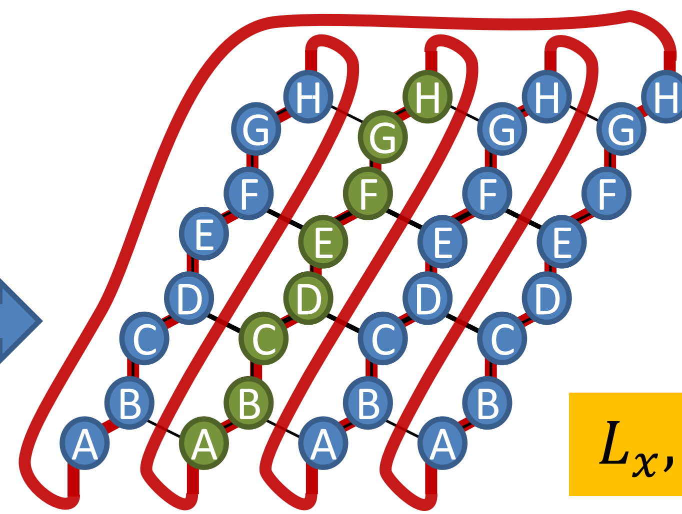


finite torus

complete set of 'ground states' $\Psi_{\mathbb{I}} \longrightarrow \Psi_{\mathbb{I}}^{tor}$ complete basis of quasi-degenerate ground subspace

$\Psi_{\mathbb{S}} \longrightarrow \Psi_{\mathbb{S}}^{tor}$



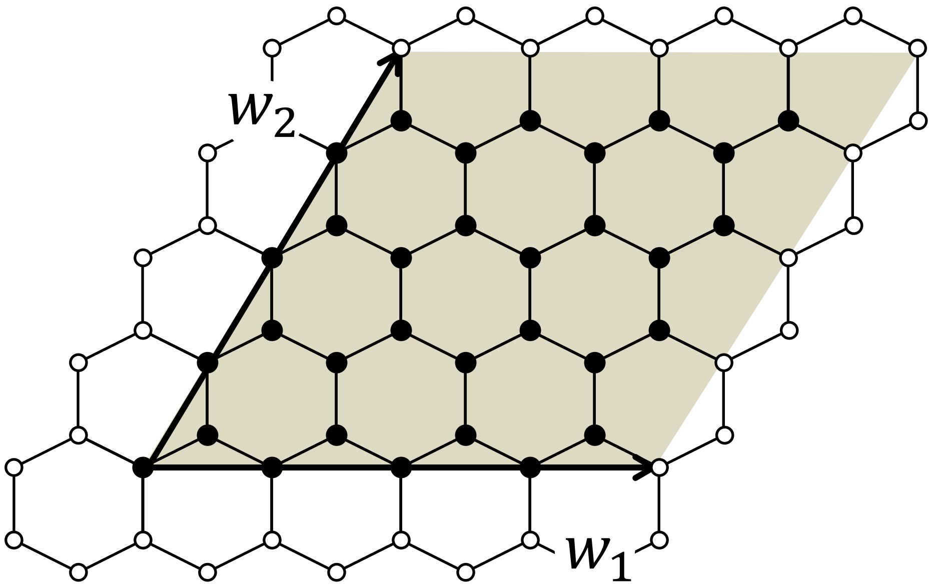


$L_x, L_y \gg \xi$

$(L_x = \infty, L_y = 4)$

$(L_x = 4, L_y = 4)$

Y. Zhang, T. Grover, A. Turner, M. Oshikawa, A. Vishwanath, PRB 2012



- torus: two vectors w_1, w_2

- modular transformations $SL(2,\mathbb{Z})$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \longrightarrow \begin{bmatrix} w_1{}' \\ w_2{}' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

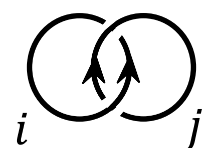
$$a, \mathrm{b}, \mathrm{c}, \mathrm{d} \in \mathbb{Z}; \quad ad - bc = 1$$

- generators

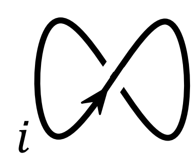
$$\mathscr{s} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \mathscr{u} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- ground space of H is a representation of the modular group

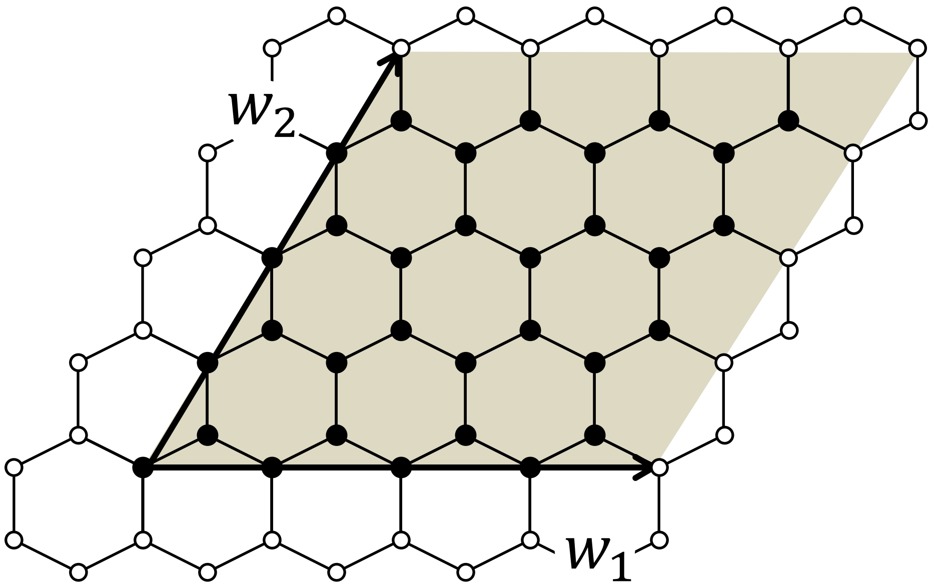
$\mathscr{s} \longrightarrow \mathrm{S}$ topological S matrix $S_{ij} = \dfrac{1}{D}$



$\mathscr{u} \longrightarrow \mathrm{U}$ topological U matrix $U_{ii} = \dfrac{1}{d_i}$



Y. Zhang, T. Grover, A. Turner, M. Oshikawa, A. Vishwanath, PRB 2012



- ${}^{\pi}/_{3}$ rotation $R_{\pi/3}$ is a symmetry of H on torus
- it corresponds to US^{-1}
- matrix of overlaps

$$V_{ij} = \langle \Psi_i^{tor} | R_{\pi/3} | \Psi_j^{tor} \rangle$$

$$V = DUS^{-1}D^{\dagger}$$

$$S = \begin{bmatrix} S_{\mathbb{I}\mathbb{I}} & S_{\mathbb{I}\mathbb{s}} \\ S_{\mathbb{s}\mathbb{I}} & S_{\mathbb{s}\mathbb{s}} \end{bmatrix} \qquad U = e^{-i\frac{2\pi}{24}c} \begin{bmatrix} 1 & 0 \\ 0 & \theta_{\mathbb{s}} \end{bmatrix} \qquad D = \begin{bmatrix} e^{i\phi_{\mathbb{I}}} & 0 \\ 0 & e^{i\phi_{\mathbb{s}}} \end{bmatrix}$$

$e^{i\phi_j}$ freedom in defining Ψ_j^{tor}

$S_{\mathbb{I}i}, S_{i\mathbb{I}} > 0$

$L_x = L_y = 6$

Numerics!

$6 \times 6 \times 2 = 72$ sites

$$V = e^{-i\frac{2\pi}{24}c} \begin{bmatrix} S_{\mathbb{I}\mathbb{I}} & S_{\mathbb{I}\mathbb{s}} e^{i(\phi_{\mathbb{s}} - \phi_{\mathbb{I}})} \\ S_{\mathbb{s}\mathbb{I}} e^{i(\phi_{\mathbb{I}} - \phi_{\mathbb{s}})} & \theta_{\mathbb{s}} (S_{\mathbb{s}\mathbb{s}})^* \end{bmatrix} = \begin{bmatrix} 0.685 + 0.181i & -0.229 + 0.669i \\ -0.693 - 0.138i & -0.183 + 0.681i \end{bmatrix}$$

$$S = \begin{bmatrix} S_{\mathbb{I}\mathbb{I}} & S_{\mathbb{I}\mathbb{s}} \\ S_{\mathbb{s}\mathbb{I}} & S_{\mathbb{s}\mathbb{s}} \end{bmatrix} \qquad U = e^{-i\frac{2\pi}{24}c} \begin{bmatrix} 1 & 0 \\ 0 & \theta_{\mathbb{s}} \end{bmatrix} \qquad D = \begin{bmatrix} e^{i\phi_{\mathbb{I}}} & 0 \\ 0 & e^{i\phi_{\mathbb{s}}} \end{bmatrix}$$

$$S_{\mathbb{I}i}, S_{i\mathbb{I}} > 0$$

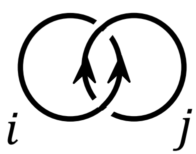
Numerics!

$$V = e^{-i\frac{2\pi}{24}c} \begin{bmatrix} S_{\mathbb{I}\mathbb{I}} & S_{\mathbb{I}\mathbb{s}} e^{i(\phi_{\mathbb{s}} - \phi_{\mathbb{I}})} \\ S_{\mathbb{s}\mathbb{I}} e^{i(\phi_{\mathbb{I}} - \phi_{\mathbb{s}})} & \theta_{\mathbb{s}} (S_{\mathbb{s}\mathbb{s}})^* \end{bmatrix} = \begin{bmatrix} 0.685 + 0.181i & -0.229 + 0.669i \\ -0.693 - 0.138i & -0.183 + 0.681i \end{bmatrix}$$

$$\Downarrow$$

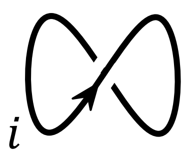
topological S matrix

$$S_{ij} = \frac{1}{D}$$



topological U matrix

$$U_{ii} = \frac{1}{d_i}$$



$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{10^{-3}}{\sqrt{2}} \begin{bmatrix} -1.4 & 0.2 \\ -1.4 & 4 + 4i \end{bmatrix}$$

$$U = e^{-i\frac{2\pi}{24}1} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \times e^{-i\frac{2\pi}{24}0.01} \begin{bmatrix} 1 & 0 \\ 0 & e^{-i0.007} \end{bmatrix}$$

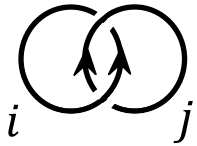
chiral semion

(Monte Carlo statistical error $< 10^{-4}$)

Numerics!

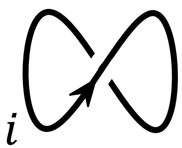
topological S matrix

$$S_{ij} = \frac{1}{D}$$



topological U matrix

$$U_{ii} = \frac{1}{d_i}$$



$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{10^{-3}}{\sqrt{2}}\begin{bmatrix} -1.4 & 0.2 \\ -1.4 & 4+4i \end{bmatrix}$$

$$U = e^{-i\frac{2\pi}{24}1}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \times e^{-i\frac{2\pi}{24}0.01}\begin{bmatrix} 1 & 0 \\ 0 & e^{-i0.007} \end{bmatrix}$$

chiral semion

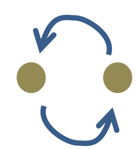
(Monte Carlo statistical error $< 10^{-4}$)

Numerics!

- from topological S matrix
  - quantum dimensions $d_{\mathbb{I}} = d_{\mathbb{s}} = 1, \quad D = \sqrt{2}$
  - $\mathbb{Z}_2$ fusion rules

$\mathbb{I} \times \mathbb{I} = \mathbb{I}$	$\mathbb{I} \times \mathbb{s} = \mathbb{s}$
$\mathbb{s} \times \mathbb{I} = \mathbb{s}$	$\mathbb{s} \times \mathbb{s} = \mathbb{I}$

- from topological U matrix
  - central charge $c = 1$
  - topological spin $\Theta_{\mathbb{s}} = i$ semion!

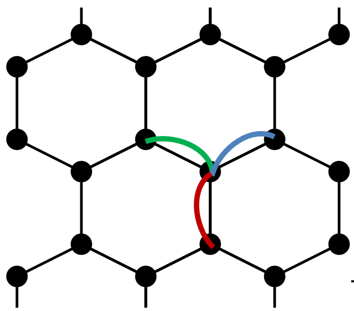


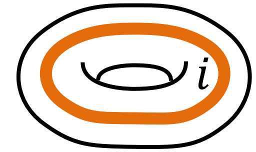
self statistics

$$|\Psi\rangle \to i|\Psi\rangle$$

Kitaev Honeycomb
(non-Abelian phase with magnetic field)

A. Kitaev , Annals of Physics 2006





h=0.01

$+h\sum_r (\sigma^X{}_r + \sigma^Y{}_r + \sigma^z{}_r)$

$$H = \sum_{\langle rr' \rangle_x} \sigma^x{}_r \sigma^x{}_{r'} + \sum_{\langle rr' \rangle_y} \sigma^y{}_r \sigma^y{}_{r'} + \sum_{\langle rr' \rangle_z} \sigma^z{}_r \sigma^z{}_{r'}$$

H Numerics $\rightarrow$ Ψ_1, Ψ_2, Ψ_3 $\rightarrow$ $\Psi_1{}^{tor}$, $\Psi_2{}^{tor}$, $\Psi_3{}^{tor}$

'ground states'
infinite cylinder

'ground states'
finite torus

Numerics!

$L_x = L_y = 4$

$$S = \frac{1}{2}\begin{bmatrix} 1.02 & 1.40 & 1.01 \\ 1.41 & 0.03 & -1.41 \\ 1.04 & -1.36 & 1.04 \end{bmatrix}$$

$$\approx \frac{1}{2}\begin{bmatrix} 1.00 & 1.41 & 1.00 \\ 1.41 & 0.00 & -1.41 \\ 1.00 & -1.41 & 1.00 \end{bmatrix}$$

$\sqrt{2} \approx 1.41$

5% !

Ising anyon model

$$S = \frac{1}{2}\begin{array}{c} \begin{array}{ccc} \mathbb{I} & \sigma & \varepsilon \end{array} \\ \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \end{array}$$

- quantum dimensions

$$d_{\mathbb{I}} = 1 \quad d_\sigma = \sqrt{2} \quad d_\varepsilon = 1; \quad D = 2$$

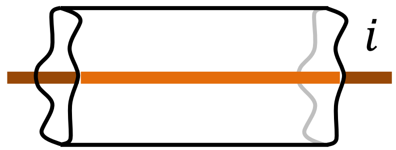
- fusion rules

$$\sigma \times \varepsilon = \sigma \quad \sigma \times \sigma = \mathbb{I} + \varepsilon \quad \varepsilon \times \varepsilon = \mathbb{I}$$

OUTLINE

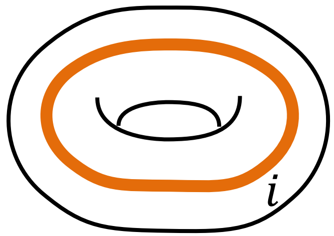
1) GROUND STATES

Infinite cylinder



- edge spectrum
  - quantum dimensions
  - chiral CFT

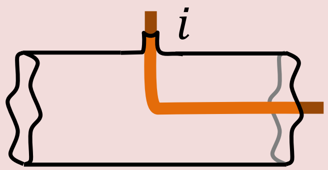
Finite torus



- S matrix
  - mutual statistics
  - quantum dimensions
  - fusion rules
- U matrix
  - central charge
  - topological spins

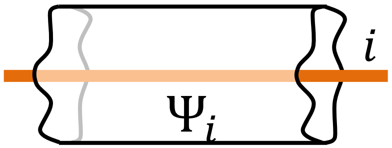
2) QUASIPARTICLE EXCITATIONS

Infinite cylinder

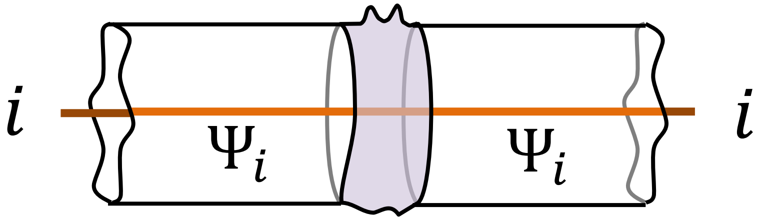


- integer excitations
- fractionalized excitations

- ground states:

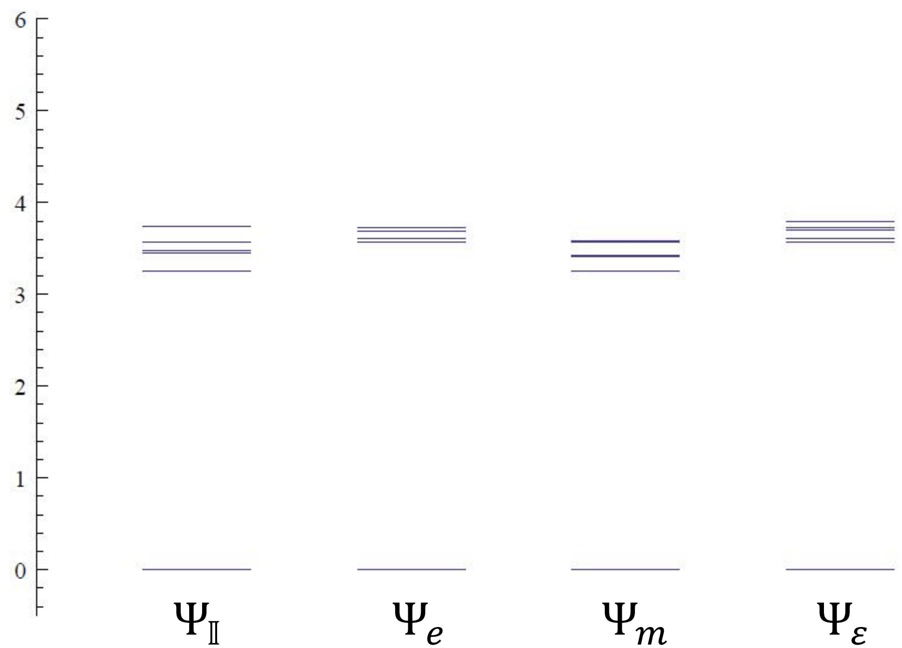


- integer excitations

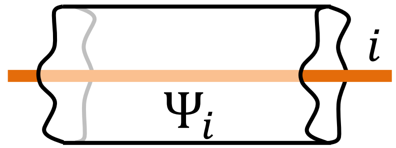


Example: toric code with magnetic field $0.1\sigma_z + 0.05\sigma_x$

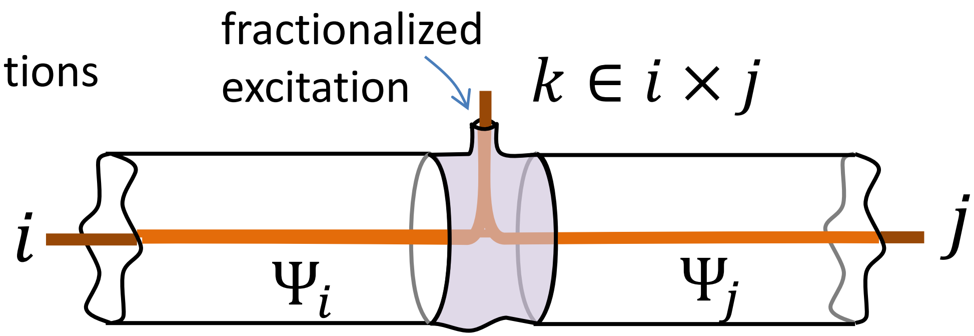
$\mathrm{i} = \mathbb{I}, e, m, \varepsilon$



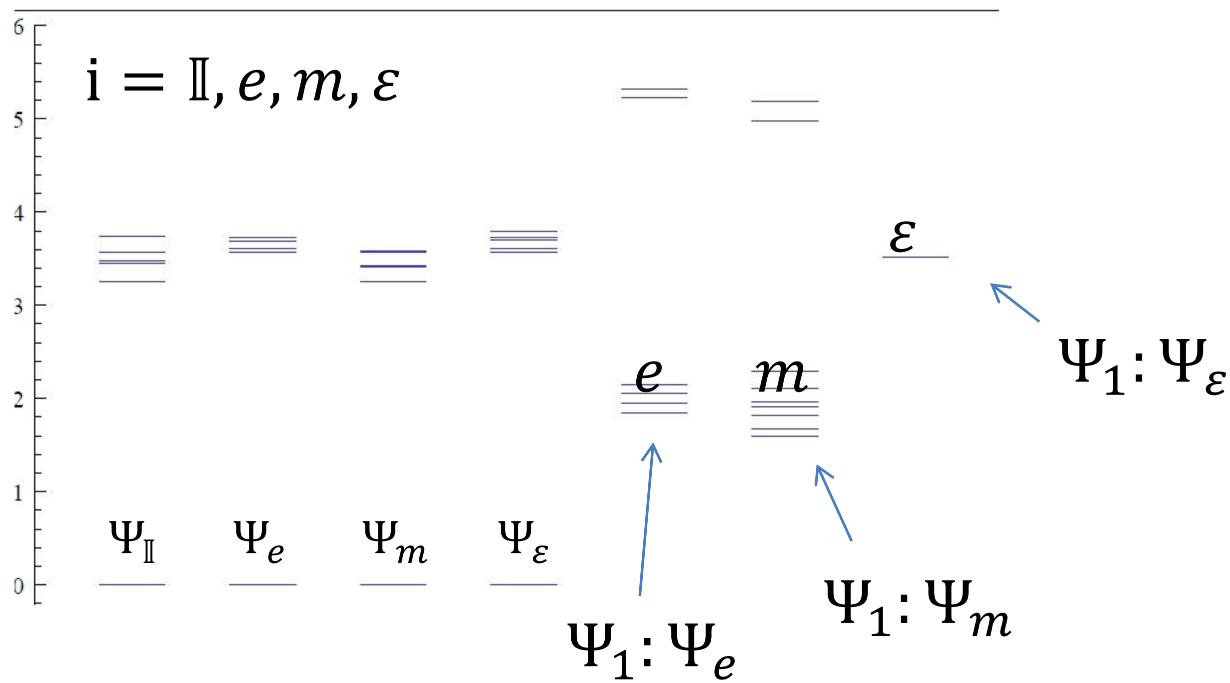
- ground states:

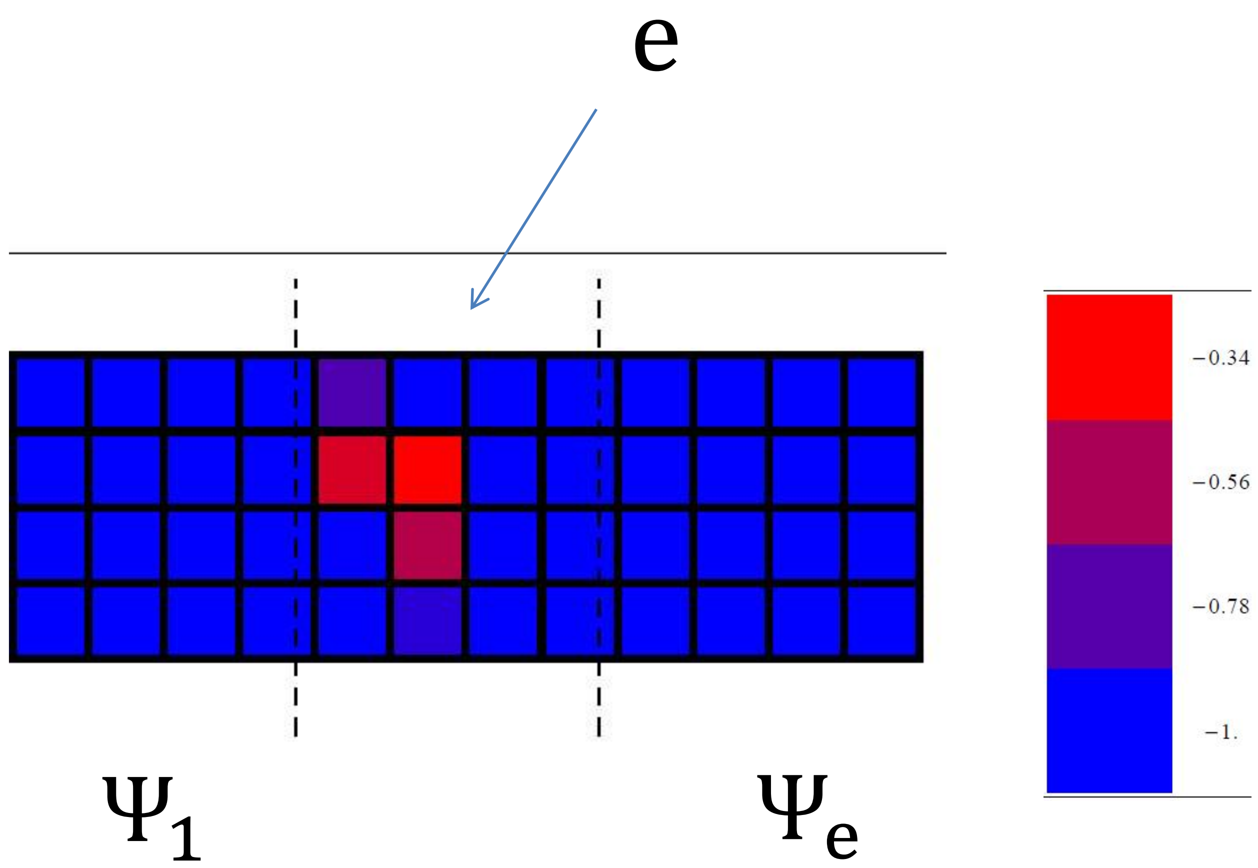


- fractionalized excitations



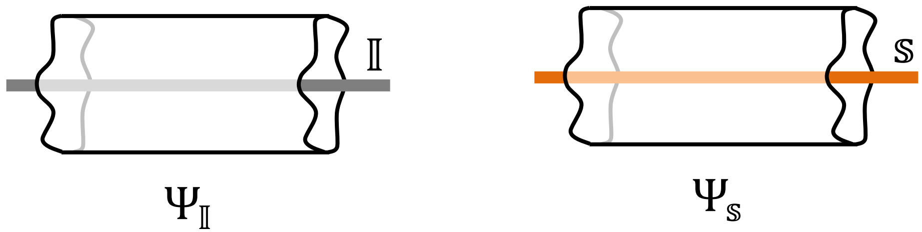
Example: toric code with magnetic field $0.1\sigma_z + 0.05\sigma_x$



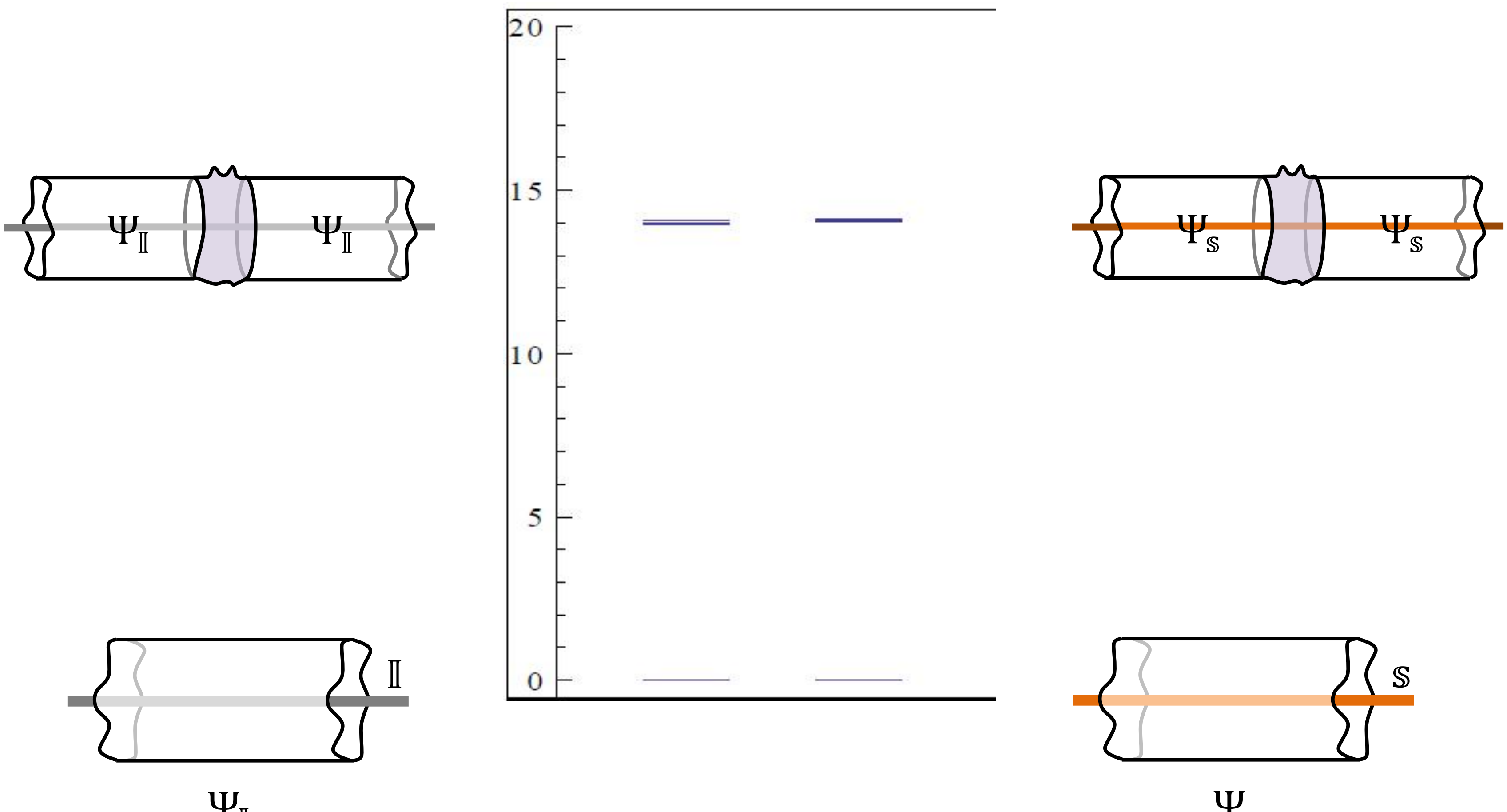

e
−0.34
−0.56
−0.78
−1.
Ψ1
Ψe

Haldane model (hard-core bosons)

- ground states:

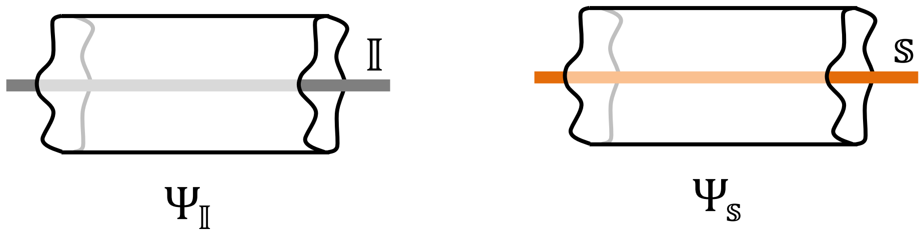


- integer excitations

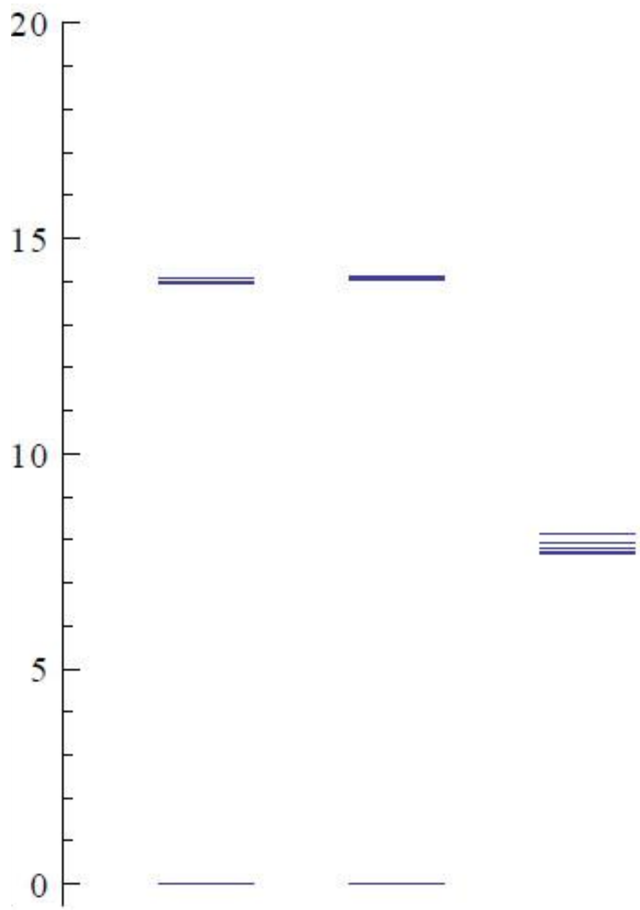


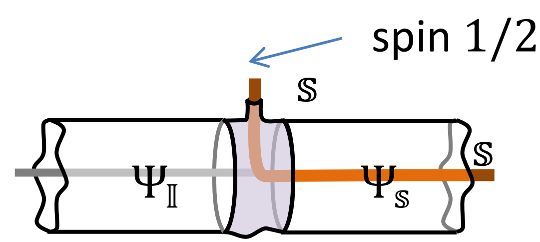
Haldane model (hard-core bosons)

- ground states:



- fractionalized excitations



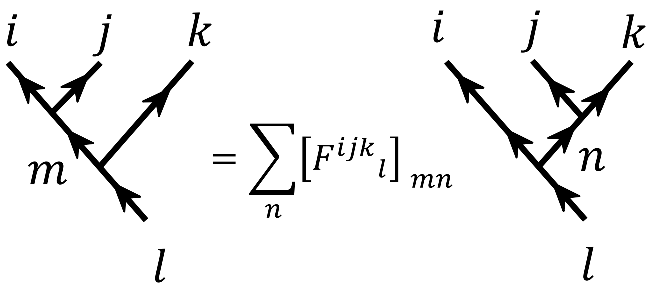


microscopic Hamiltonian

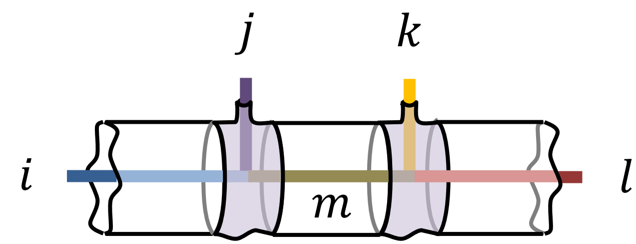
H

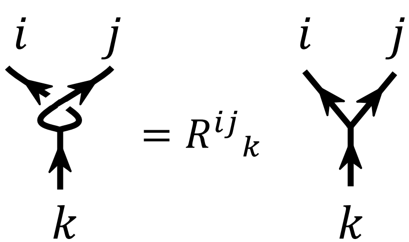
on infinite cylinder

ultimate goal:
complete characterization

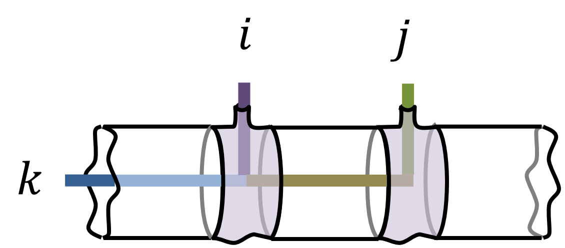


- F – symbols





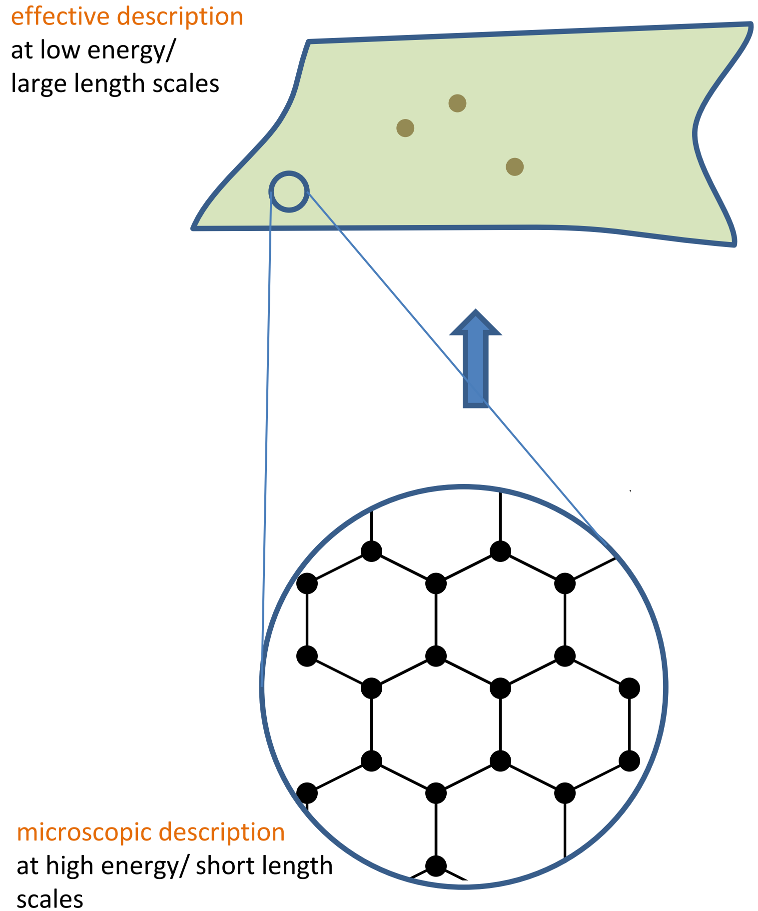
- R – symbols

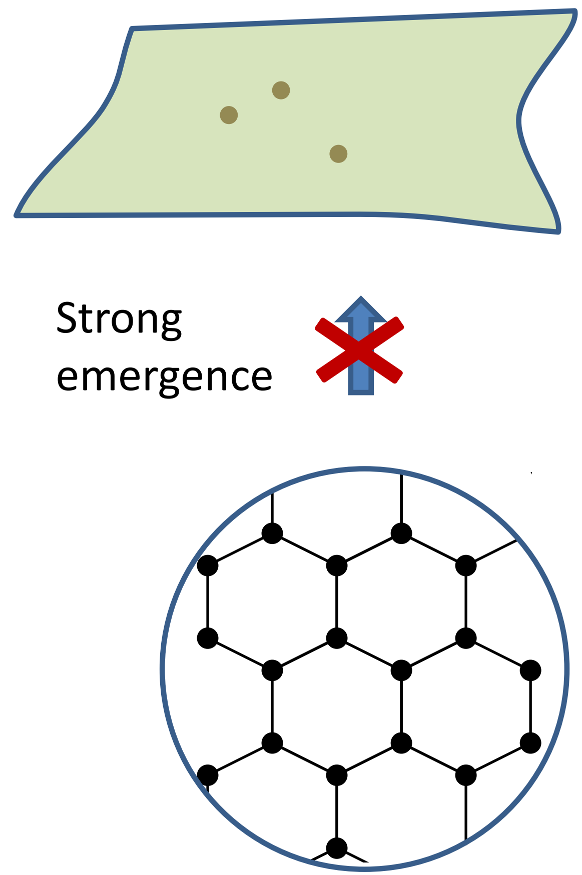


Conclusions

Reductionism versus Emergence

(bottom up)





P. W. Anderson

R. B. Laughlin

Emergent topological order can be derived from microscopic models alone.
It is not an example of strong emergence!