Topological Entanglement Entropy in 3D Walker-Wang models

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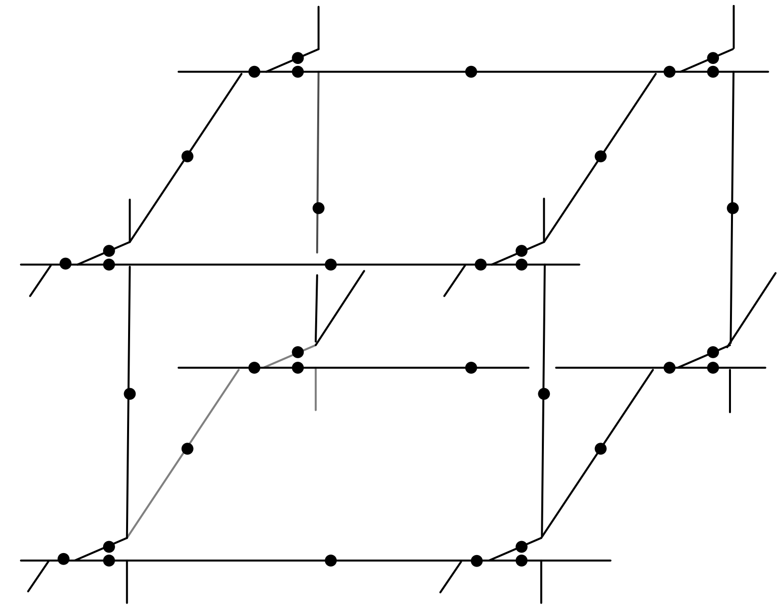
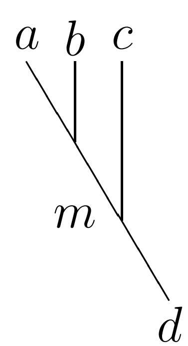
- New physics in 3D models
  - Trivial order and confined particles in bulk
  - Topological order and deconfined anyons on boundary
- Potentially new types of particle excitations in bulk

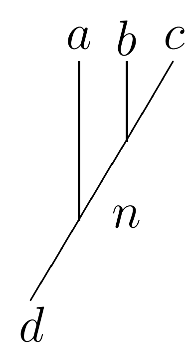
Walker-Wang Models*



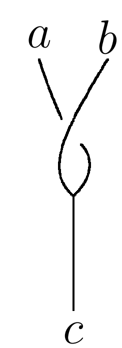
- Input
  - 3-manifold discretized on a lattice
    - stick to cubic lattice with or without boundaries
    - decorate so all vertices are 3-valent
  - Unitary Braided Fusion Category (UBFC)
    - Finite label set $L = \{a, b, c, \ldots\}$
    - Creation and annihilation structures $t_a \in \{\pm 1\} \quad \forall a \in L$
    - F-matrices



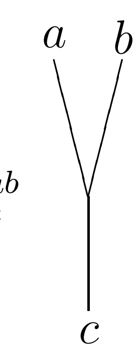
$= \sum_n F^{abc}_{d;nm}$



and R-matrices



$= R^{ab}_c$



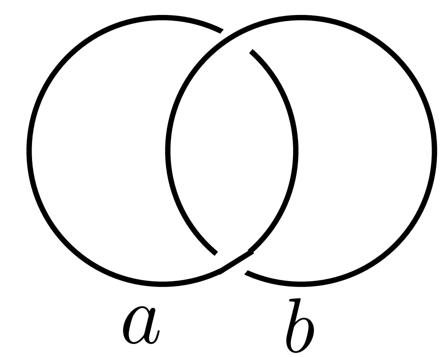
    - Assume multiplicity free models and self dual charges

*K. Walker and Z. Wang, Front. Phys. **7**, 150 (2012).

A few words on categories (no more!)

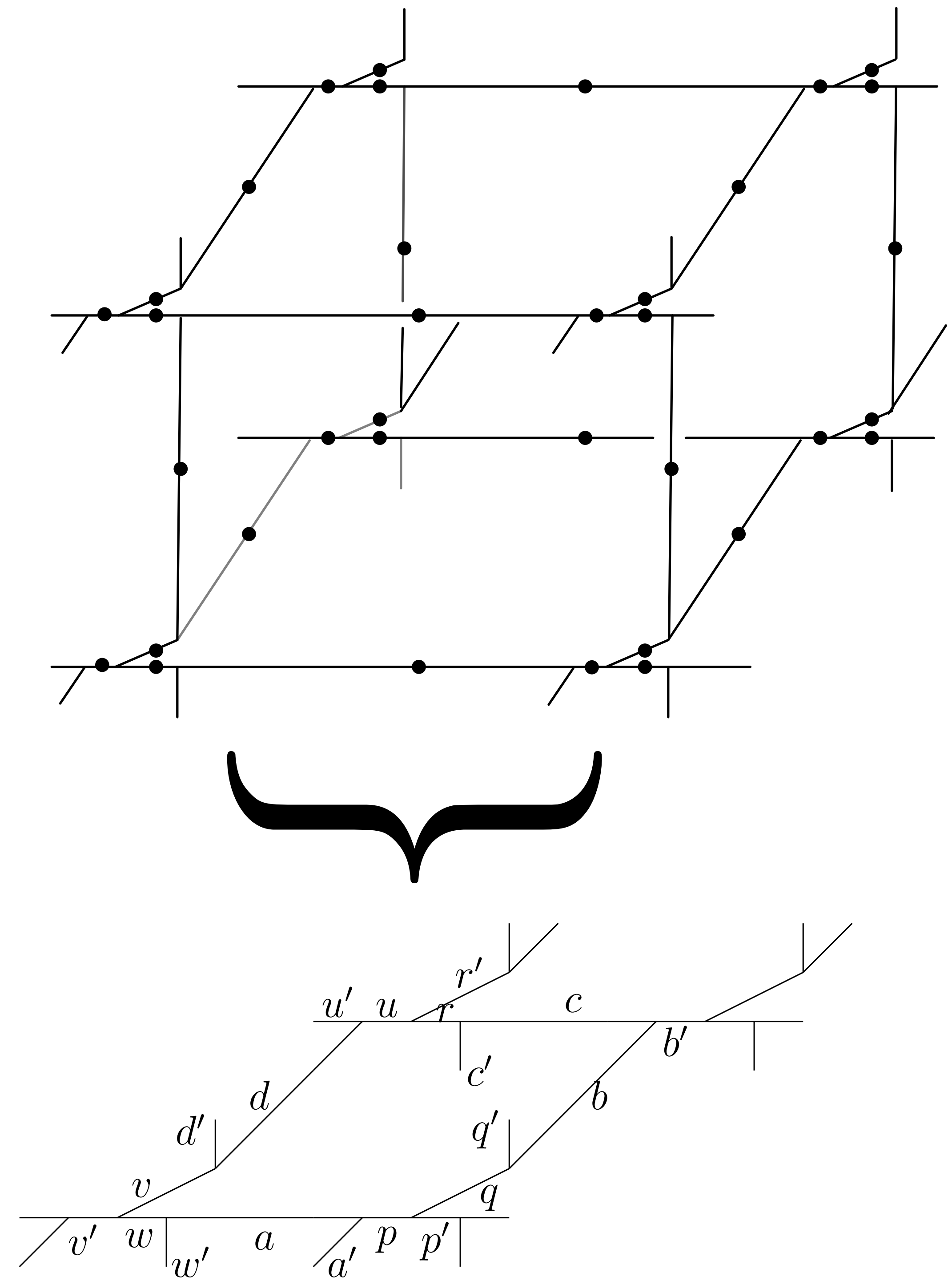
- UBFCs
  - Unitary symmetric fusion categories
    - 3D Levin-Wen models (they realize all discrete gauge theories coupled to bosons or fermions)
    - 3D Toric-code
  - Modular Tensor Categories [unitary S-matrix]
    - quantum doubles of spherical fusion categories
      - Kitaev toric code
    - quantum group categories
    - 2D Levin-Wen models

$$S_{a,b} = \frac{1}{\mathcal{D}}$$



- Unlike 3D Levin-Wen models Walker-Wang models can describe MTCs
- Note: any MTC leads to a TQFT (converse unknown)

Walker-Wang model Hamiltonian

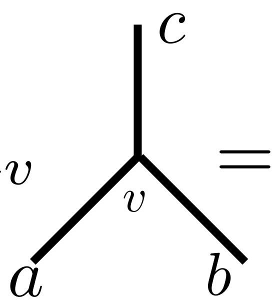


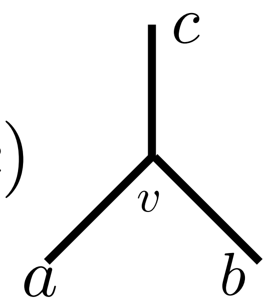
- Exactly solvable model

$$H = -\sum_v A_v - \sum_p B_p$$

$$[A_v, B_p] = [A_v, A_{v'}] = [B_p, B_{p'}] = 0$$

- Vertex operators

A_v 

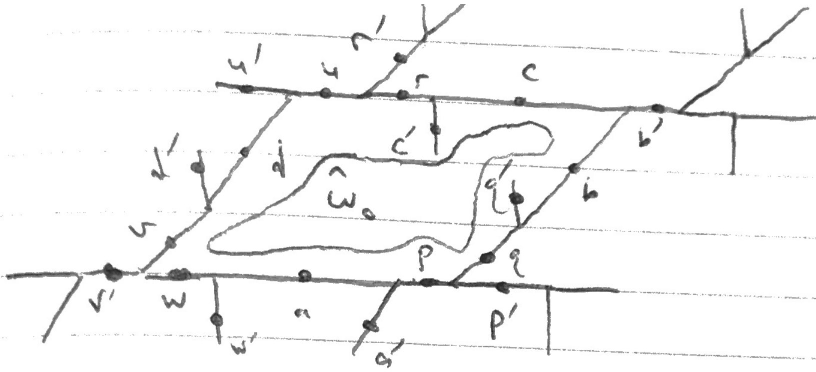
$= \delta(a \times b \to c)$ 

- Face operators

$$B_p = \frac{1}{\mathcal{D}^2} \sum_{s \in L} d_s B_p^s$$

$$[B_p^s]^{a'',b'',c'',d'',p'',q'',r'',u'',v'',w''}_{a,b,c,d,p,q,r,u,v,w} =$$

$$R_q^{q'b} \overline{R_c^{c'r}} \overline{R_{q''}^{q'b''}} R_{c''}^{c'r''} F^{a''sp}_{a';ap''} F^{p''sq}_{p';pq''} F^{q''sb}_{q';qb''} F^{b''sc}_{b';bc''} F^{c''sr}_{c';cr''} F^{r''su}_{r';ru''} F^{u''sd}_{u';ud''} F^{d''sv}_{d';dv''} F^{v''sw}_{v';vw''} F^{w''sa}_{w';wa''}$$



$$H = -\sum_v A_v - \sum_f B_f$$

$$B_f = \sum_{s \in L} \frac{d_s}{D^2} B_f^s$$

A_v is a projector
B_f is a projector

A_v: (vertex with legs a, b, c) $= \delta(a \times b \to c)$ etc

Ground states are invariant under insertion of loop projector $\hat{\omega}_0$ inside face

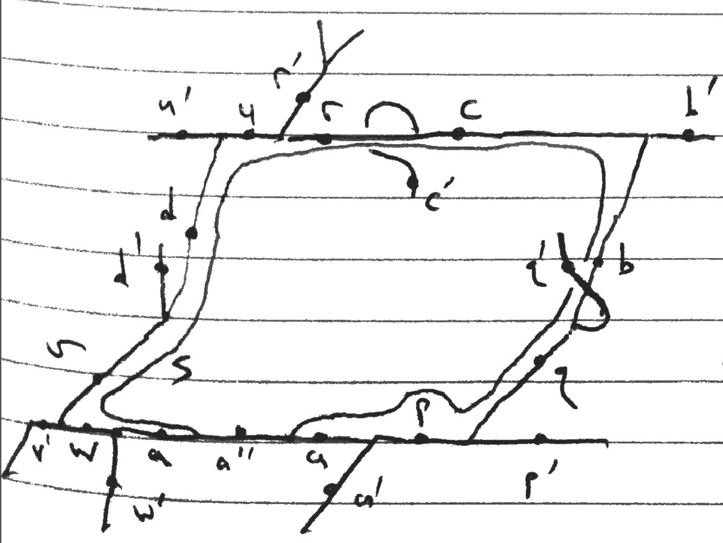
$$\hat{\omega}_0 = \sum_{s \in L} \frac{d_s}{D^2} \hat{s}$$

↑ loop of types

Of course each loop s is contractible (with amplitude d_s)

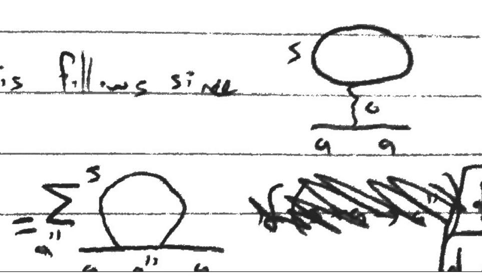
$$\sum_{s \in L} \frac{d_s^2}{D^2} = 1$$

Evaluated on a basis $|\Psi_f^s, abcdpqruvw\rangle$ we need to twist 2 vertical strands out of the way to fit the loop s inside

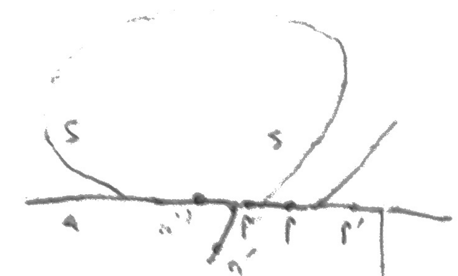


Multiplies $|\Psi_f^s, abcdpqruvw\rangle$ by $\overline{R}_q^{q'b} R_c^{c'r}$

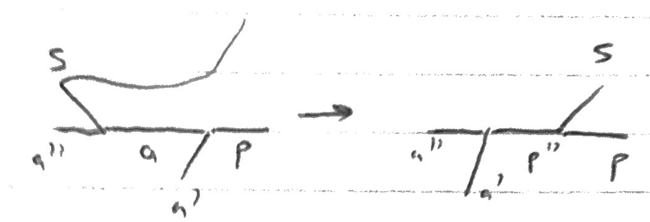
← This follows since



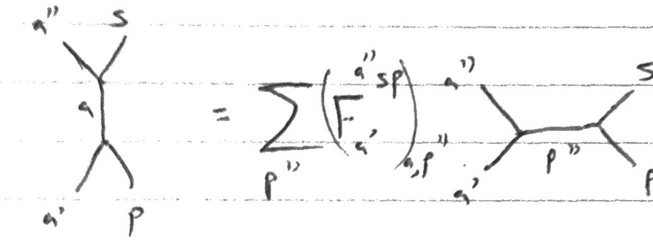
Now drag right end of strand s counterclockwise



In doing so we needed the move

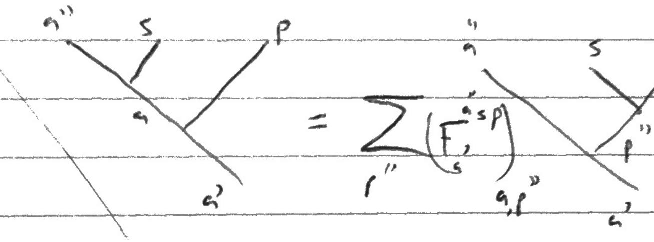


or



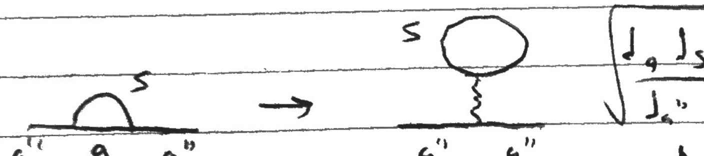
$|\Psi_f^s, abcdpqruvw\rangle$
$\to \overline{R}_q^{q'b} R_c^{c'r} \sum_{p''} (F_{a'}^{a''sp})_{a,p''}$
$|\Psi_f^s, abcdp''qruvw\rangle$

i.e looks like



Repeat this 9 more times (each time picking up a sum over an F symbol)

until end up with bubble



Now Undo both twists

Then $\langle \Psi_f^s, a''b''c''d''p''q''r''u''v''w'' | B_f^s | \Psi_f^s, abcdpqruvw\rangle$

$$= \overline{R}_q^{q'b} R_c^{c'r} \overline{R}_{q''}^{q'b''} R_{c''}^{c'r''} (F_{a'}^{a''sp})_{a p''} (F_{p'}^{p''sq})_{p q''} (F_{q'}^{q''sb})_{q b''} (F_{b'}^{b''sc})_{b c''} (F_{c'}^{c''sr})_{c r''}$$
$$(F_{r'}^{r''su})_{r u''} (F_{u'}^{u''sd})_{u d''} (F_{d'}^{d''sv})_{d v''} (F_{v'}^{v''sw})_{v w''} (F_{w'}^{w''sa})_{w a''}$$

Monday, 20 January 14

A non-modular WW model: 3D Toric code

$$H = -\sum_{v} \underbrace{\prod_{i \in N(v)} \sigma^z_i}_{B_v} - \sum_{f} \underbrace{\prod_{i \in \partial p} \sigma^x_i}_{B_f}$$

($i \in N(v)$: 3 valent; $i \in \partial p$: 10 edges)

$[B_v, B_f] = 0$

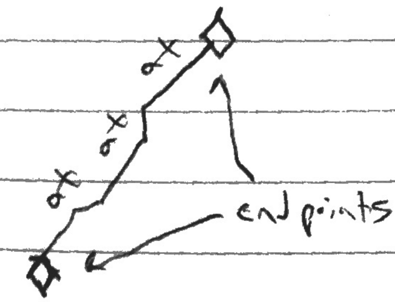
$[B_v, B_{v'}] = [B_f, B_{f'}] = 0$

Excitations:

Vertex defects $B_v = -1$

Face defects $B_f = -1$

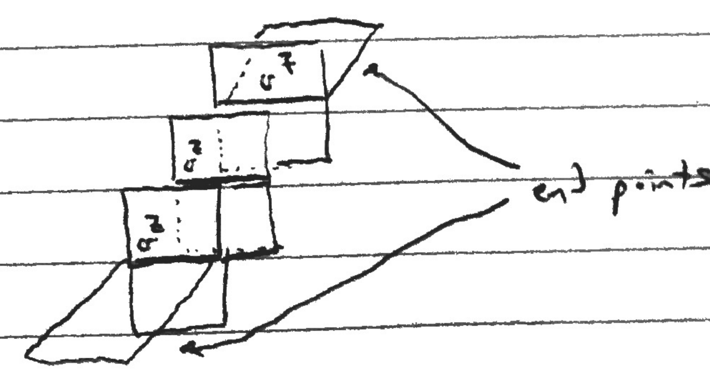
Vertex defects created by $\hat{W}_v(C_{AB}) = \prod_{i \in C_{AB}} \sigma^x_i$



$\hat{W}_v(C_{AB})$ commutes with H except at end points so defects are deconfined

Face defects created by $\hat{W}_p(S) = \prod_{i \in S} \sigma^z_i$

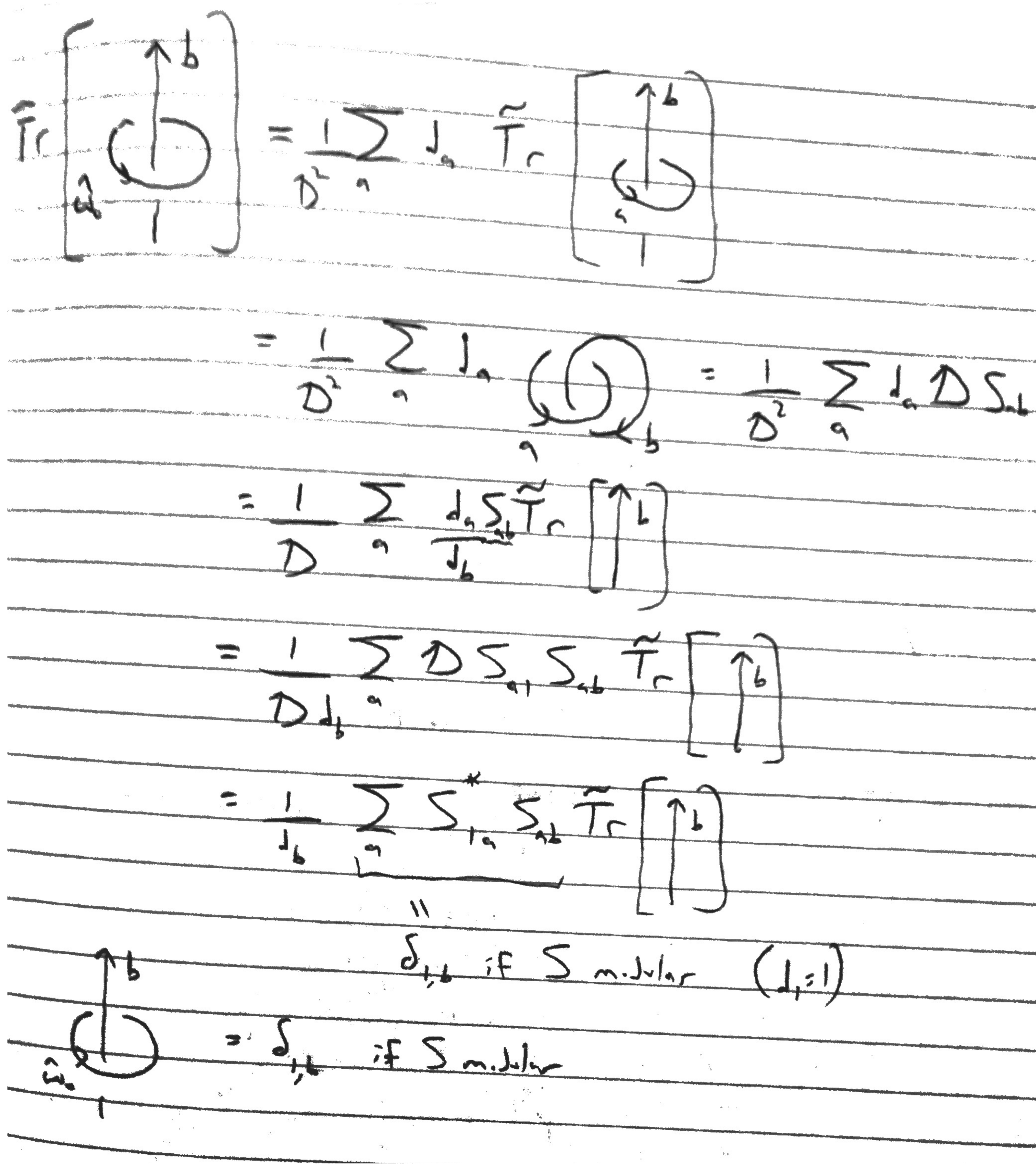
Energy cost along trajectory at all indicated faces



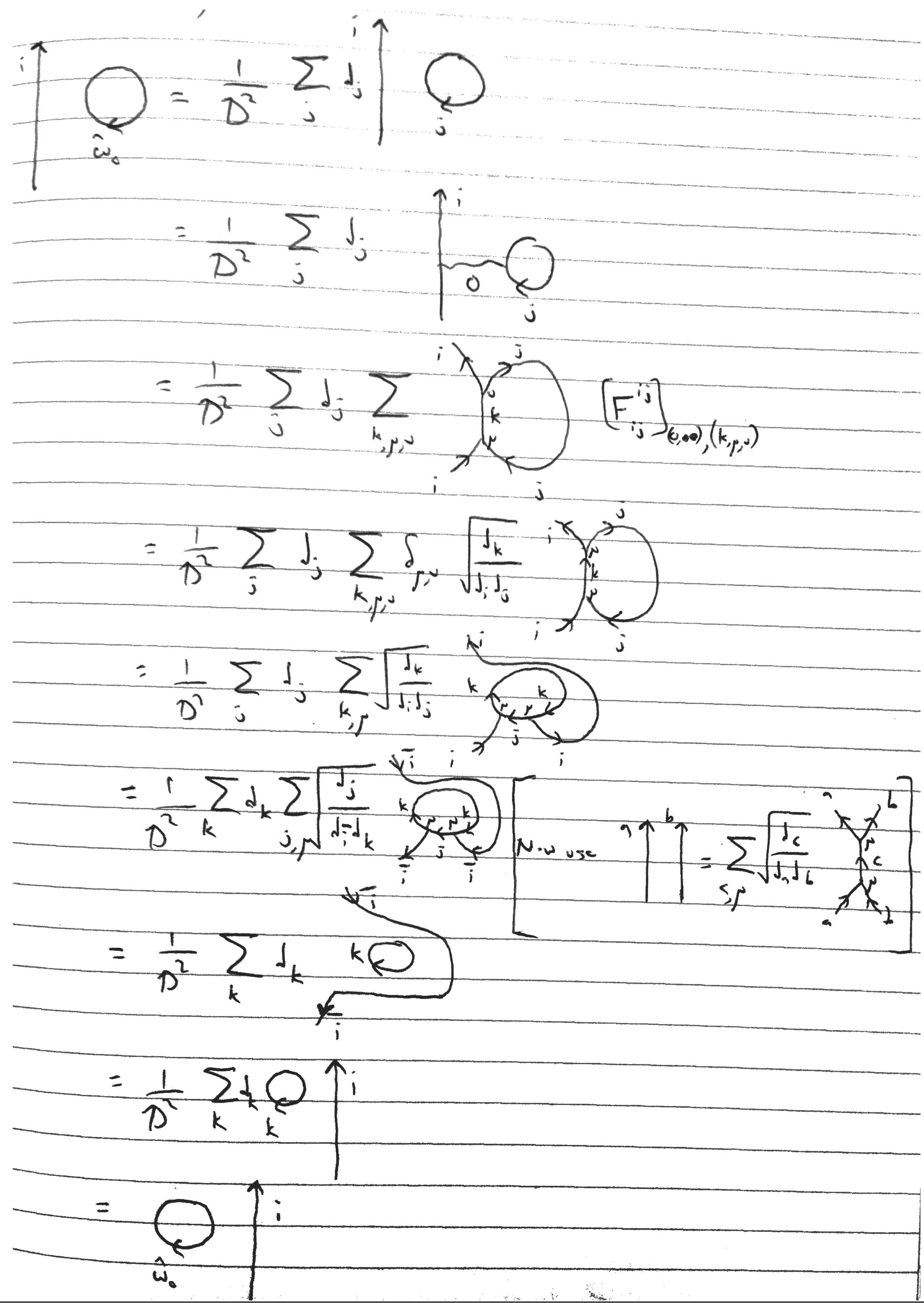
On 3D Torus 8 degenerate ground states generated by 3 noncontractible $\hat{W}_v$ operators in the 3 periodic directions

Two identities

1. Modular trap



2. Handle sliding

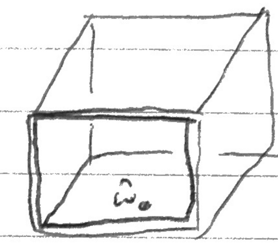


Together they imply MTCs have confined particles

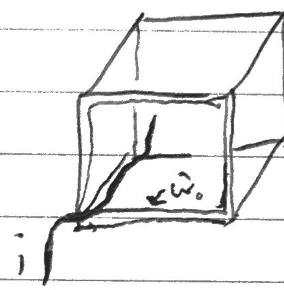
$$B_p = \frac{1}{D^2}\sum_{s\in L} d_s \underbrace{W_V^s(\partial_p)}_{\text{loop operator around boundary of plaquette } p}$$

Show handle slide and string confinement proofs

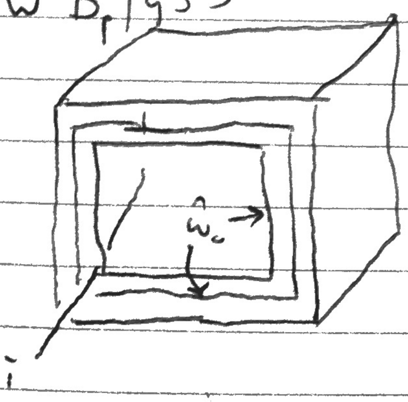
$|GS\rangle = B_p|GS\rangle$



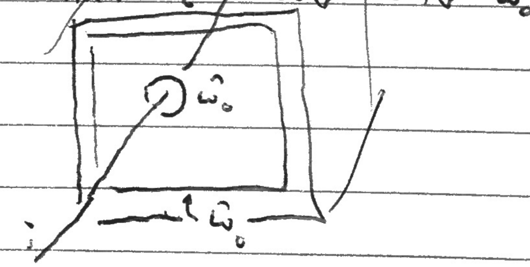
$\hat{W}^i B_p|GS\rangle$

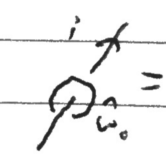


$B_p \hat{W}^i B_p|GS\rangle$



handle slide inner $\hat{\omega}_0$ over outer $\hat{\omega}_0$



If a MTC then  $= \delta_{i,0}\, i$

and $B_p \hat{W}^i B_p|GS\rangle = \delta_{i,0} \hat{W}^i B_p |GS\rangle$

So any nontrivial string operator $\hat{W}^i$ violates a plaquette operator that it pierces

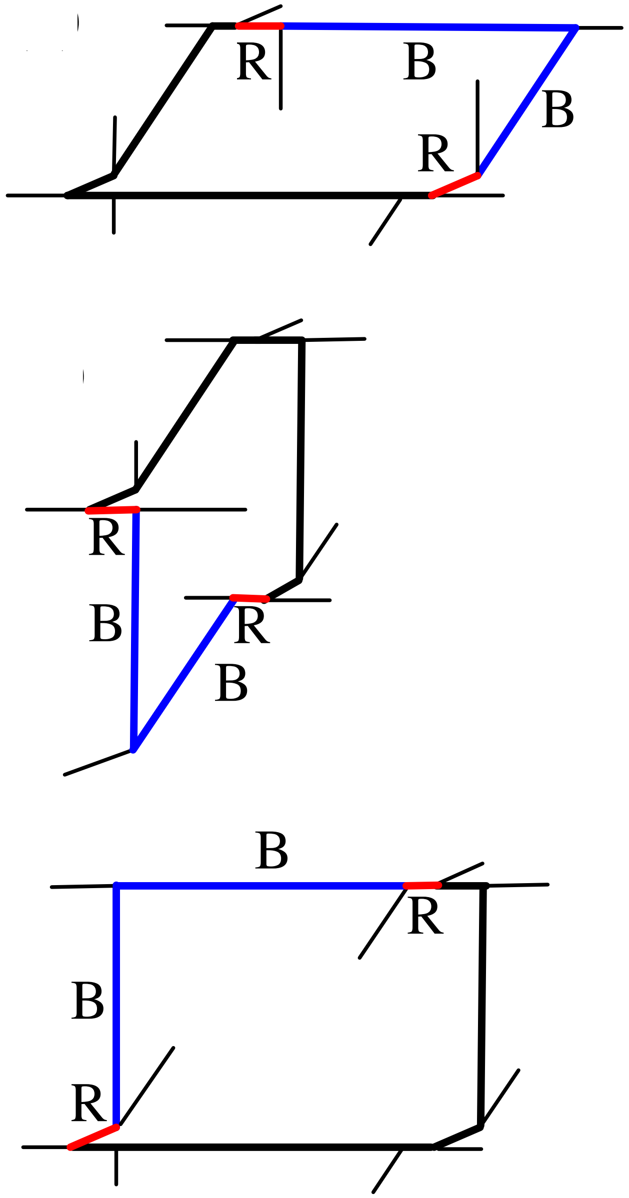
$\Rightarrow$ Confinement

Walker-Wang models from MTC: examples

- 3D-Semion model

$$H = -\sum_v \prod_{s(v)} \sigma_i^z + \sum_p (\prod_{i\in\partial p} \sigma_i^x)(\prod_{j\in s(p)} i^{n_j})\ i^{\sum_{j\,\mathrm{red}} n_j - \sum_{j\,blue} n_j}$$

  - broken time reversal symmetry
  - parity acts as complex conjugate (opposite parity projection of lattice) so Parity x Time is a good symmetry
  - Abelian semions on boundary



- 3D-Fibonacci model
  - Particles $\{1, \tau\}$
  - Fusion rule $\tau \times \tau = 1 + \tau$
  - Non-Abelian anyons on boundary

- Overview of Walker Wang models with MTC input
  - All particles confined in the bulk
    - may be deconfined on a boundary
  - Non-degenerate ground states on a system without boundaries
  - Explicitly broken time-reversal symmetry in the bulk
  - Boundary modes are gapped
    - Boundaries act like fractional topological insulators with topological properties of 2D fractional quantum Hall systems
    - This in contrast to fractional topological insulators which have protected gapless boundary modes

C.W. von Keyserlingk, F.J. Burnell, and S.H. Simon **87**, 045107 (2013).

- What about topological entanglement entropy?

Topological Entanglement Entropy

- Constant correction to area law behaviour of entropy of a subsystem A

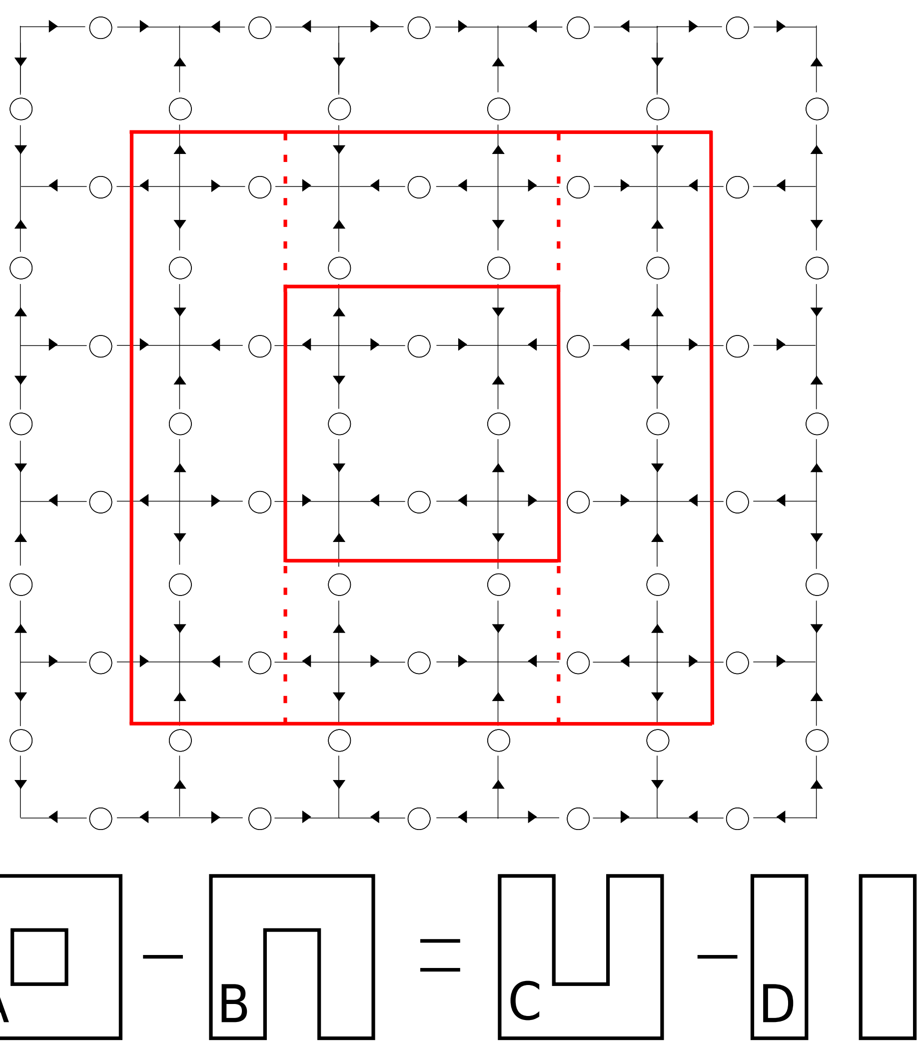
$$S(\rho_A) \equiv -\mathrm{tr}[\rho_A \log_2(\rho_A)] = \alpha|\partial A| - \boxed{\gamma} + \varepsilon$$

ε ← Goes to zero for large boundary

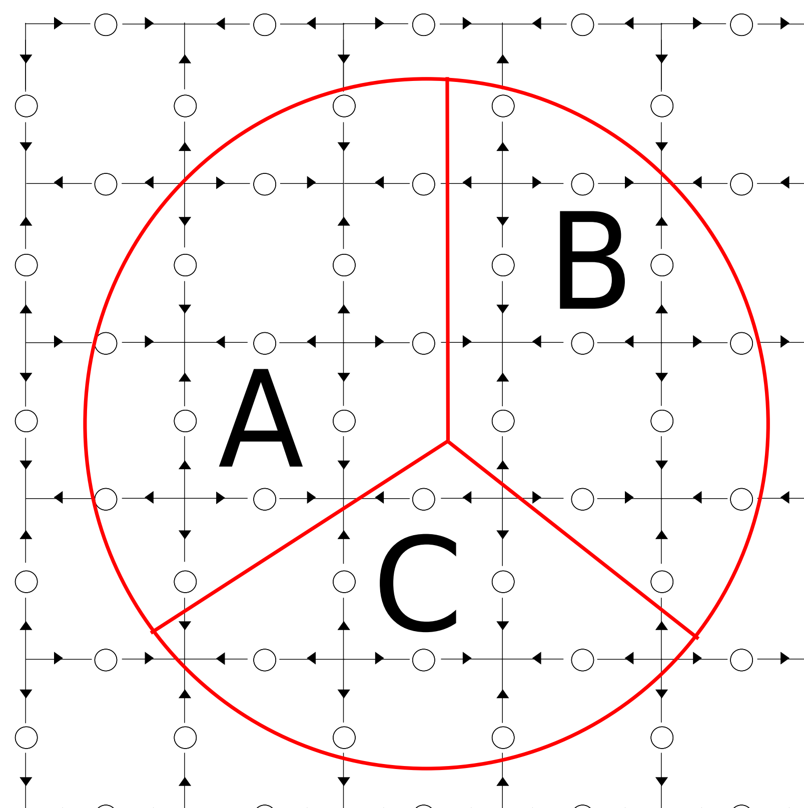
γ ← S_{topo}

- Two standard methods to compute in 2D

Levin&Wen

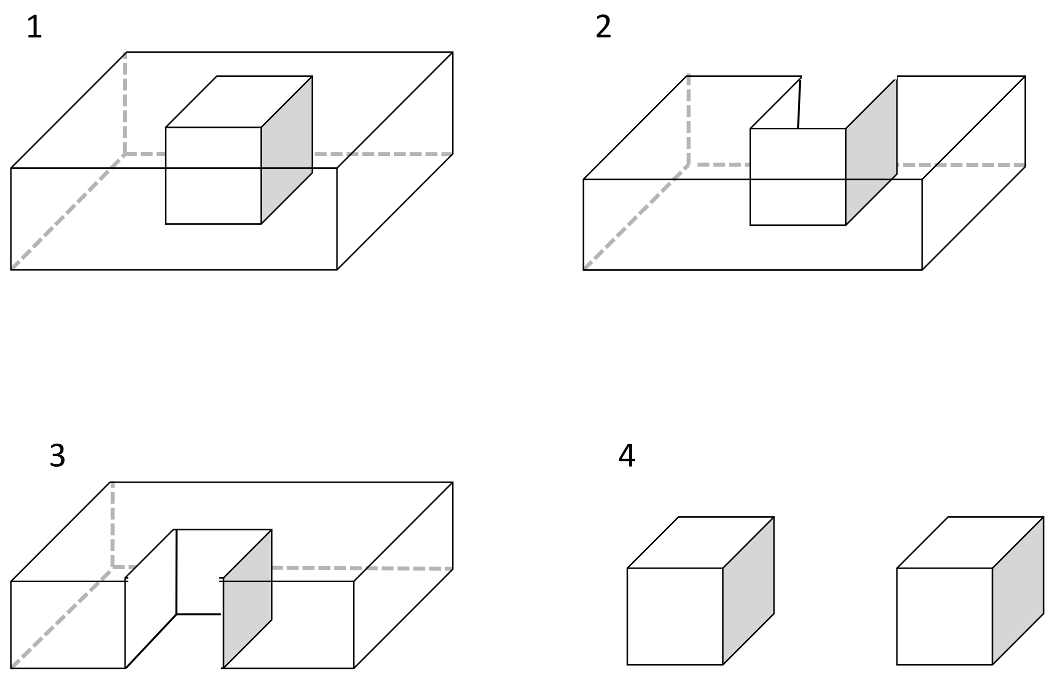


Kitaev&Preskill



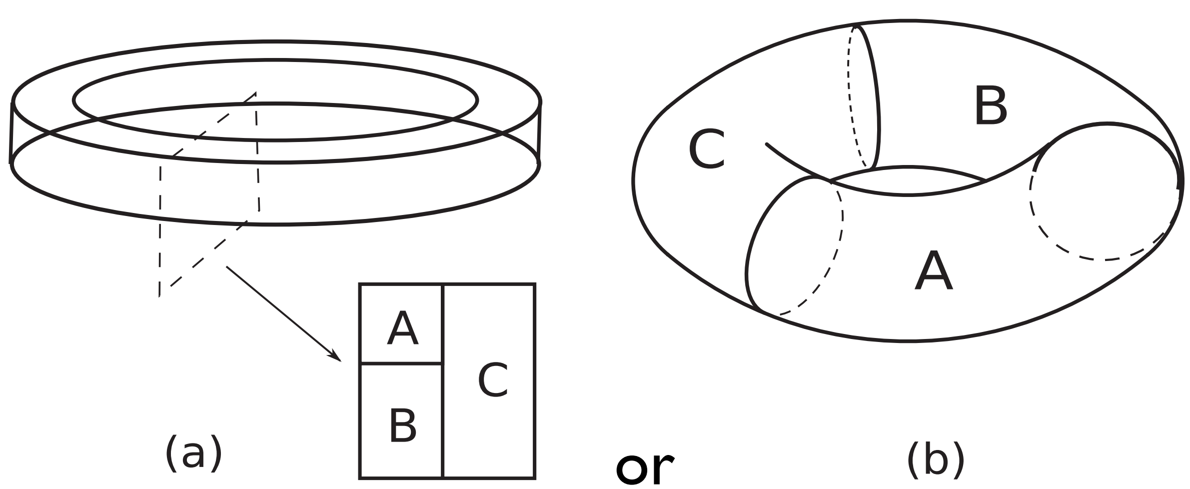
$$S_{\mathrm{topo}}^{\mathrm{KP}} \equiv -(S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC})$$

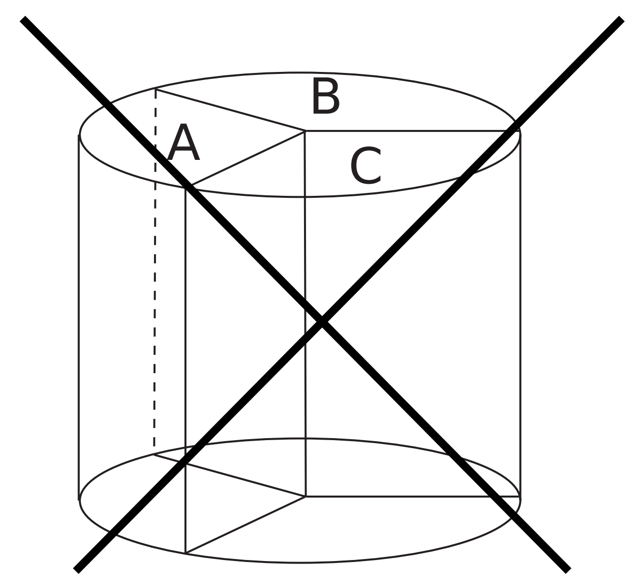
- Extrusion to 3D
  - Levin&Wen type decomposition



C.W. von Keyserlingk, F.J. Burnell, and S.H. Simon **87**, 045107 (2013).

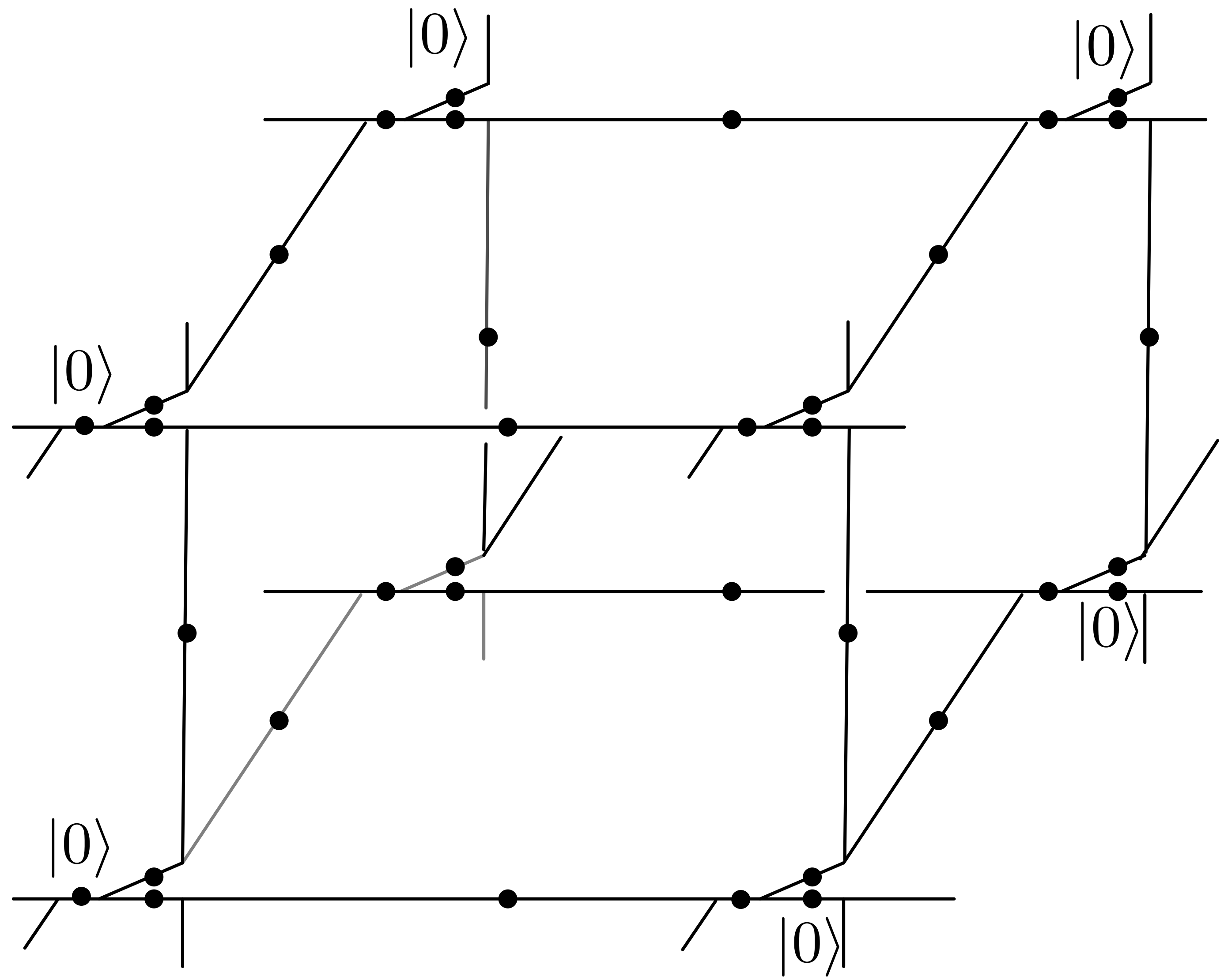
  - Kitaev&Preskill type decomposition





T. Grover, A.M. Turner, and A. Vishwanath, Phys. Rev. B **84**, 195120 (2011).

- Alternative approach
    - Choose elementary cube and freeze boundaries to $|0\rangle$
    - Taking into account fusion constraints at corners problem reduces to a $|L|^{12}$ dimensional Hilbert space. Plaquette operators are 8 body
    - Compute ground state $|G\rangle$
    - Compute system entropy for subsystems
    - Plot entropy as a function of number of boundaries produced by cuts
    - Intercept is the top. ent. entropy
- Results
    - 3D Toric code $S_{\text{topo}} = \log(2) = 1$
        - non modular
    - 3D Semion model $S_{\text{topo}} = 0$
        - modular
    - 3D Fibonacci model $S_{\text{topo}} = 0$
        - modular



Conclusions

- Physics of topological lattice models is richer in 3D vs. 2D
- All particles are confined when the theory is modular
- Evidence that the topological entanglement entropy in the ground state is trivial in the bulk
- To do: Make the argument for TEE general
  - Use the modular structure directly (perhaps in terms of lack of symmetry contraints on Schmidt coefficients of a bipartite decomposition)
  - Investigate TEE for excited states
    - Can one deform the Hamiltonian to allow for loop like excitations in the bulk with dynamic stability (mass independent of loop size)?