Quantum Yang-Mills theory

an overview of a programme

Cedric Beny, Ash Milsted, and Tobias J. Osborne

What this talk is

What this talk is

- Problem description

What this talk is

- Problem description
- A candidate wavefunction

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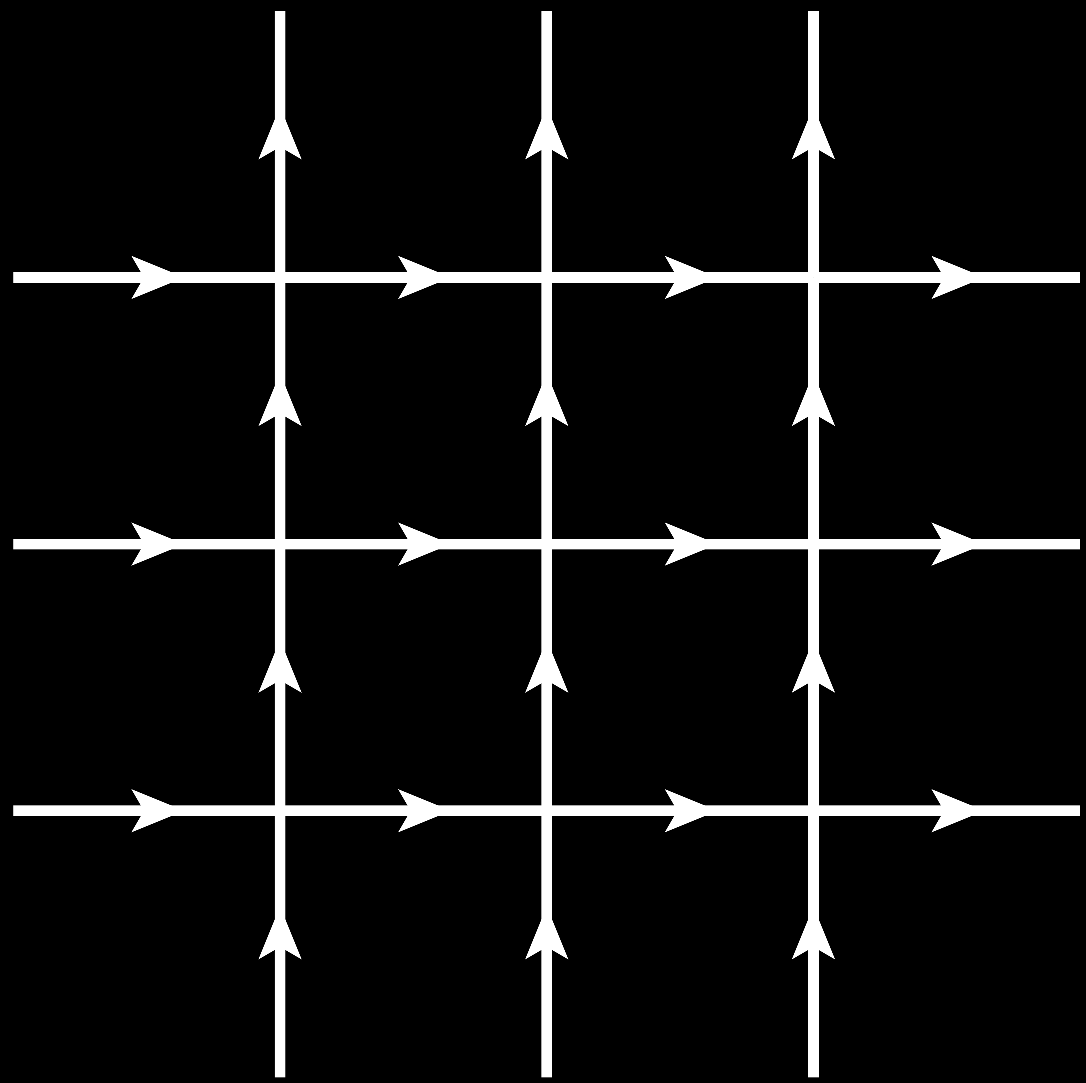
- Problem description
- A candidate wavefunction
- Continuum limit

What this talk is

- Problem description
- A candidate wavefunction
- Continuum limit
- Renormalisation

Hilbert space

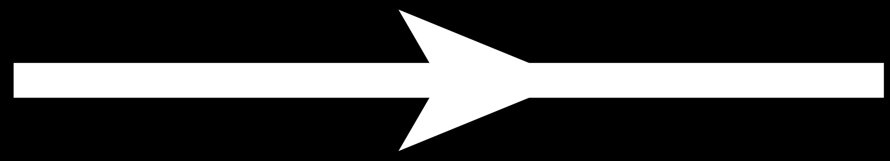
Like the toric code:



what lives on the edges?

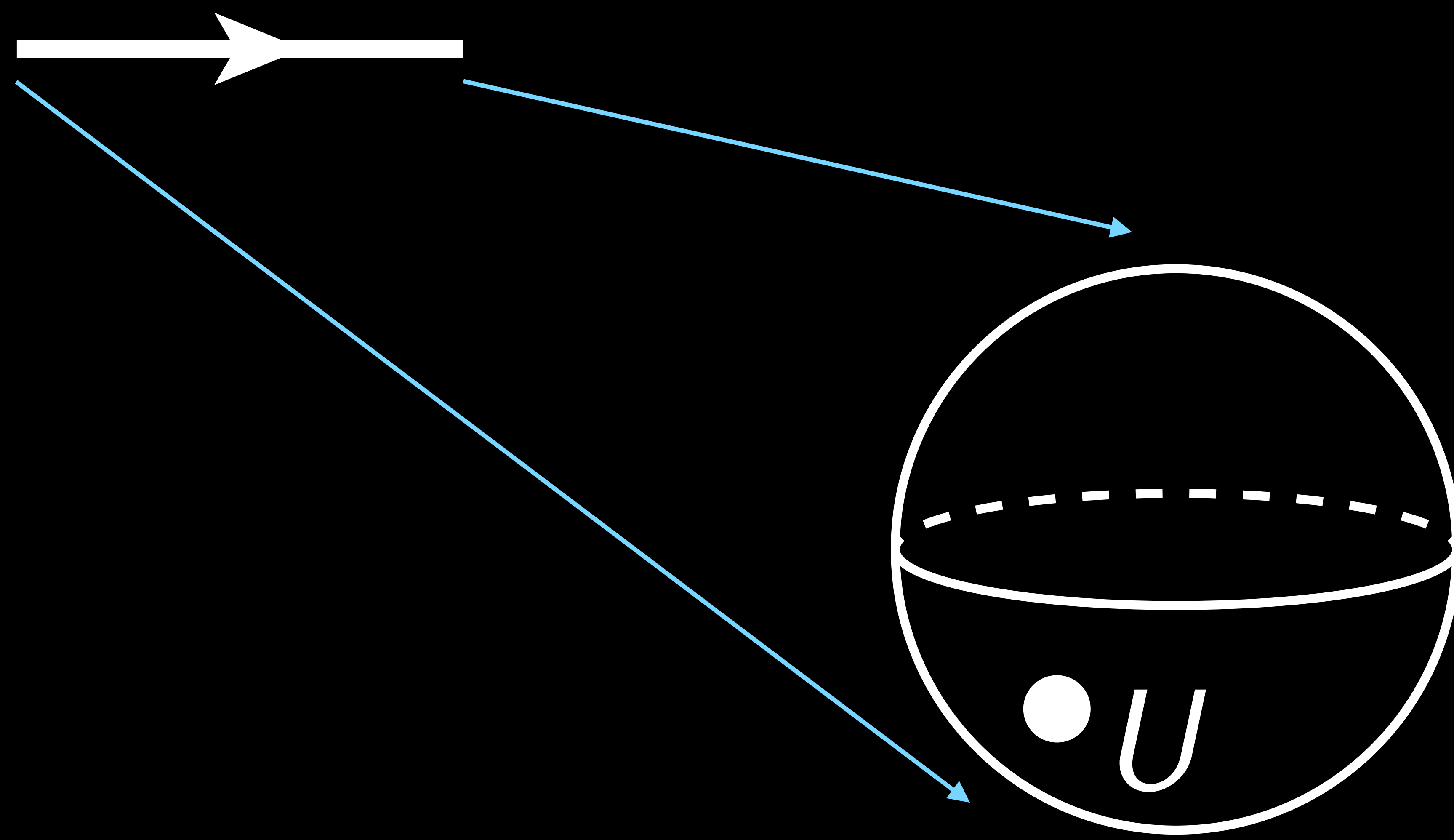
Position basis

$$|U\rangle$$



$$U \in SU(2)$$

Particle on a sphere



($SU(2)$ is diffeomorphic to the sphere S^3)

(cf. toric code: G = $\mathbf{Z}_2$)

Hilbert space: edge

$$|\psi\rangle = \int \psi(U)|U\rangle \, dU$$

$$\mathcal{H}_e \cong L^2(SU(2))$$

cf. toric code

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

$$\mathcal{H}_e = L^2(\{0,1\}) = \mathbb{C}^2$$

Momentum space

$$L^2(SU(2)) \cong \bigoplus_{l \in \frac{1}{2}\mathbb{Z}^+} V_l \otimes V_l^*$$

(Peter-Weyl theorem)

Momentum space

$$L^2(SU(2)) \cong \bigoplus_{l \in \frac{1}{2}\mathbb{Z}^+} \mathbb{C}^{2l+1} \otimes \mathbb{C}^{2l+1}$$

➡ mixture of qudit pairs

Momentum space: basis

$$|j\rangle_l |k\rangle_l \cong \sqrt{2l+1}\, t^l_{jk}$$

$l \in \frac{1}{2}\mathbb{Z}^+$ and $j, k = -l, -l+1, \ldots, l$

$t^{l}_{jk} =$ (j, k) - matrix element of spin-l representation of $SU(2)$

Examples

$$t^{0}_{jk}(U) = 1$$

$$t^{\frac{1}{2}}_{jk}(U) = [U]_{jk}$$

$$t^{1}_{11}(U) = \left([U]_{\frac{1}{2}\frac{1}{2}}\right)^2$$

$$|\psi\rangle = \sum_{l} \sum_{j,k=-l}^{l} \widehat{\psi}^{l}_{jk} |j\rangle_l |k\rangle_l,$$

where

$$\widehat{\psi}^{l}_{jk} = \sqrt{2l+1} \int \overline{t^{l}_{jk}}(U)\psi(U)\, dU$$

$$L^2(SU(2))$$

is FAPP **bipartite**

$$|\psi\rangle = \sum_{l} \sum_{j,k=-l}^{l} \widehat{\psi}^{l}_{jk} |j\rangle_{l} |k\rangle_{l},$$

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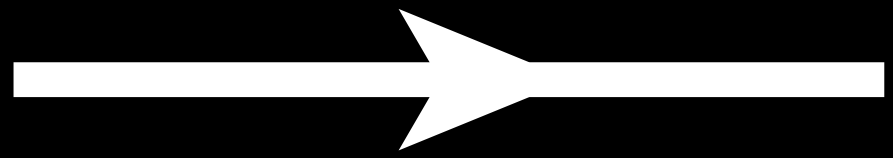
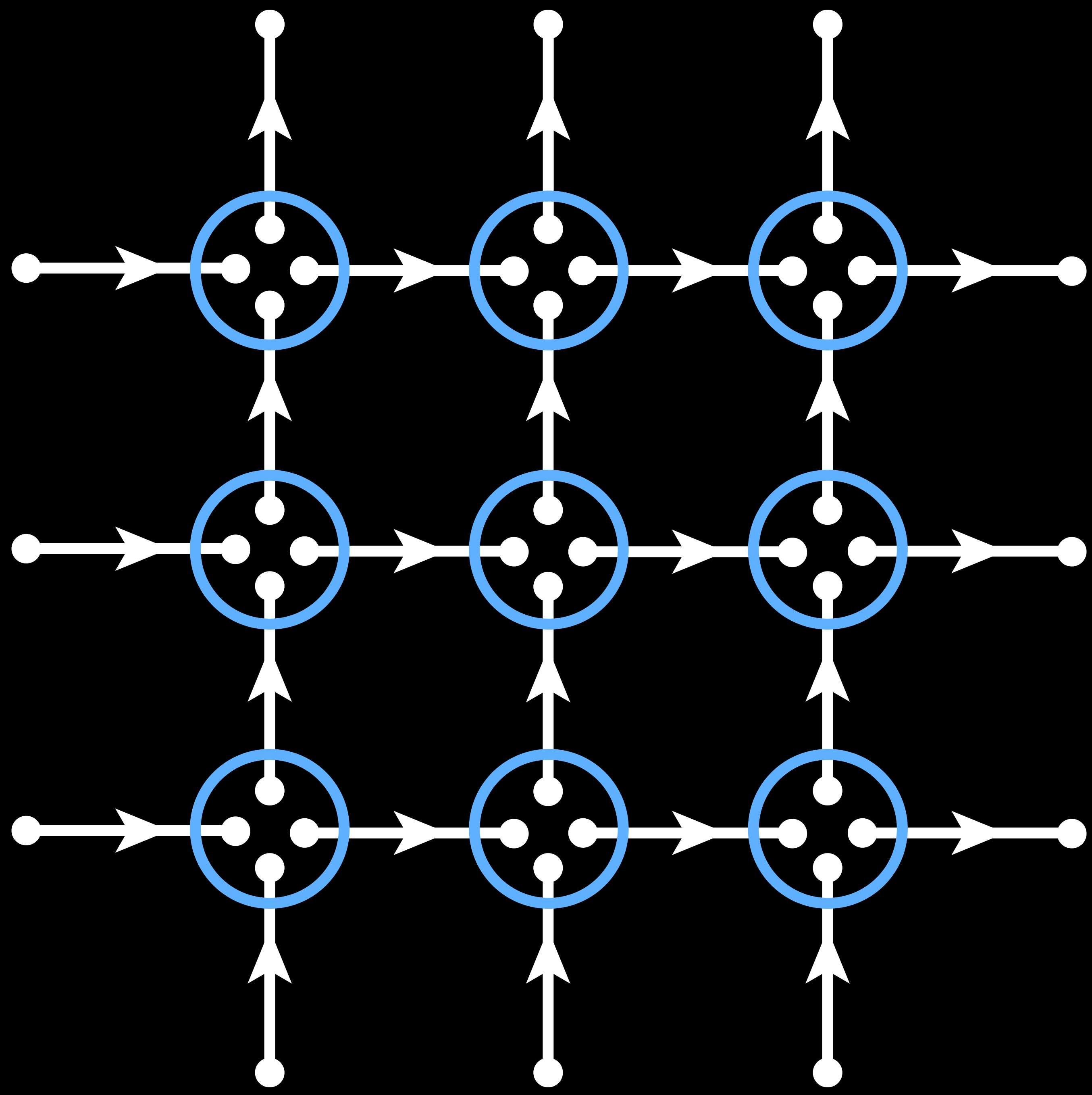


$$|\psi\rangle = \sum_{l} \sum_{j,k=-l}^{l} \widehat{\psi}^{l}_{jk} |j\rangle_l \qquad |k\rangle_l$$

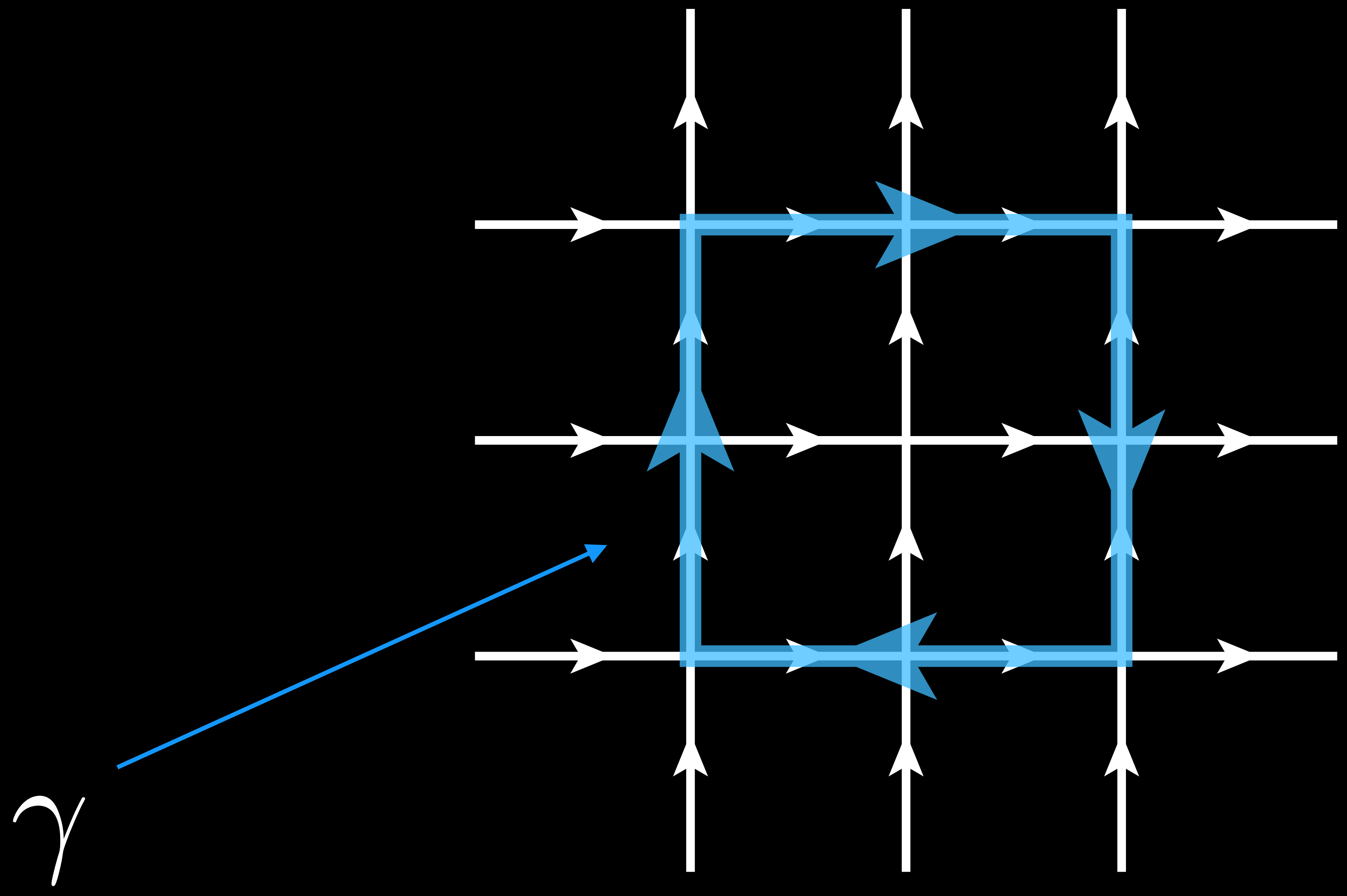


J. Baez, Adv. Math. **117**, 253-272 (1996)

Position observables

$$\widehat{u}_{jk}|U\rangle \equiv t_{jk}^{\frac{1}{2}}(U)|U\rangle$$

(classical) Wilson loops



$$\mathrm{tr}(\widehat{u}_\gamma) \equiv \sum_j \widehat{u}_{jj_1}(e_1)\widehat{u}_{j_1j_2}(e_2)\cdots\widehat{u}_{j_{n-1}j}(e_n)$$

where

$$\gamma = (e_1, e_2, \ldots, e_n)$$

Rotations

$$L_V|U\rangle \equiv |VU\rangle$$

and

$$R_V|U\rangle \equiv |UV^\dagger\rangle$$

Momentum observables

$$\widehat{\ell}_L^{\alpha} \equiv \frac{d}{d\epsilon} L_{e^{\epsilon\tau^{\alpha}}}$$

and

$$\widehat{\ell}_R^{\alpha} \equiv \frac{d}{d\epsilon} R_{e^{\epsilon\tau^{\alpha}}}$$

$$\tau^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau^1 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \tau^3 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Laplacian

$$-\triangle = \sum_{\alpha=1}^{3} (\widehat{\ell_L^\alpha})^2 = \sum_{\alpha=1}^{3} (\widehat{\ell_R^\alpha})^2 = \bigoplus_{l} (d_l^2 - 1)\mathbb{I}_l$$

where

$$d_l = 2l + 1$$

Dynamics

Kogut-Susskind hamiltonian

$$H(g) = -\frac{g^2}{2a}\sum_{e\in E}\triangle_e - \frac{2}{g^2 a}\sum_{\square}\mathrm{Re}(\mathrm{tr}(\widehat{u}_{\square}))$$

J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975)

What is the ground state $|\Omega(g)\rangle$?

Ground state: strong coupling

$$|\Omega(\infty)\rangle = \bigotimes_{e \in E} |00\rangle_0$$

Ground state: weak coupling

$$|\Omega(0)\rangle = \text{superposition of all configurations s.t.} \prod_{e\in\square} U_e = \mathbb{I}, \forall\square$$

Ground state: weak coupling

$$|\Omega(0)\rangle =$$ superposition of all configurations s.t. $$\prod_{e\in\square} U_e = \mathbb{I}, \forall\square$$

(In toric code the weak-coupling case **is** the ground state)

(Any configuration:

$$|\mathcal{U}\rangle \equiv \bigotimes_{e \in E} |U_e\rangle$$

such that

$$\prod_{e \in \square} U_e = \mathbb{I}, \forall \square$$

is called **flat**.)

The **problem**

Find $|\Phi(g)\rangle$ such that

1. The family $|\Phi(g)\rangle$ is a **contractible TNS**

2. The state $|\Phi(g)\rangle$ is manifestly **gauge invariant**

3. It **interpolates** from zero coupling to strong coupling, i.e., $|\Phi(0)\rangle = |\Omega(0)\rangle$ and $|\Phi(\infty)\rangle = |\Omega(\infty)\rangle$

4. The state $|\Phi(g)\rangle$ differs from $|\Omega(g)\rangle$ only by **irrelevant** features

5. The family $|\Phi(g)\rangle$ admits a **continuum limit**, and is **Lorentz invariant**

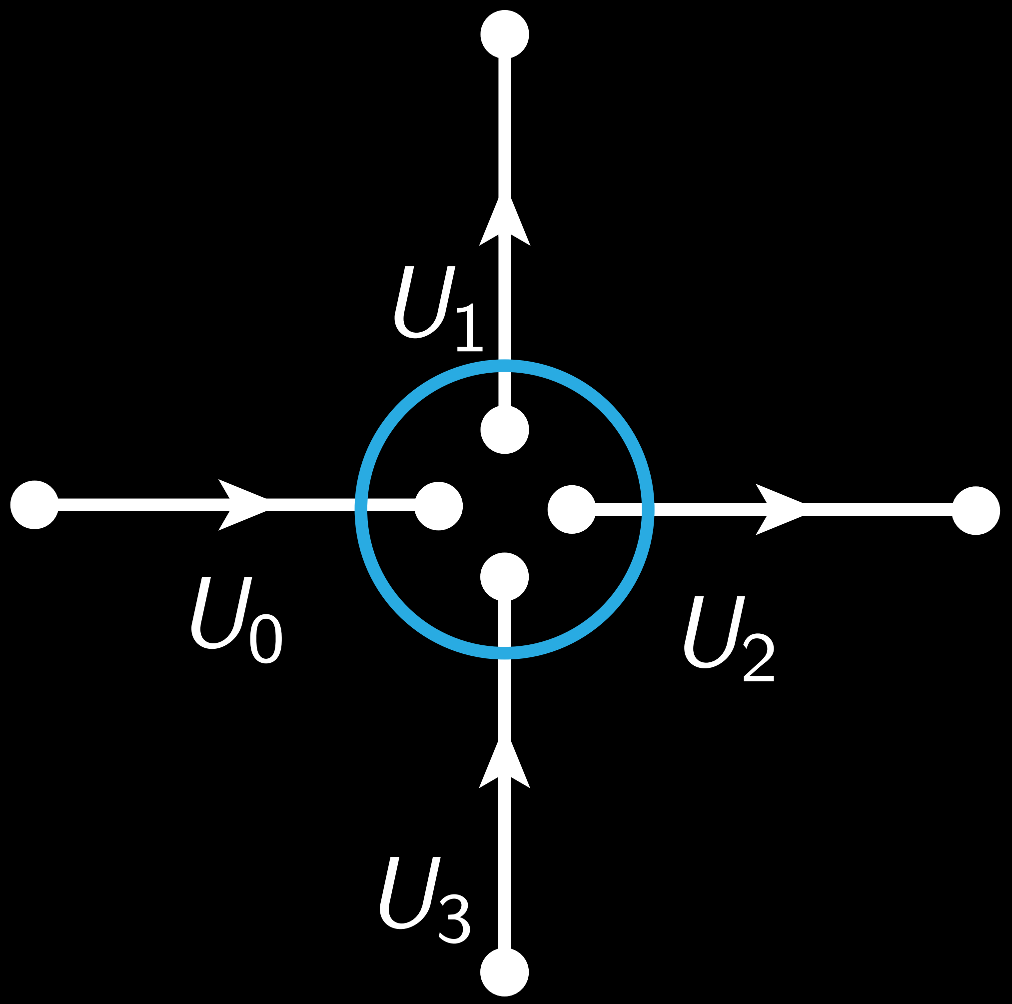
1-2. The local gauge group

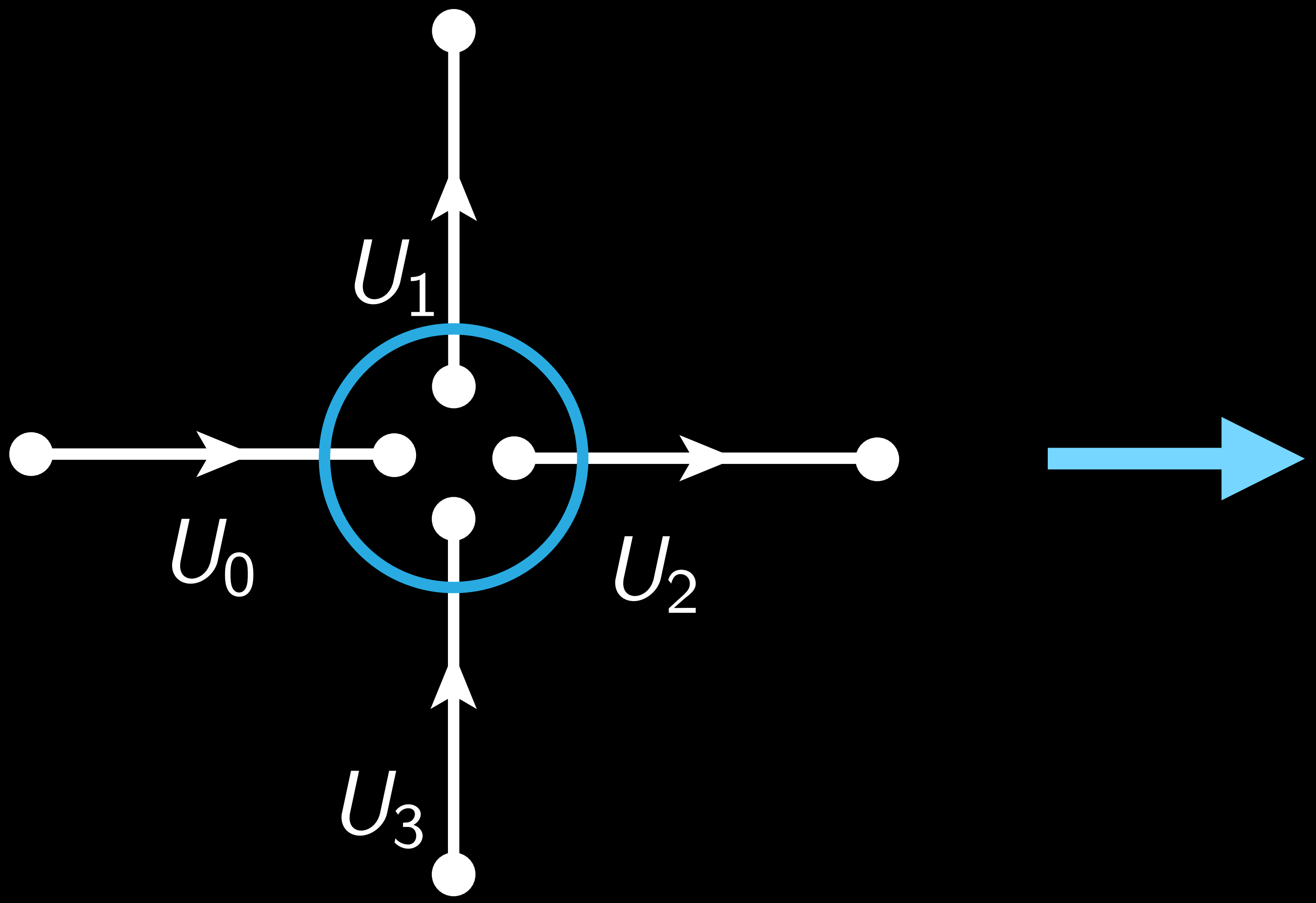
$$\mathcal{G} \cong \prod_{ax \in a\mathbb{Z}^d} SU(2)$$

One copy of $SU(2)$ per vertex

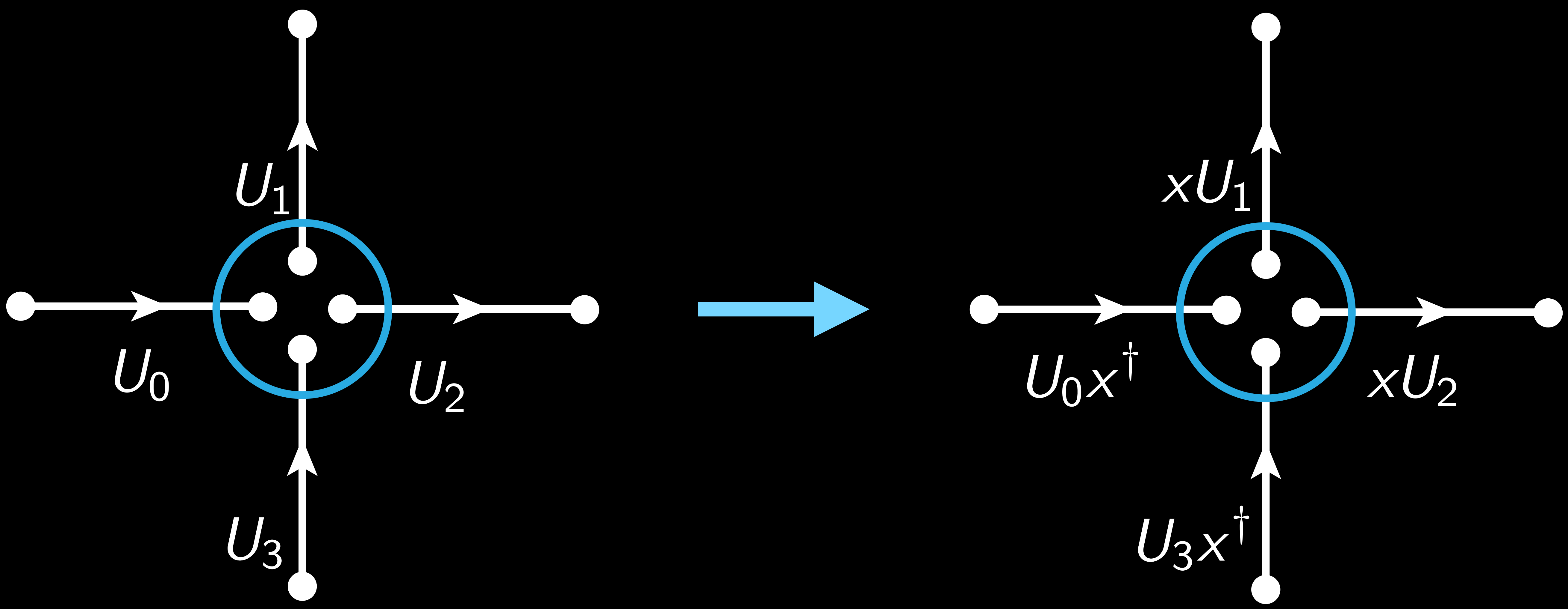
The local gauge group acts via

$$\bigotimes_{e \in E} L_{x_{e_-}} R_{x_{e_+}}, \quad x \in \mathcal{G}$$


U_1
U_0
U_2
U_3



U_1
U_0
U_2
U_3



U_1
U_0
U_2
U_3
xU_1
$U_0x^\dagger$
xU_2
$U_3x^\dagger$

Gauge-invariant sector $\mathcal{H}_{\mathcal{G}}$:

$$\bigotimes_{e \in E} L_{x_{e_-}} R_{x_{e_+}} |\psi\rangle = |\psi\rangle, \quad x \in \mathcal{G}$$

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$$\bigotimes_{e \in E} L_{x_{e_-}} R_{x_{e_+}} |\psi\rangle = |\psi\rangle, \quad x \in \mathcal{G}$$

(Toric code: star operators are exactly satisfied)

Single loop

Single loop: $\mathcal{H}_{\mathcal{G}}$

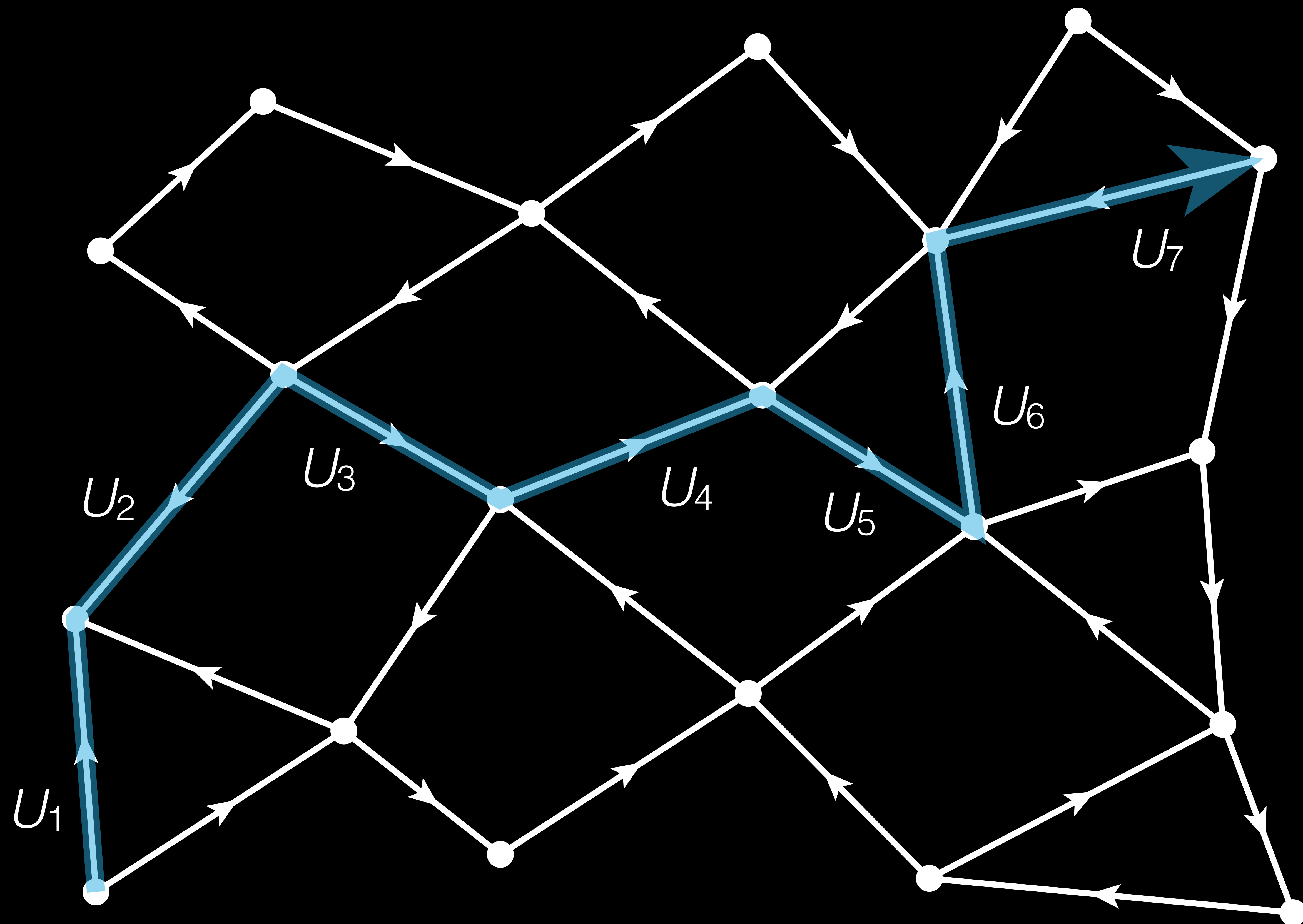
$$|\psi\rangle = \int \psi(U)|U\rangle \, dU$$

such that

$$\psi(x^{-1}Ux) = \psi(U)$$

(classical) parallel transport

$$U(\gamma) \equiv \prod_{e \in \gamma} U_e^{-\mathrm{sgn}_\gamma e}$$



$$U(\gamma) = U_7 U_6^\dagger U_5^\dagger U_4^\dagger U_3^\dagger U_2 U_1^\dagger$$

Controlled rotations

$$CU \equiv \int |U\rangle\langle U| \otimes \pi(U)\, dU$$

where $\pi(U)$ is a representation, e.g.

$$\pi(U) \cong L_U$$

or

$$\pi(U) \cong R_U$$

Quantum parallel transport

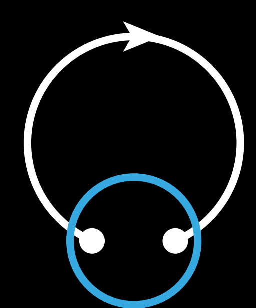
$$CU_\gamma \equiv \prod_{e \in \gamma} {CU_{es}}^{-\mathrm{sgn}_\gamma e}$$

Edge addition and subdivision

J. Baez, Adv. Math. **117**, 253-272 (1996)
E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. **43**, 4452 (2002)
M. Aguado and G. Vidal, Phys. Rev. Lett. **100**, 070404 (2008)
O. Buerschaper, M. Aguado, and G. Vidal, Phys. Rev. B **79**, 085119 (2009)
R. König, B. W. Reichardt, and G. Vidal, Phys. Rev. B **79**, 195123 (2009)

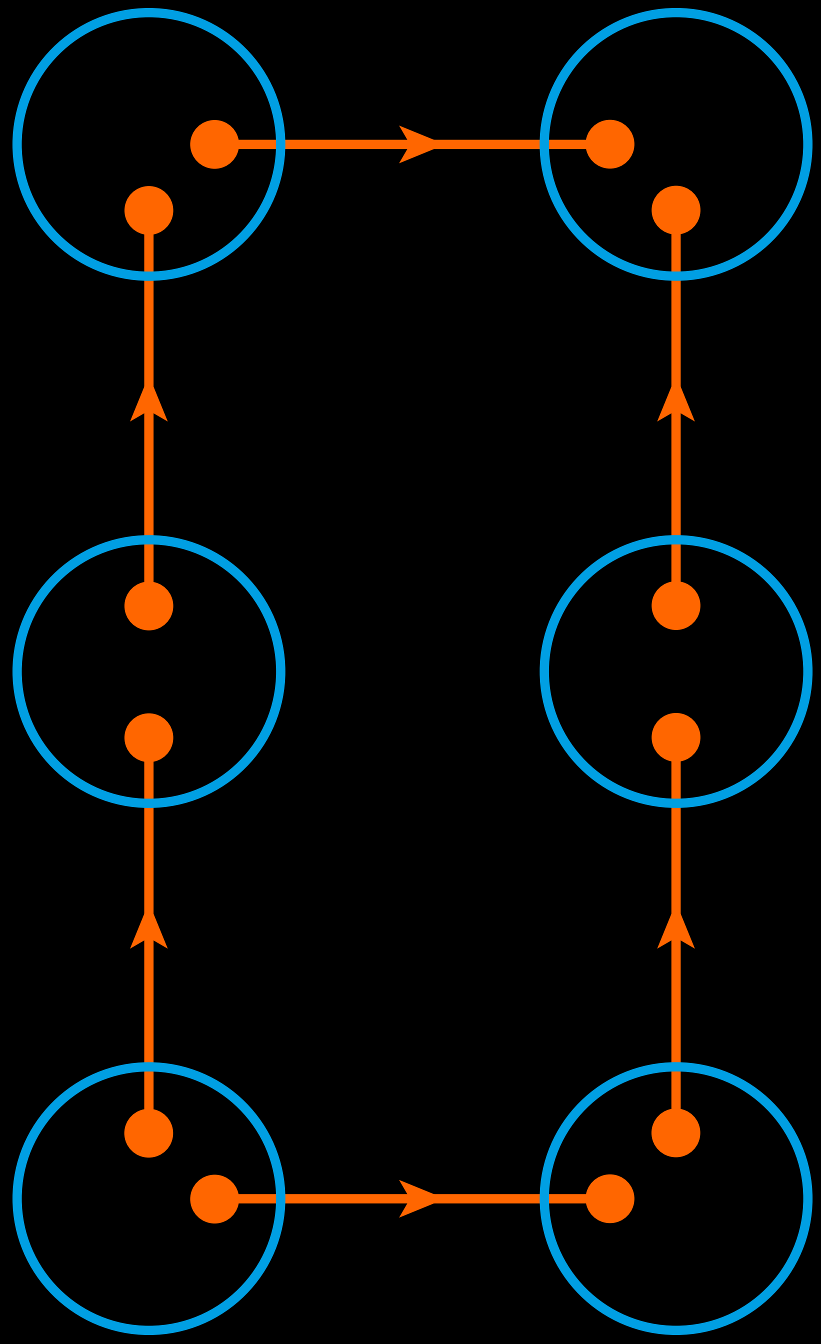
Edge addition

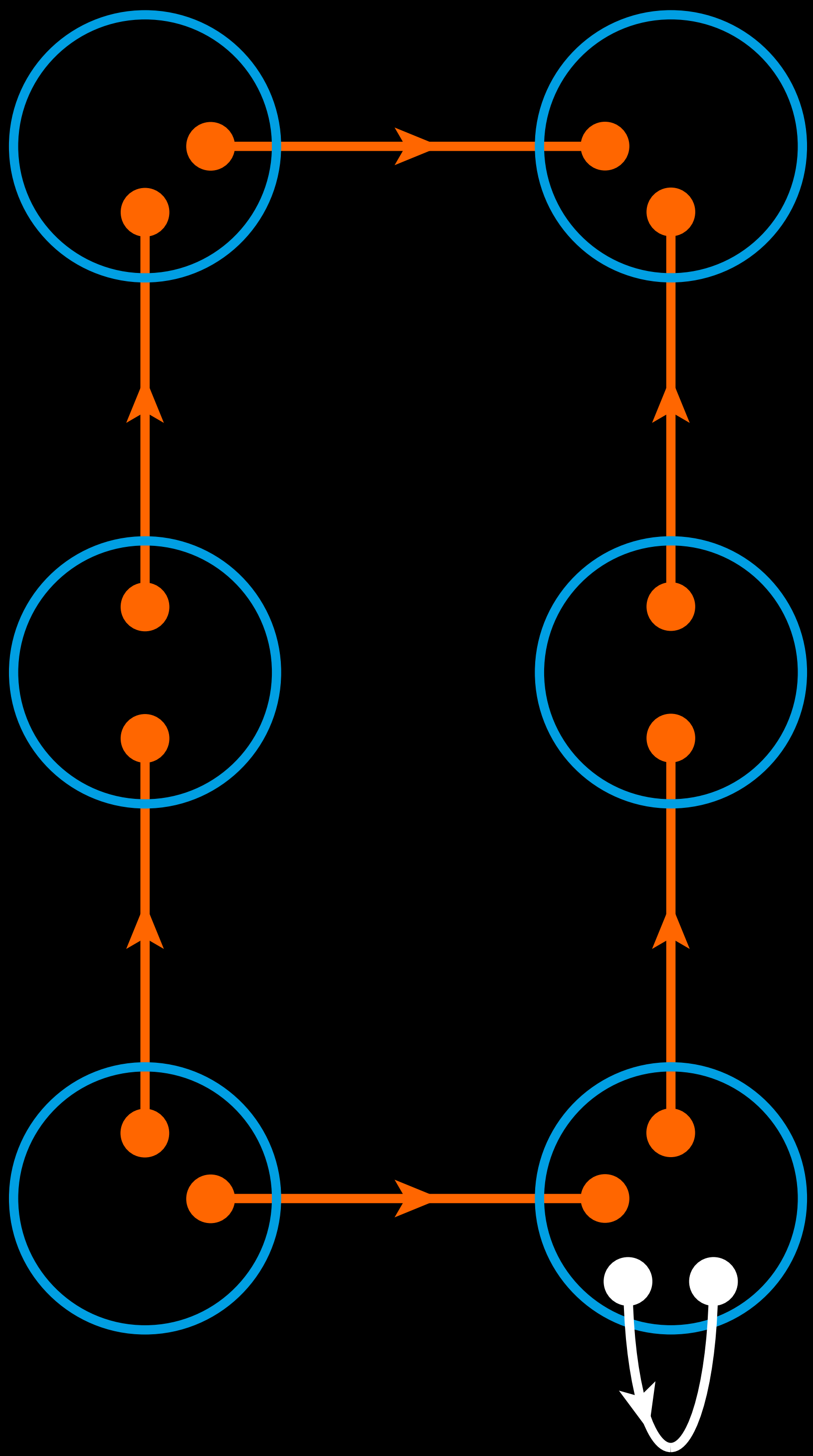
1. Add in a gauge invariant loop at a vertex:

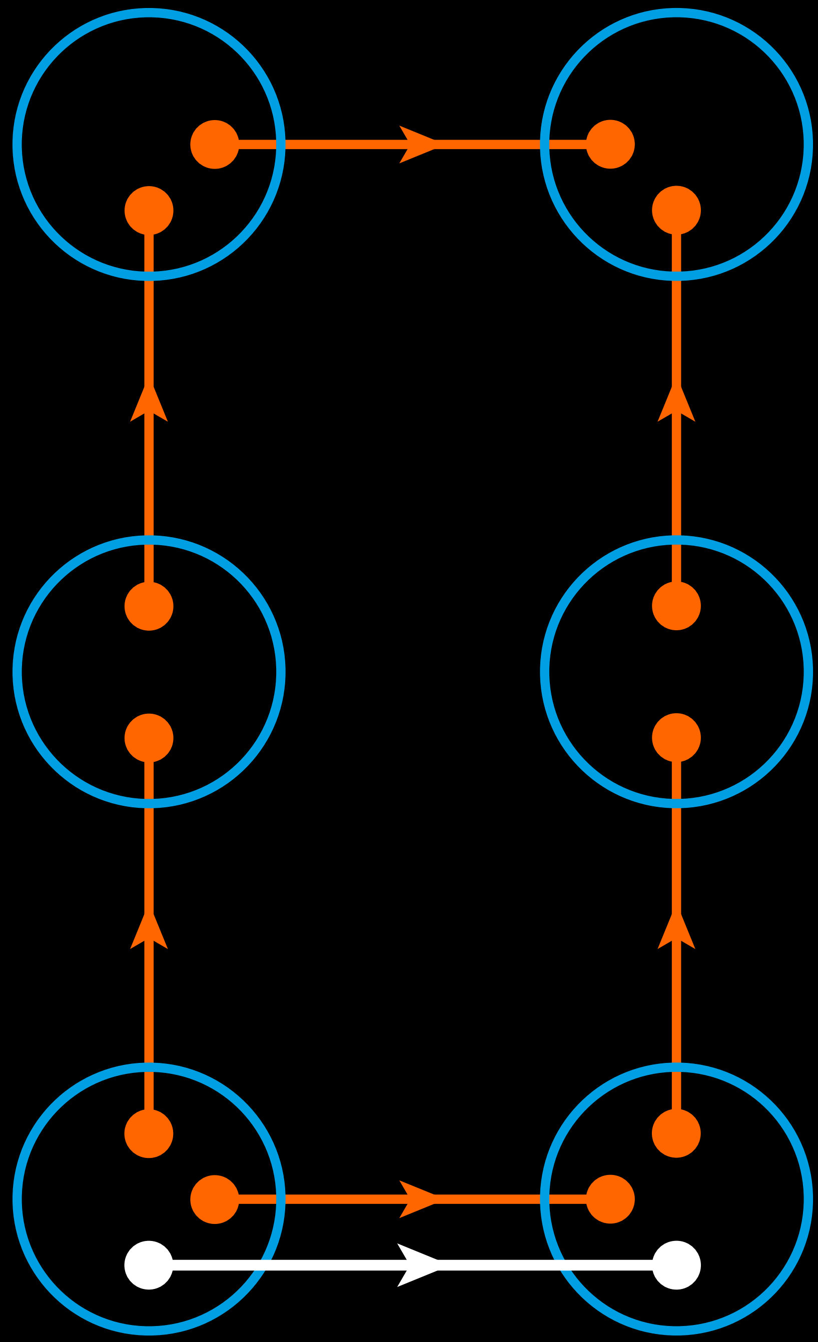


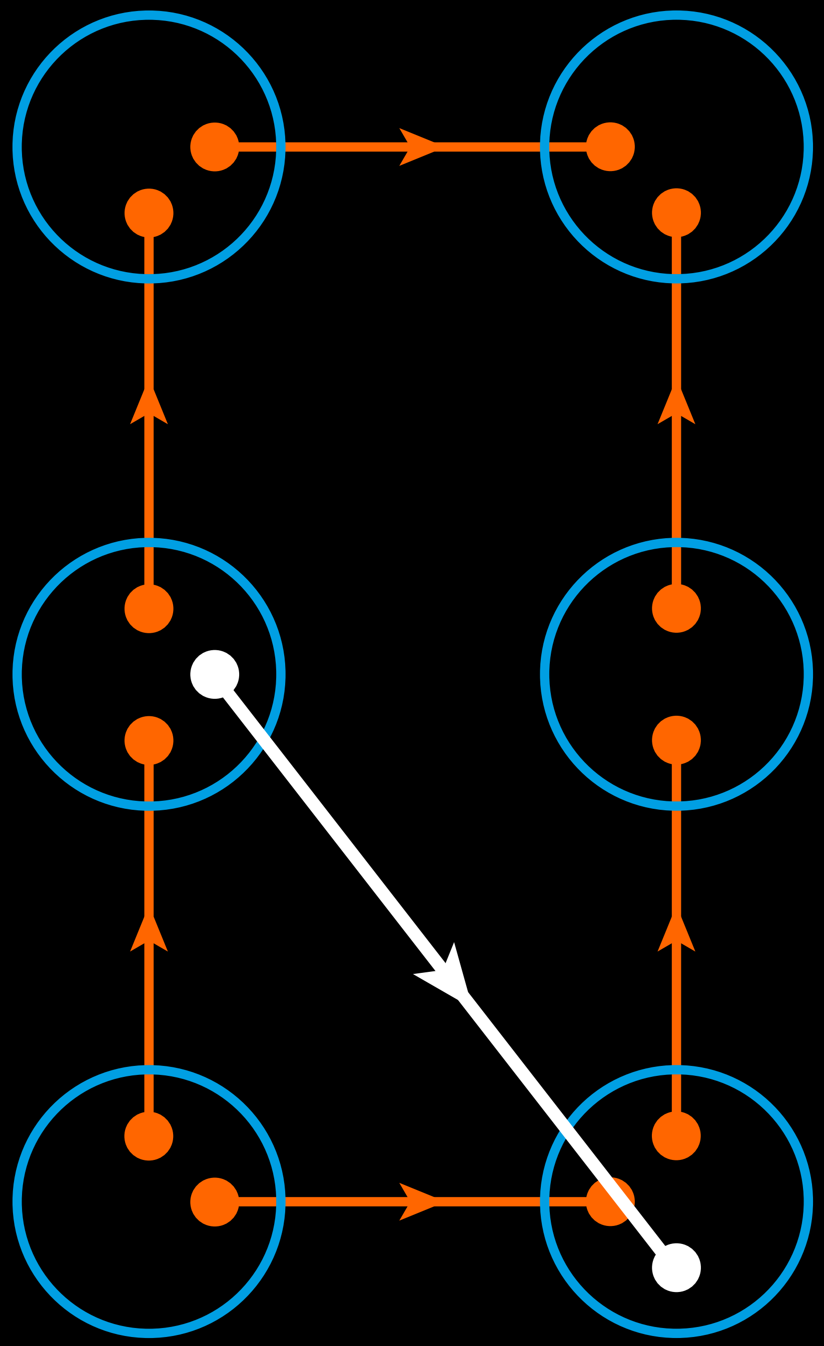
2. Parallel transport ends to destination vertices via

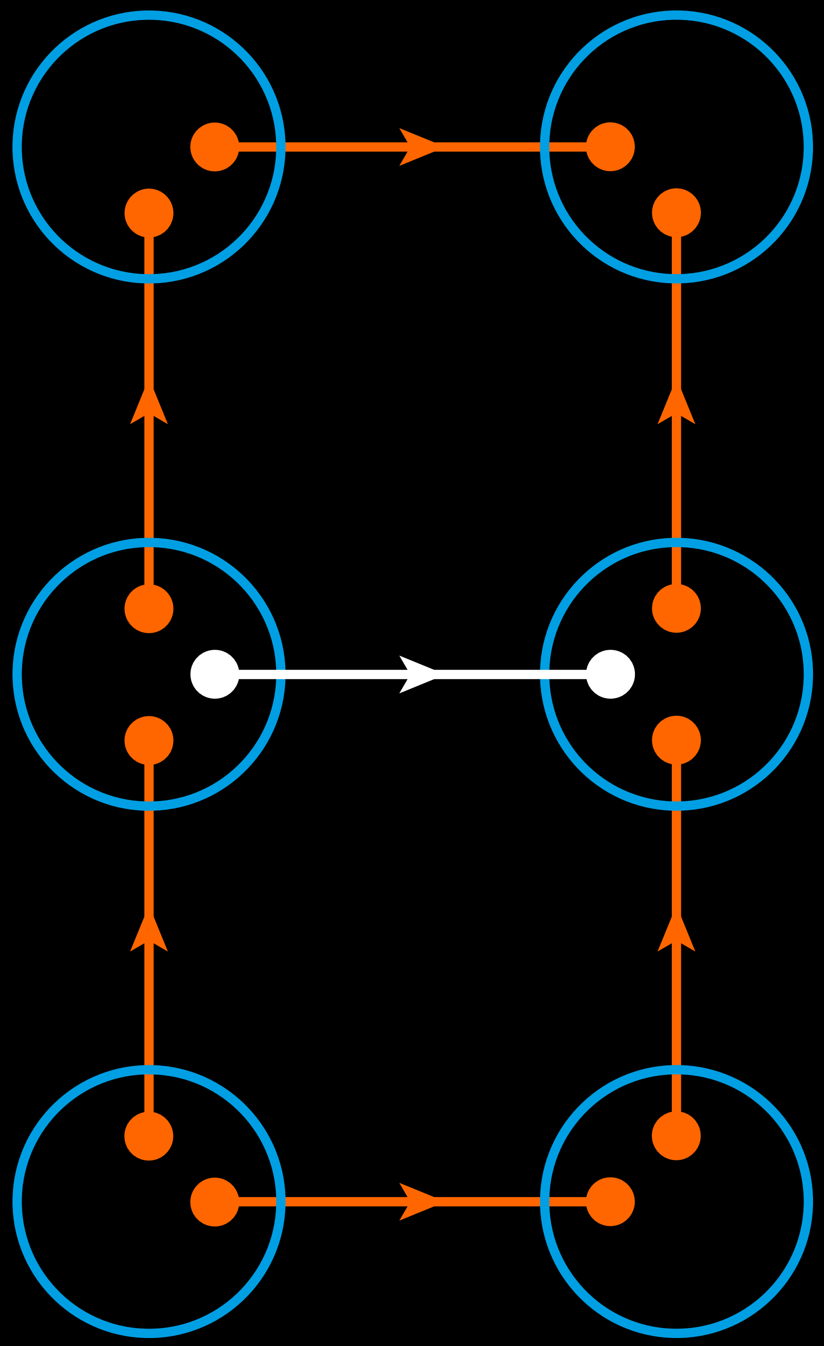
$$CL_\gamma \quad \text{and} \quad CR_\gamma$$











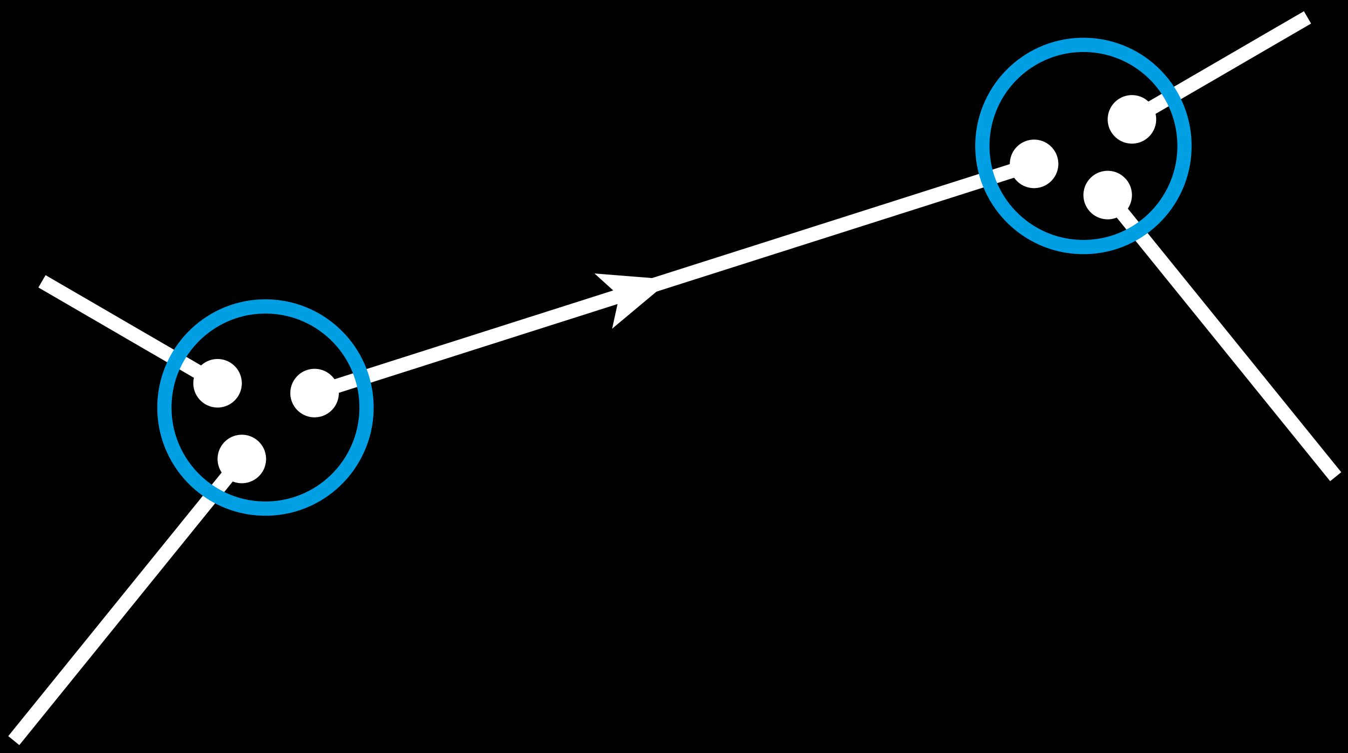
Edge subdivision

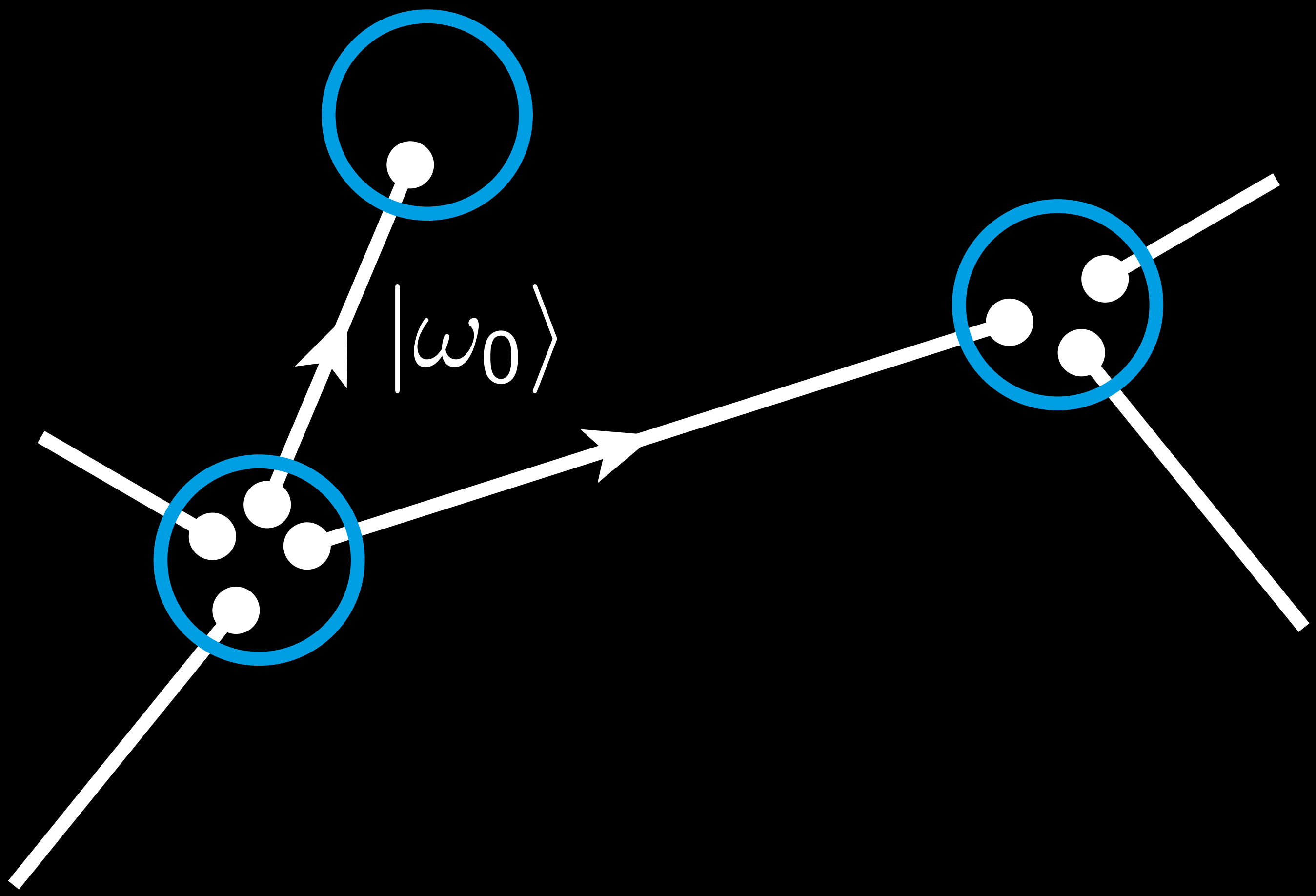
1. Add an edge in the trivial representation

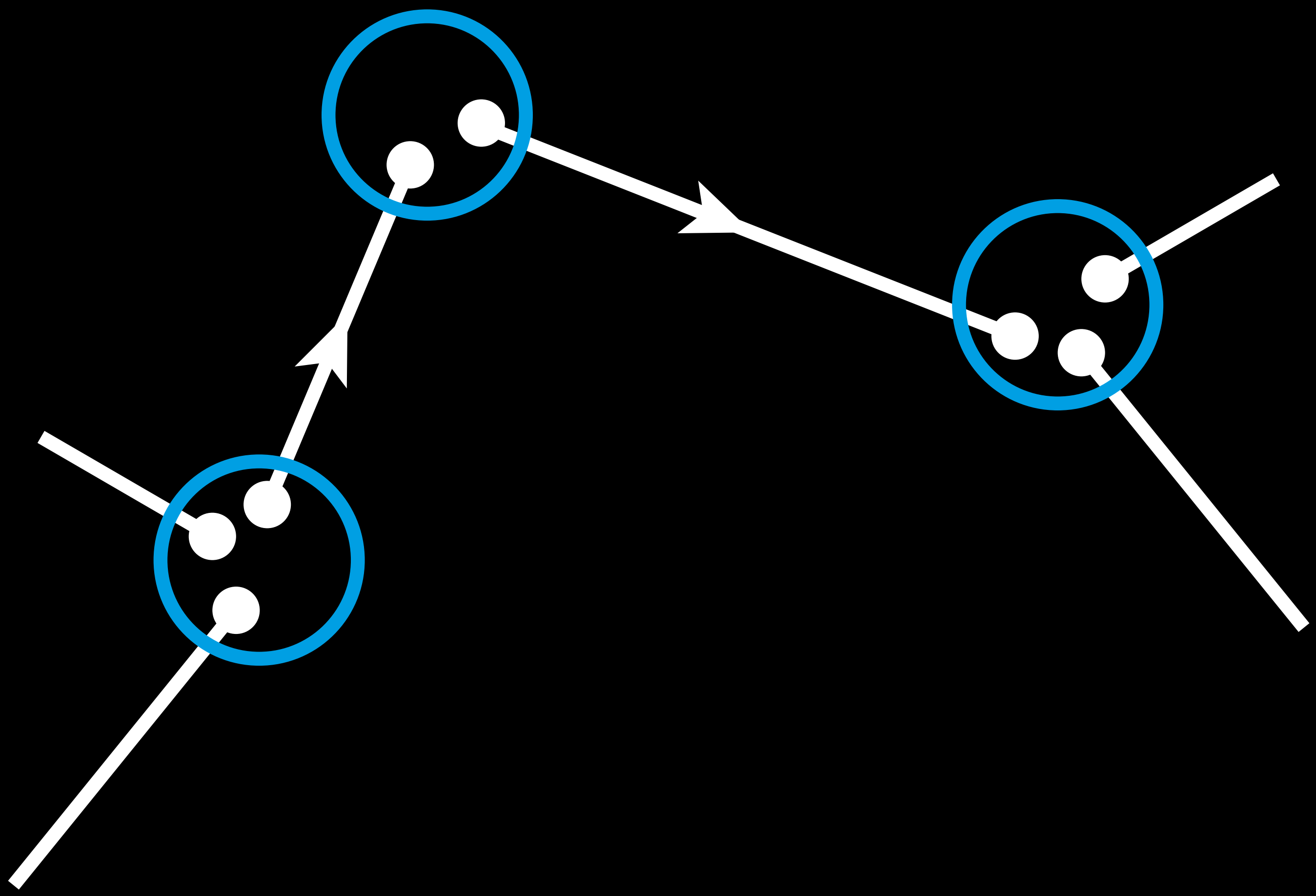
$$|\omega_0\rangle \equiv |00\rangle_0$$

2. Parallel transport the end of old edge to new location via:

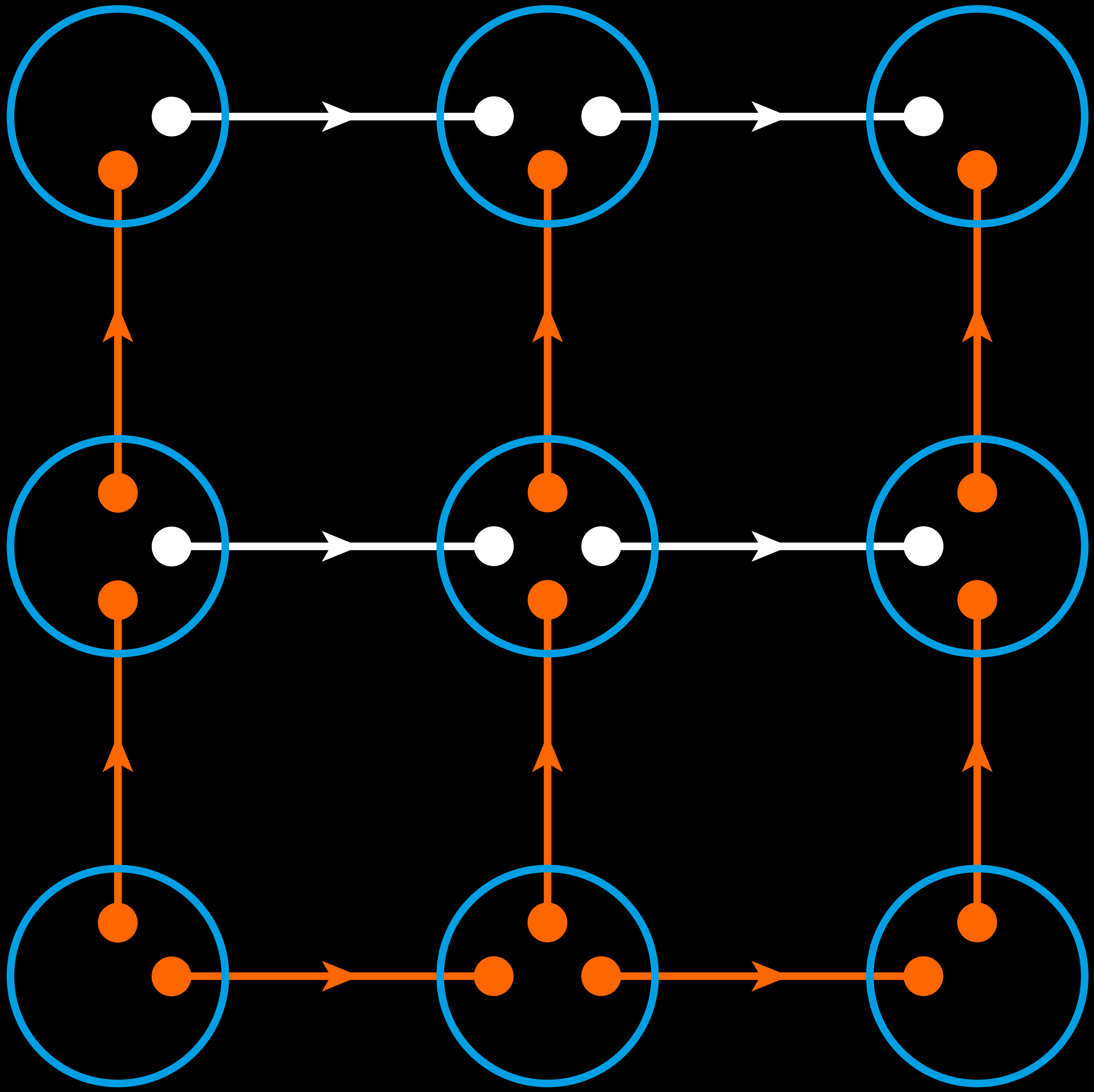
$$CL_e^{-1}$$

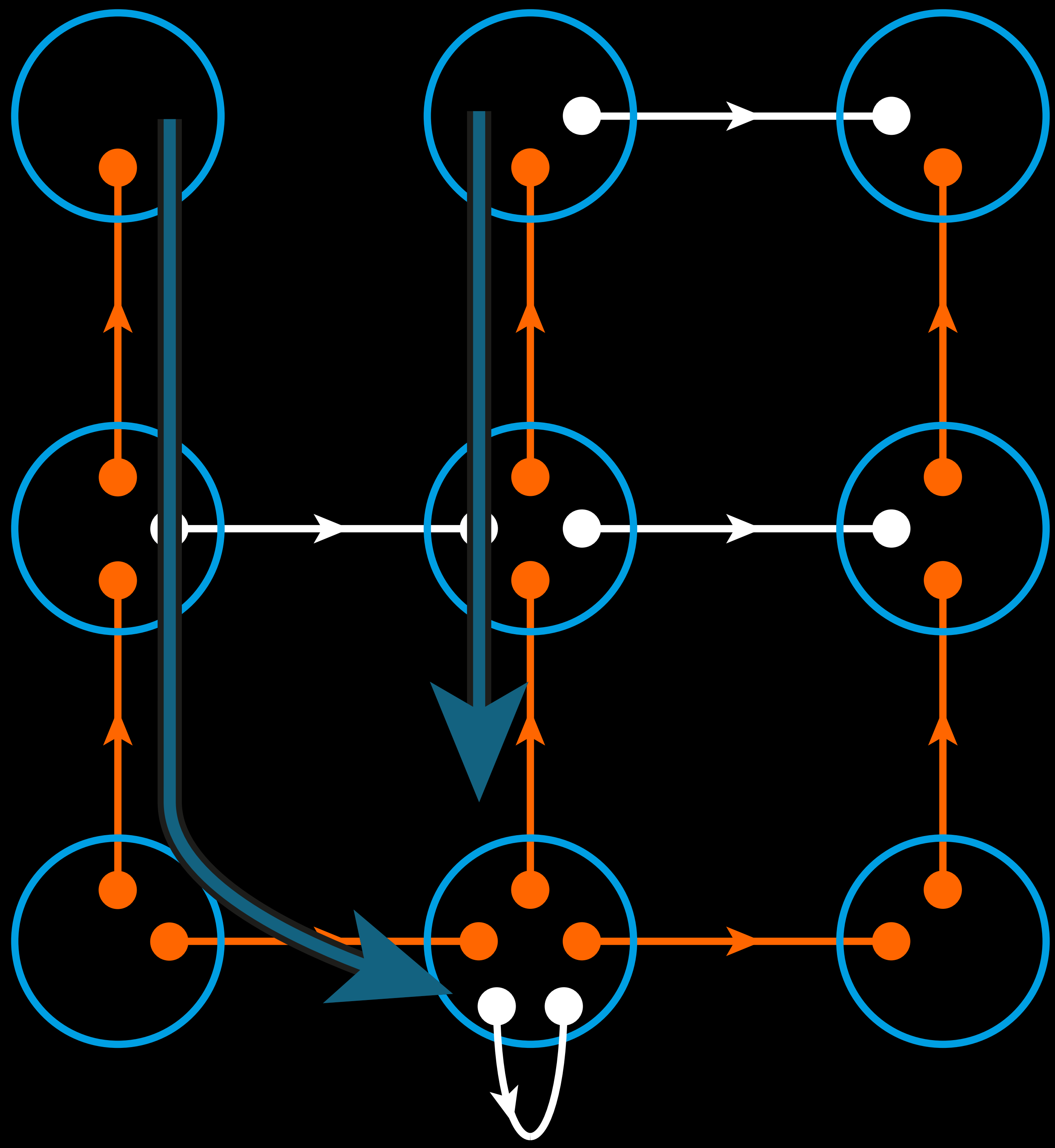


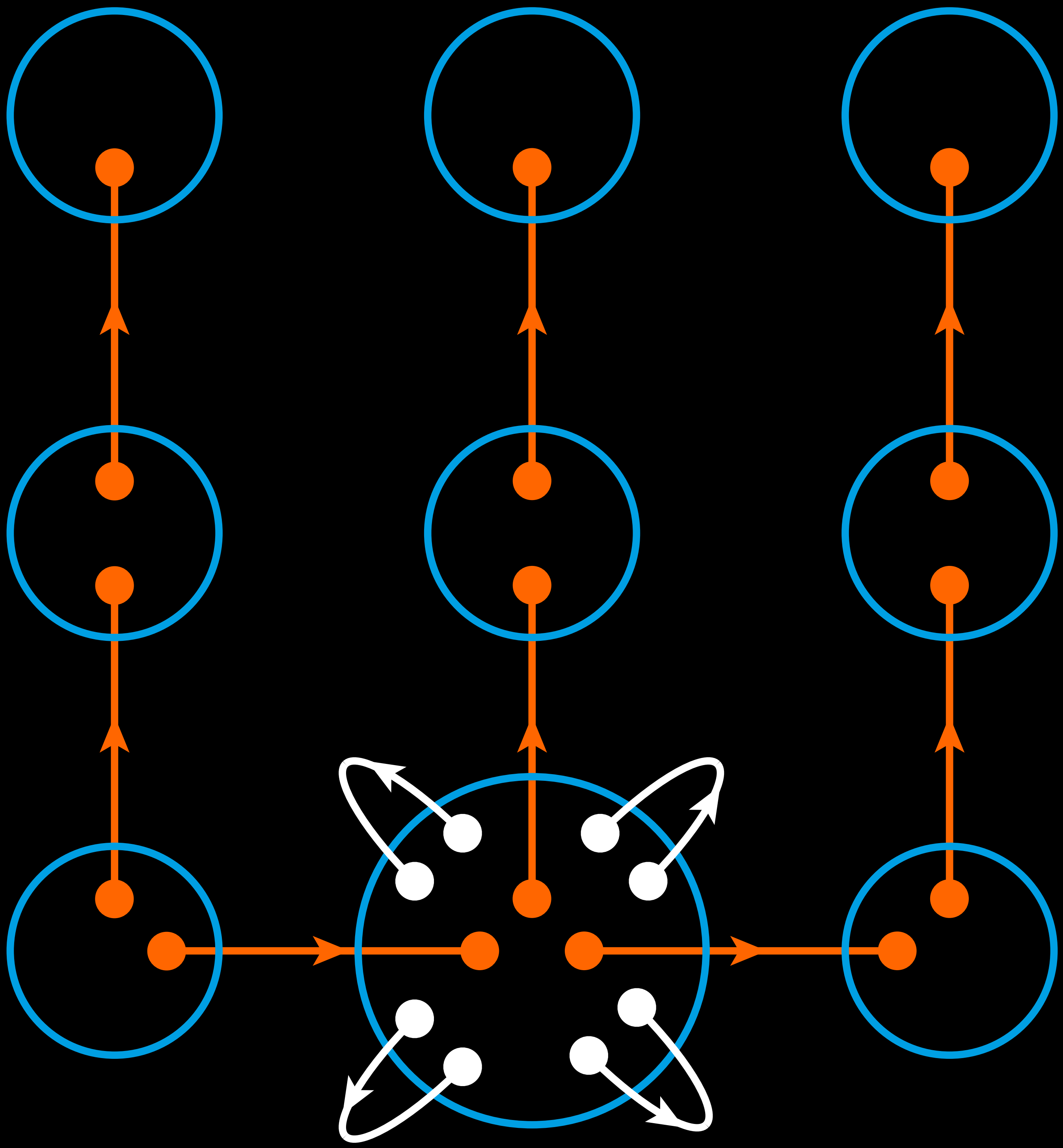

$|\omega_0\rangle$

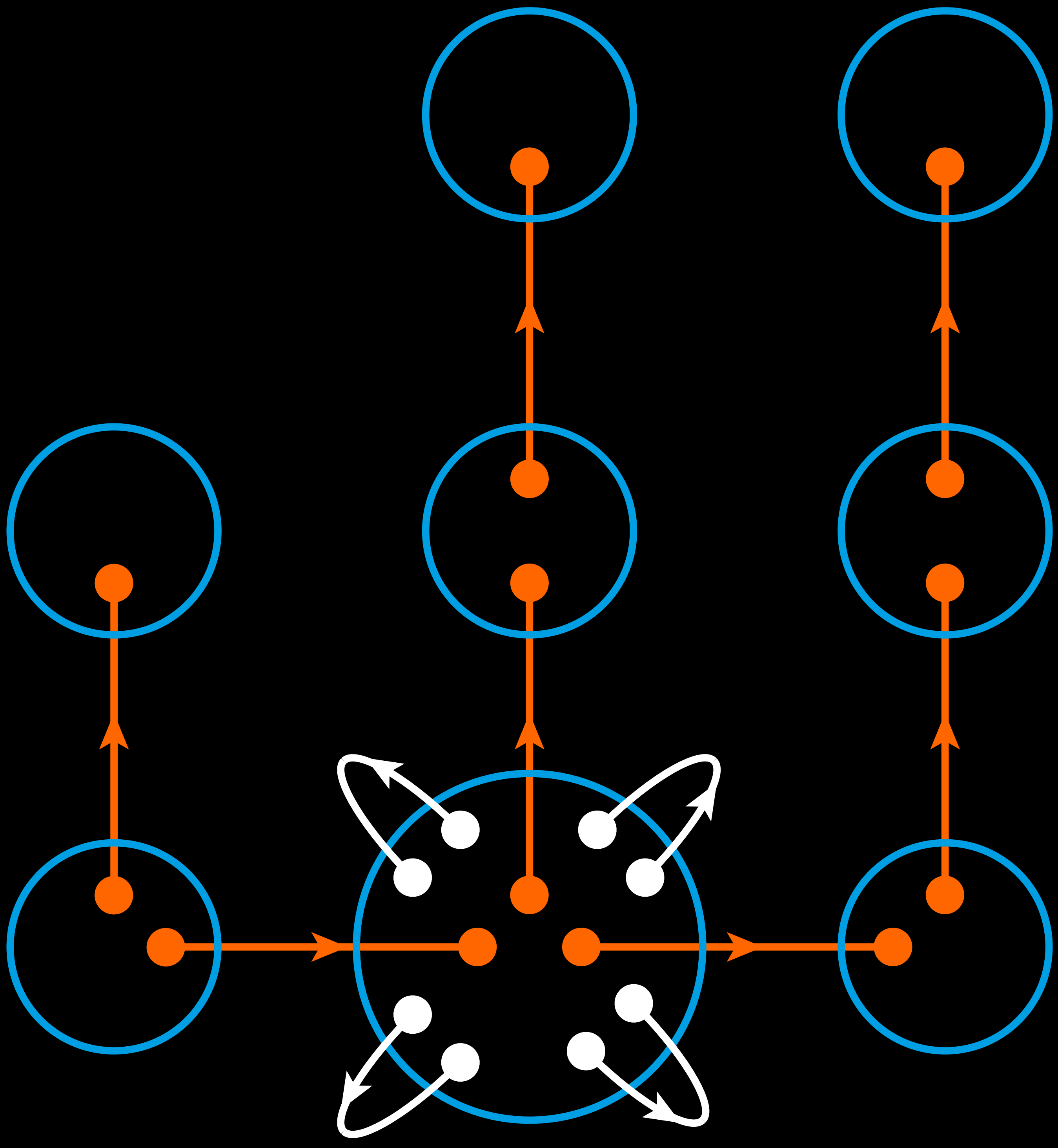


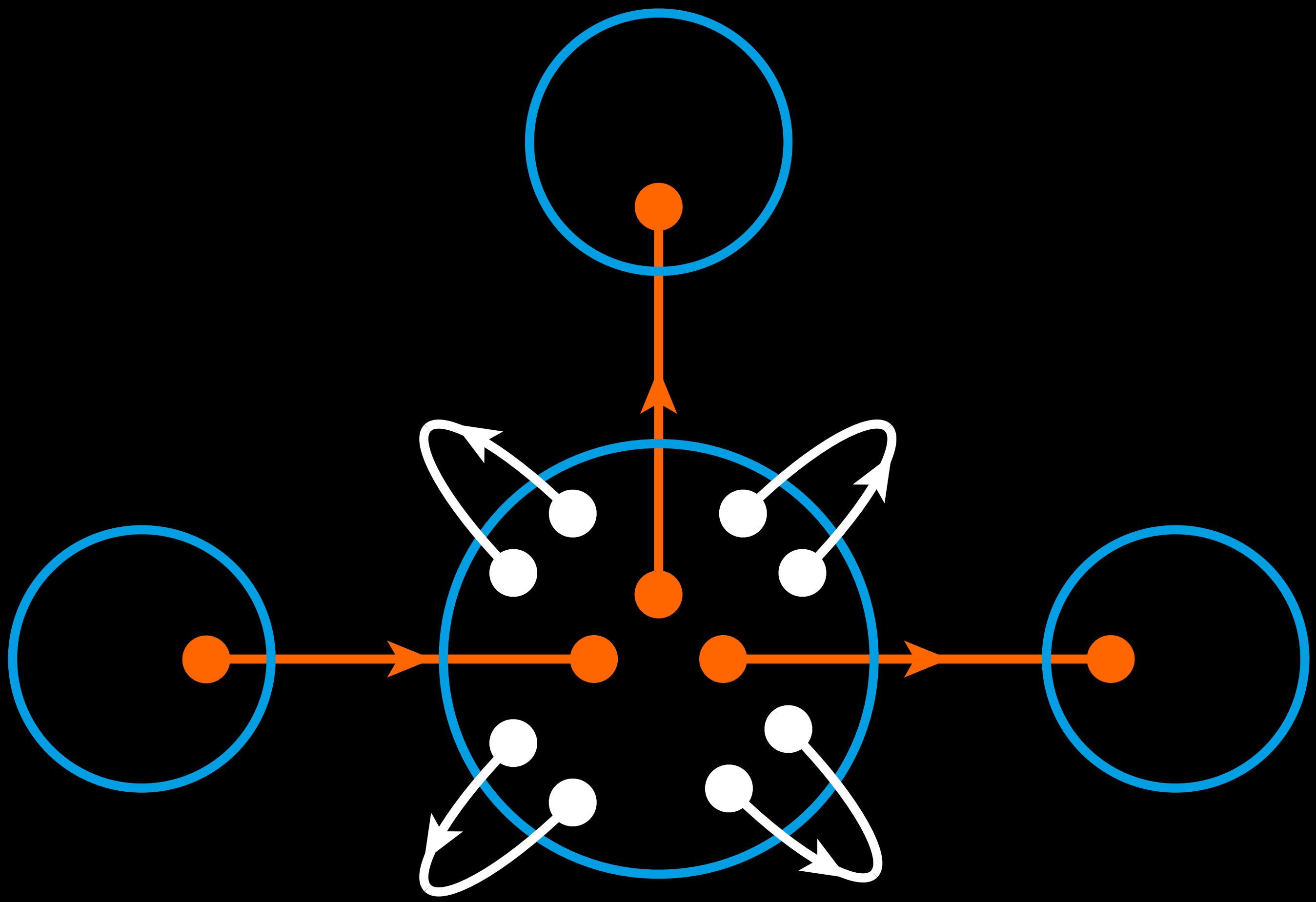
Gauge-invariant sector $\mathcal{H}_{\mathcal{G}}$:

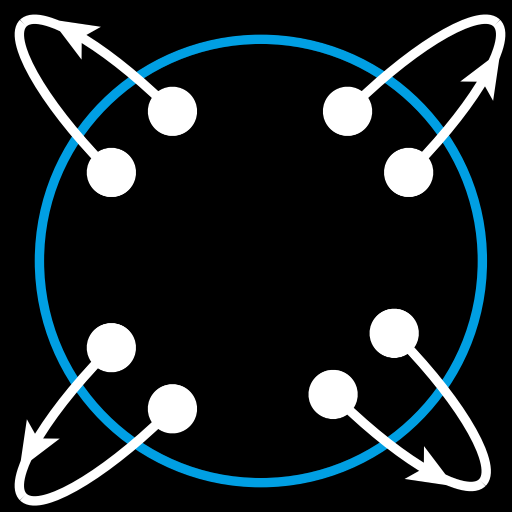












3. Interpolation

Dimensional transmutation

Lattice spacing *a* plays **no role** in diagonalising *H*(*g*):

$$H(g) = -\frac{g^2}{2a}\sum_{e\in E}\triangle_e - \frac{2}{g^2 a}\sum_{\square}\mathrm{Re}(\mathrm{tr}(\widehat{u}_{\square}))$$

How to work out a?

Ground state has correlation length:

$$\xi(g)$$

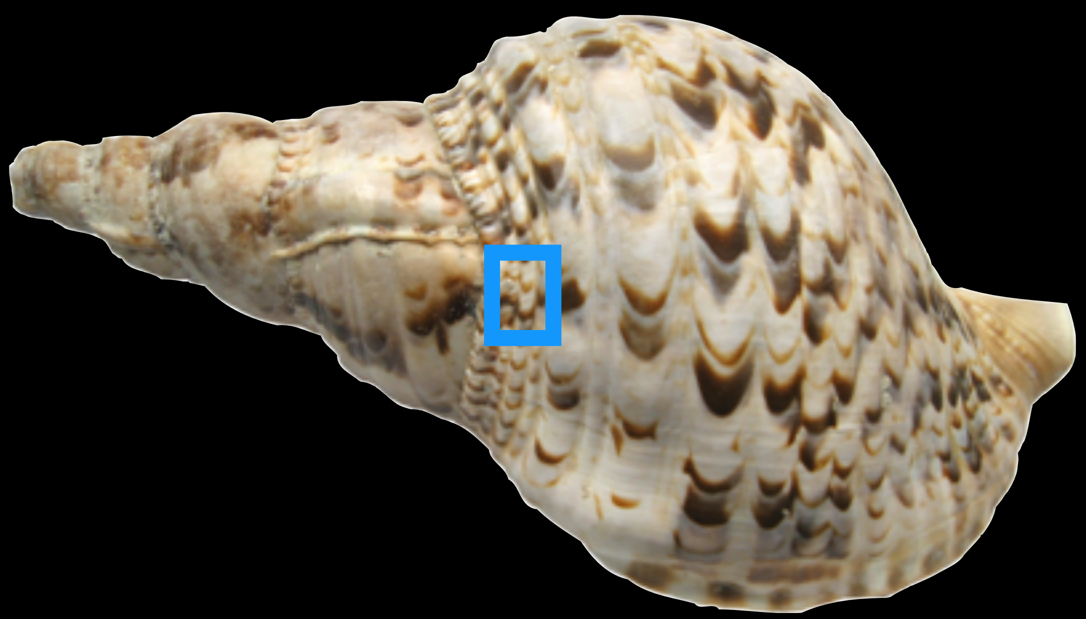
Require:

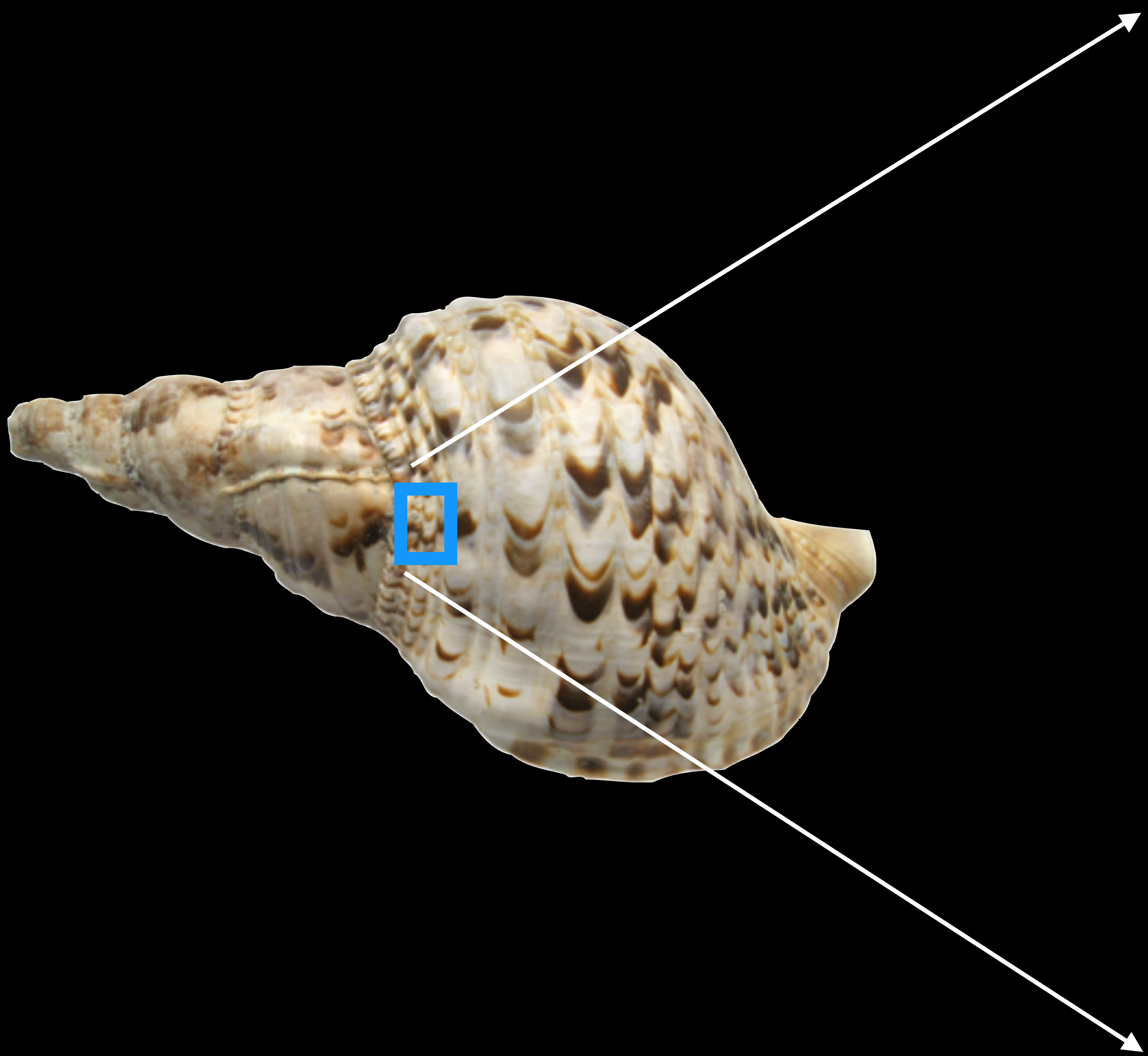
$$a\xi(g) = \text{const.}$$

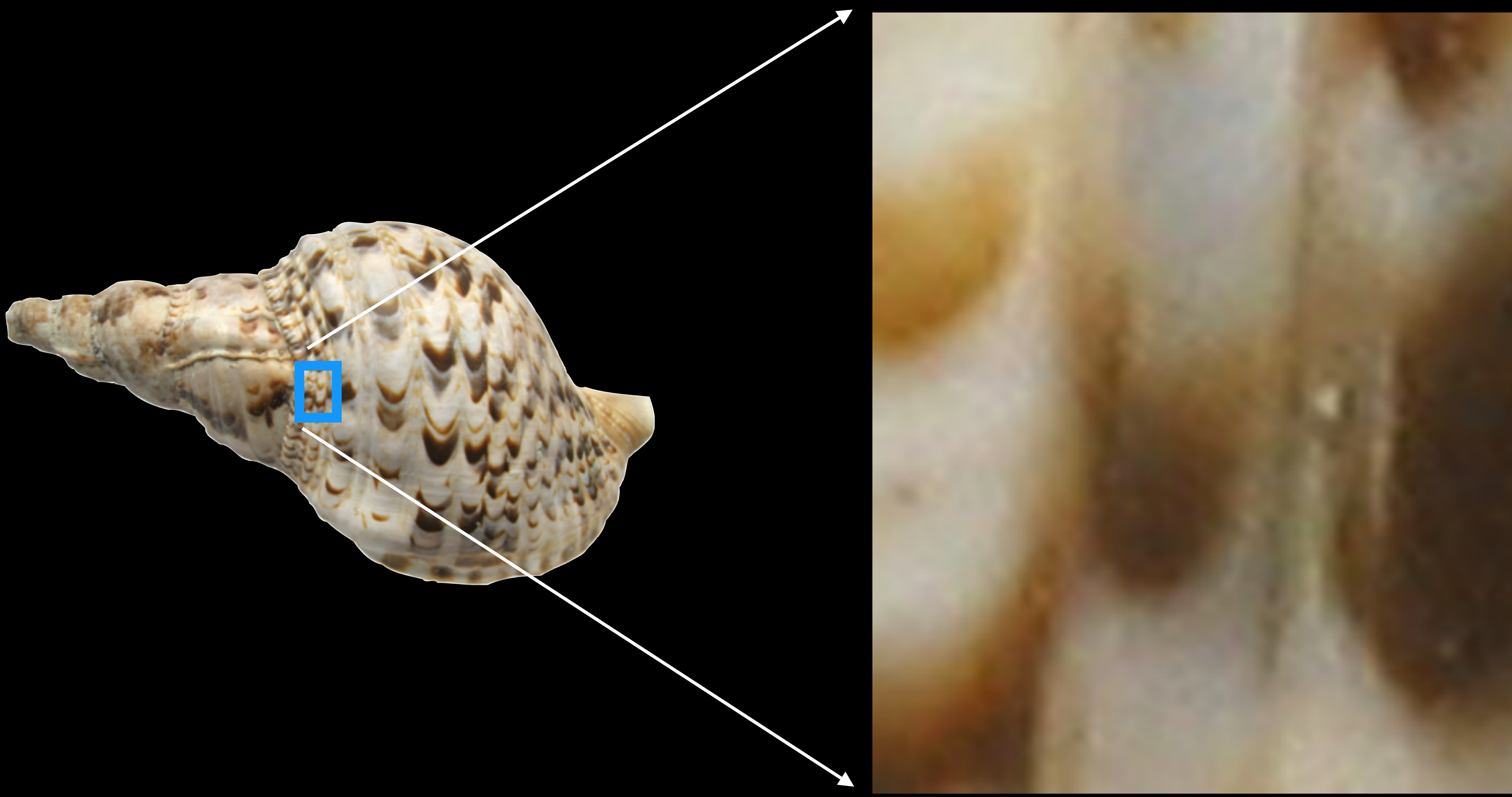
Continuum limit:

$$\xi(g) \to \infty$$

Decreasing g equivalent to **zooming in**







Laplace interpolation

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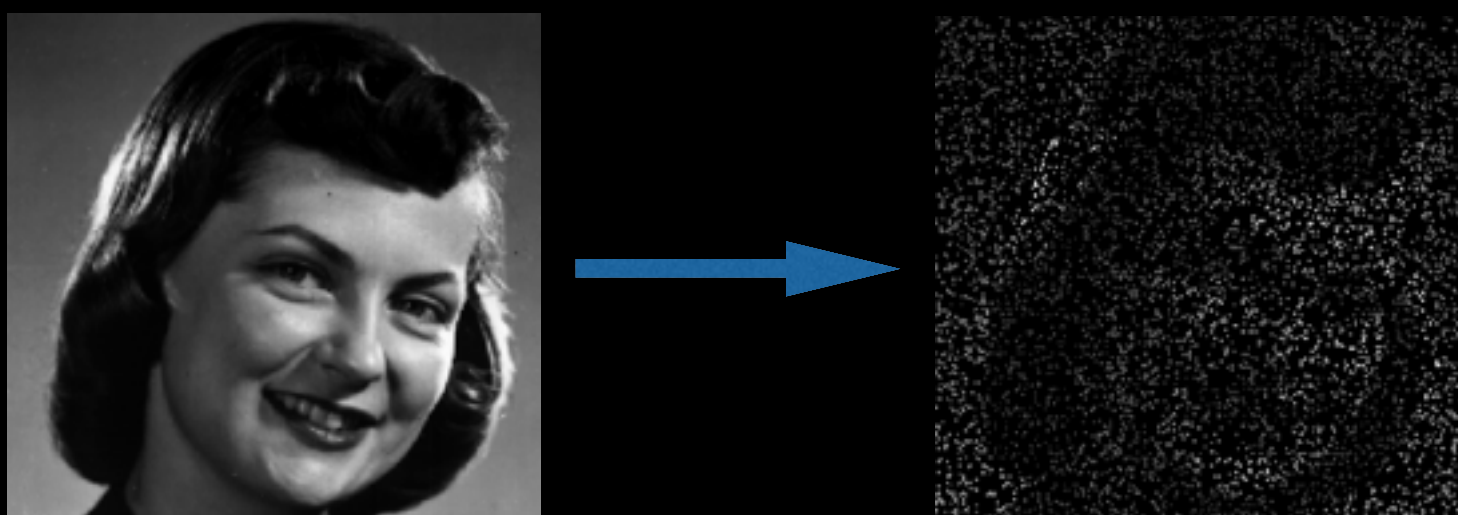
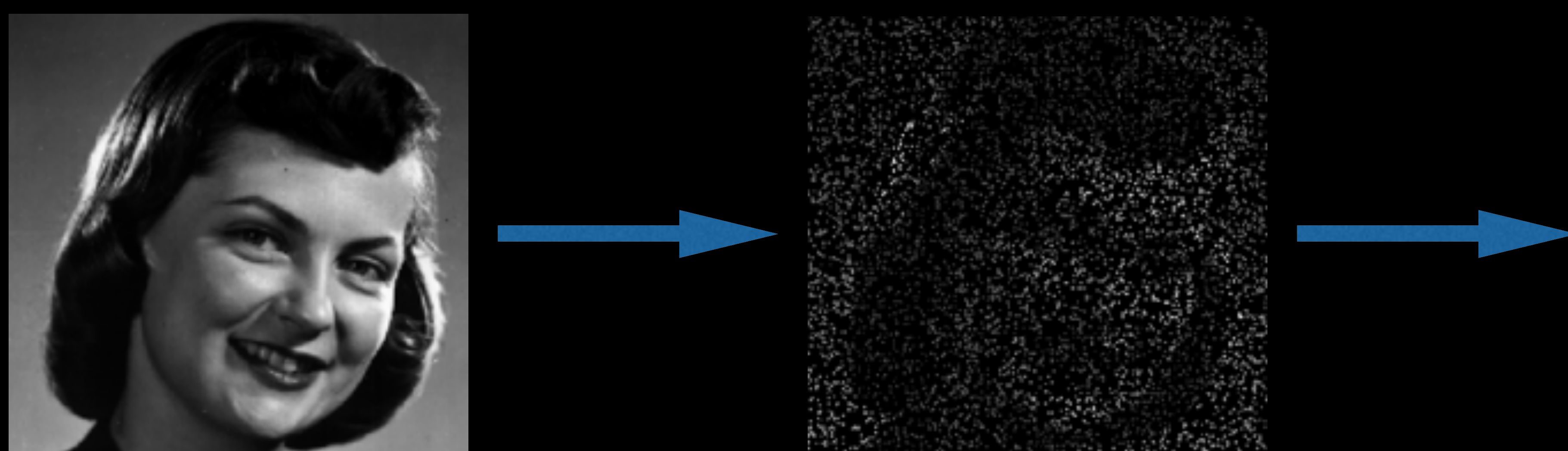
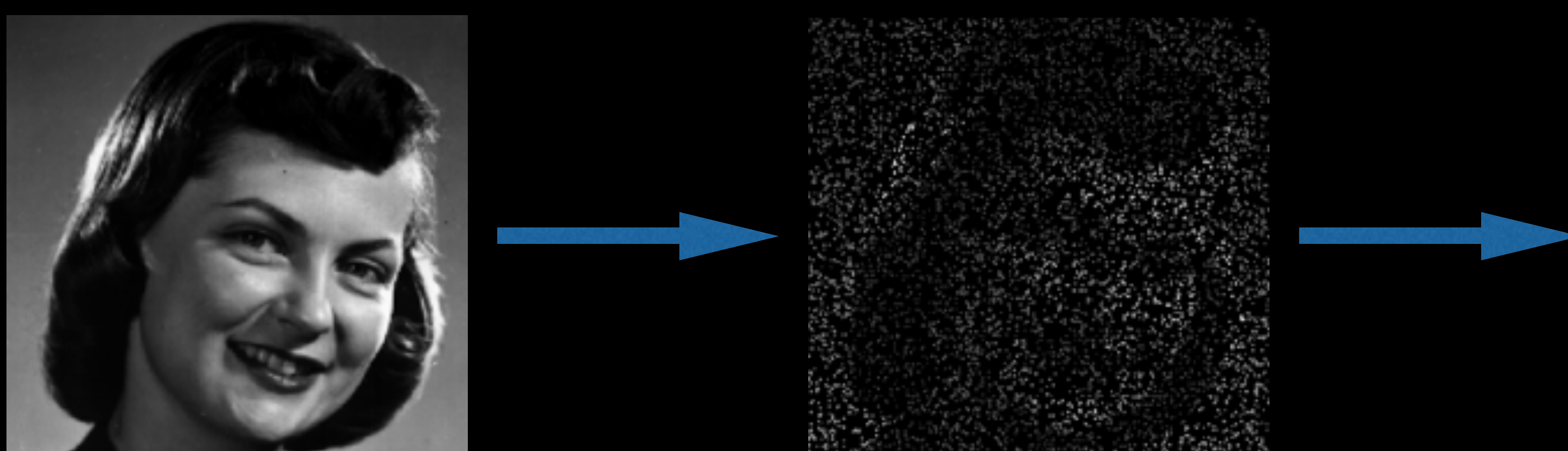


Laplace interpolation



Laplace interpolation



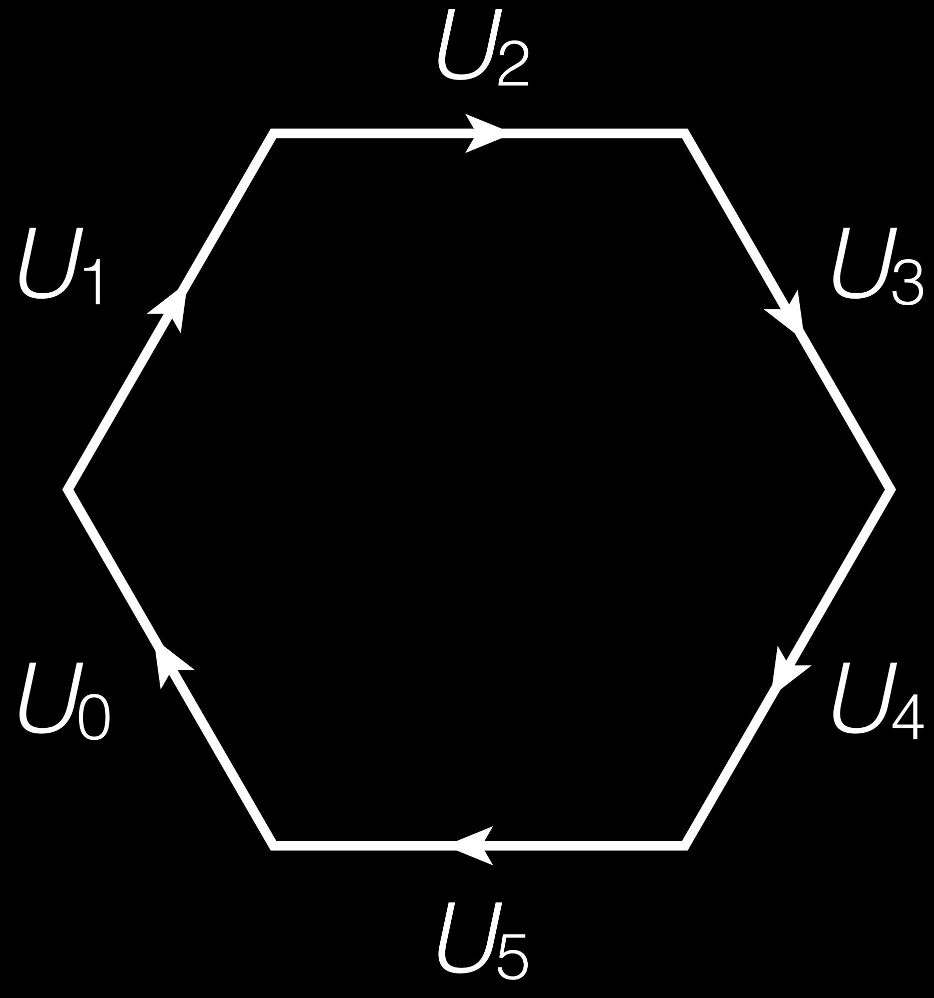
$$\nabla^2 \phi = 0$$

Laplace interpolation

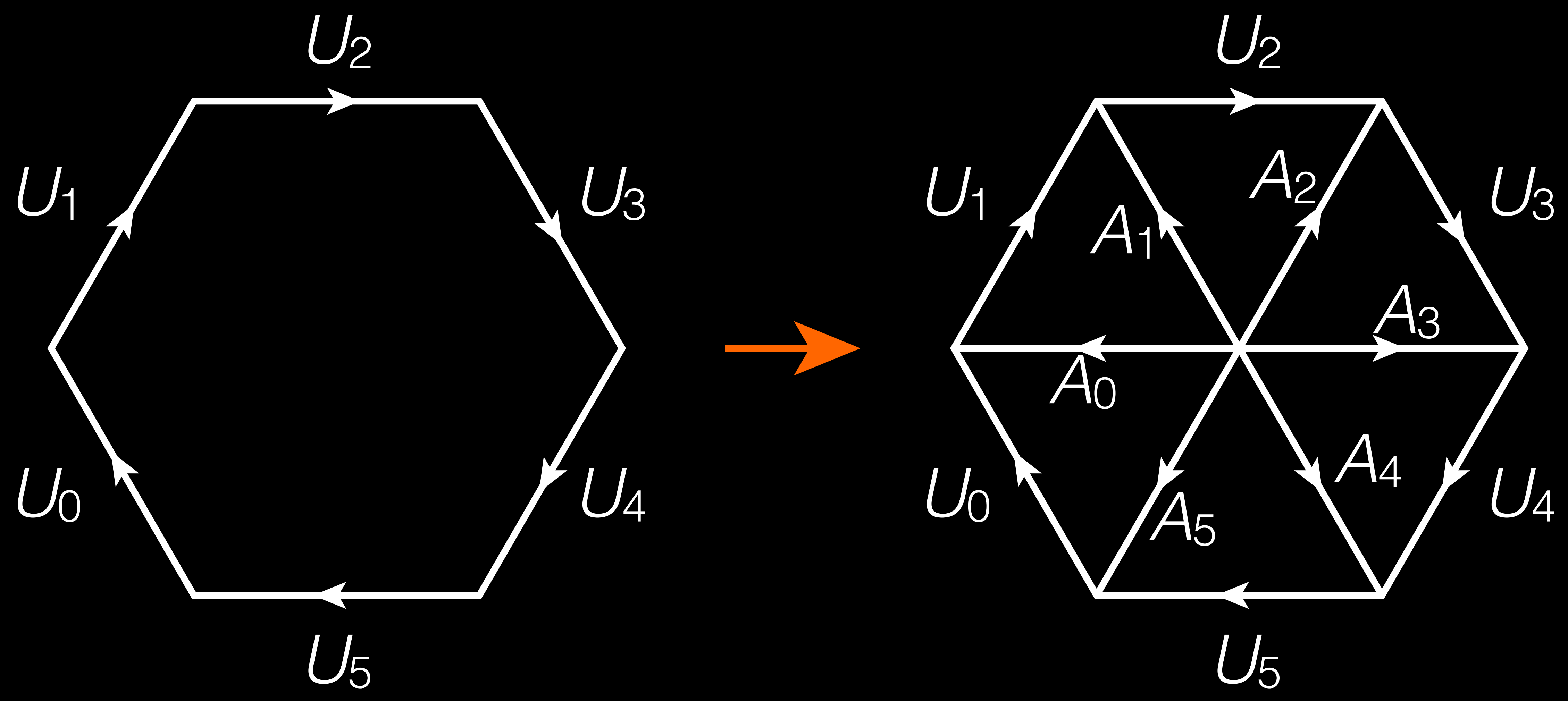
$$\nabla^2 \phi = 0$$

What is interpolation for nonabelian gauge fields?

Curvature interpolation



Curvature interpolation



Minimise:

$$-2\sum_{j=0}^{n-1} \mathrm{Re}(\mathrm{tr}(U_j A_j^\dagger A_{j-1}))$$

(classical) solution

$$A_j = \theta(j,k)^\dagger \eta^\dagger U_0 \cdots U_j, \quad k = 0, 1, \ldots, n-1$$

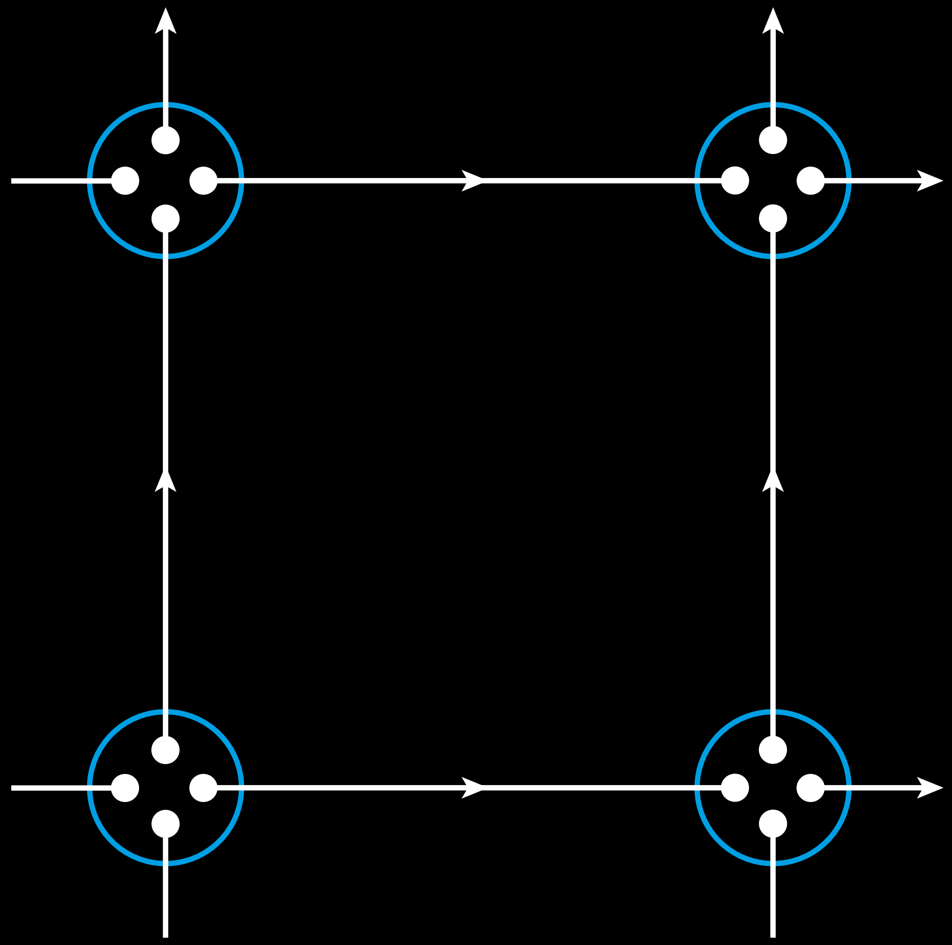
where

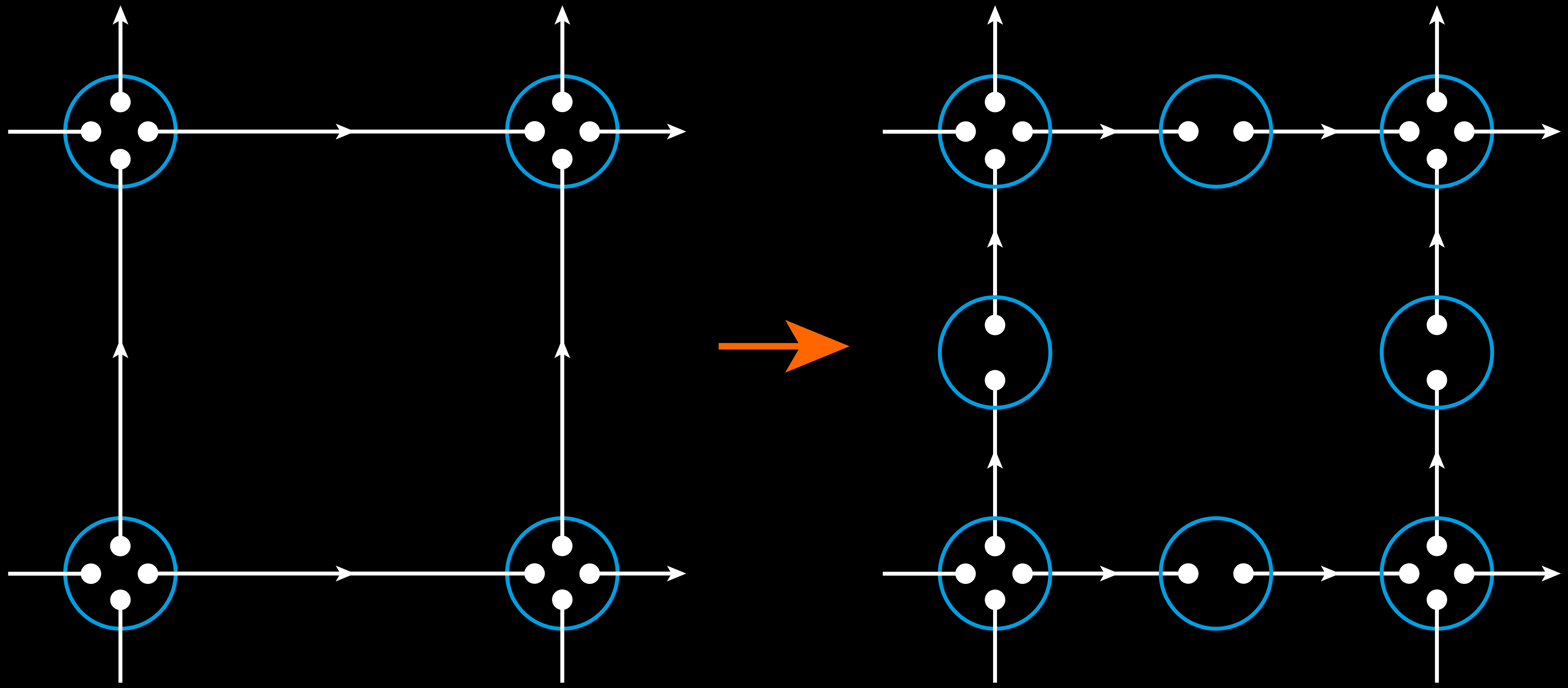
$$\eta^\dagger U_{n-1}^\dagger \cdots U_0^\dagger \eta = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

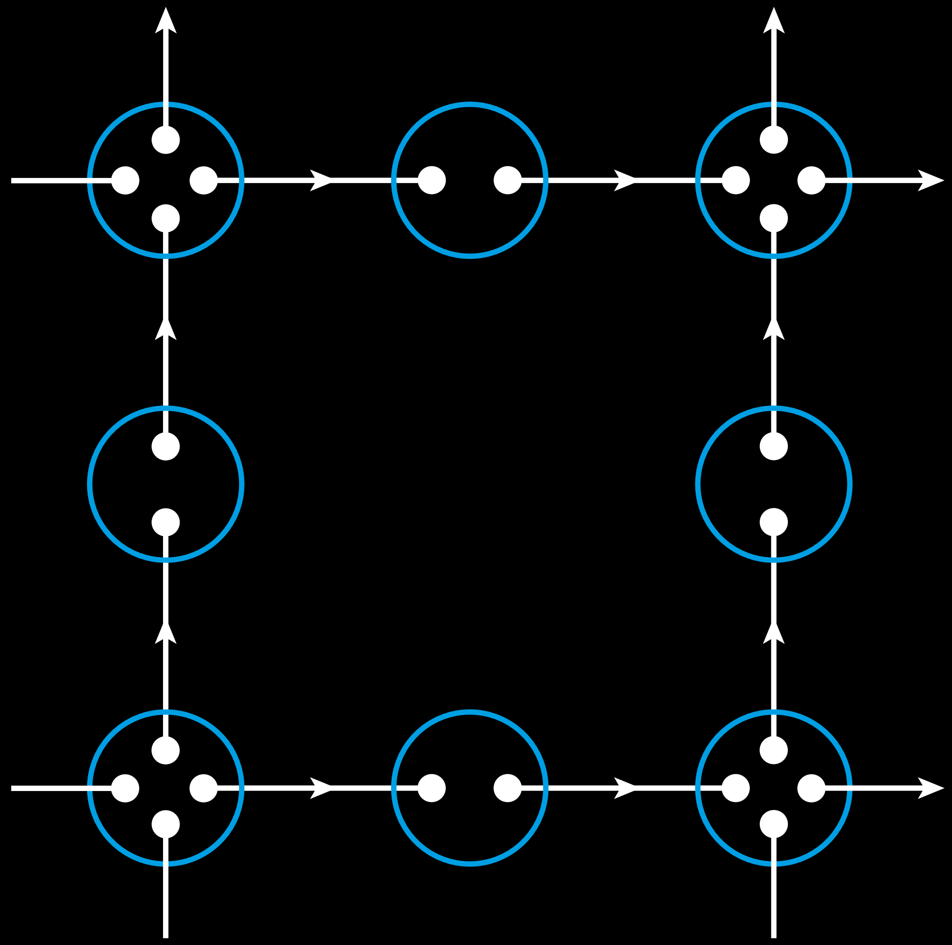
and

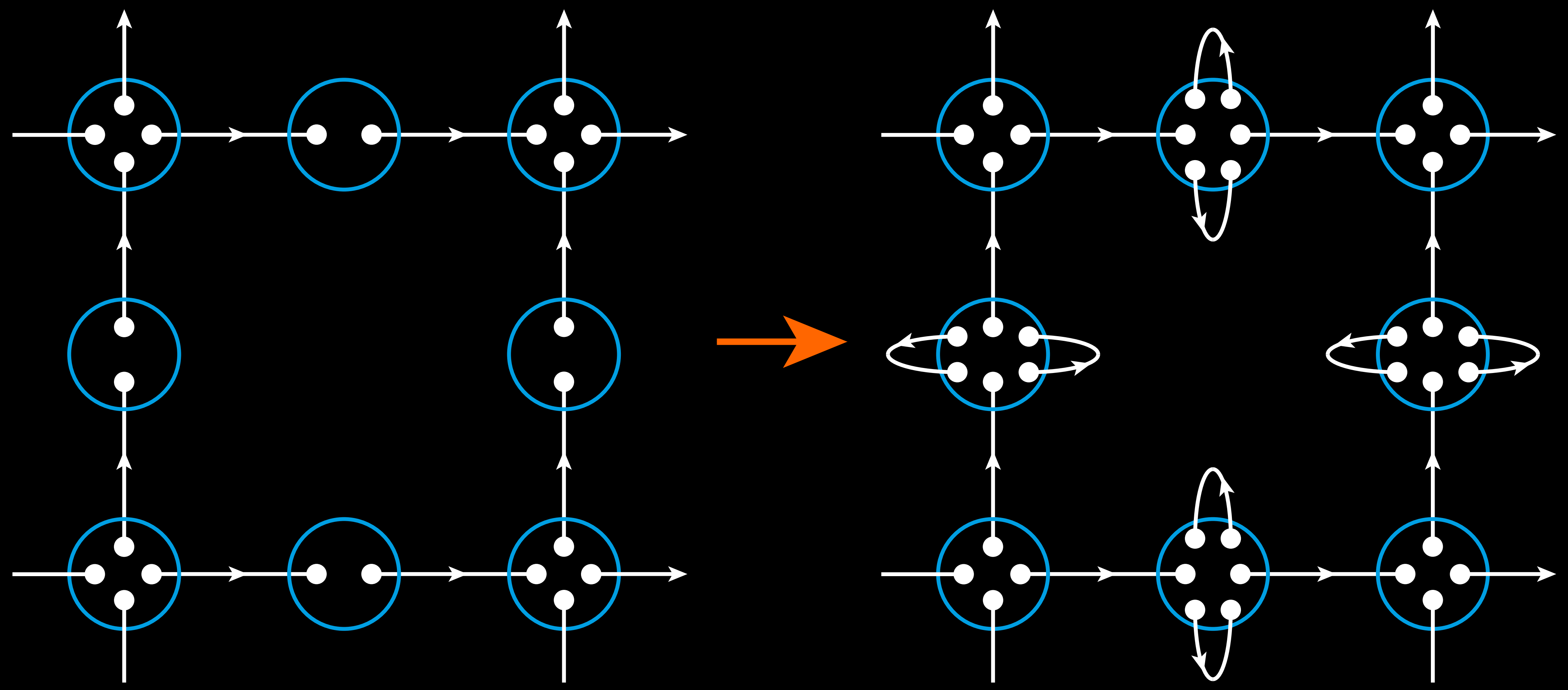
$$\theta(j,k) \equiv \begin{pmatrix} e^{-i\frac{j}{n}(\phi - 2\pi k)} & 0 \\ 0 & e^{i\frac{j}{n}(\phi - 2\pi k)} \end{pmatrix}$$

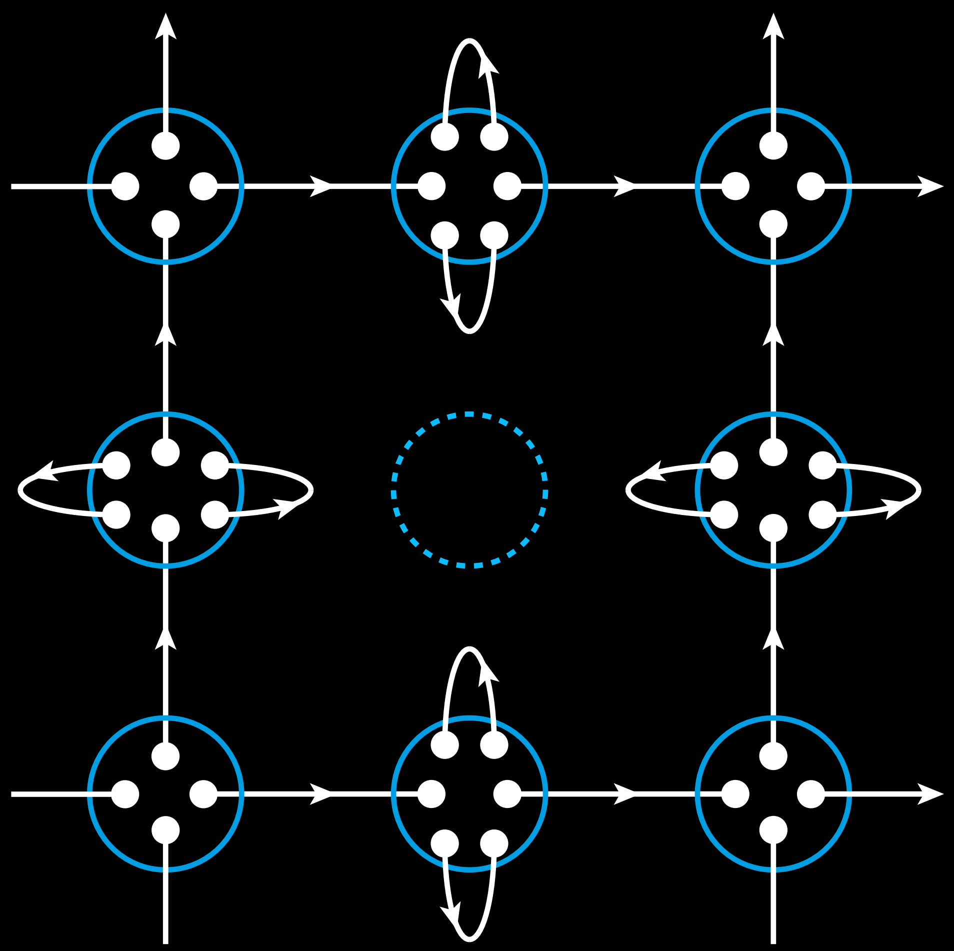
Quantum interpolation

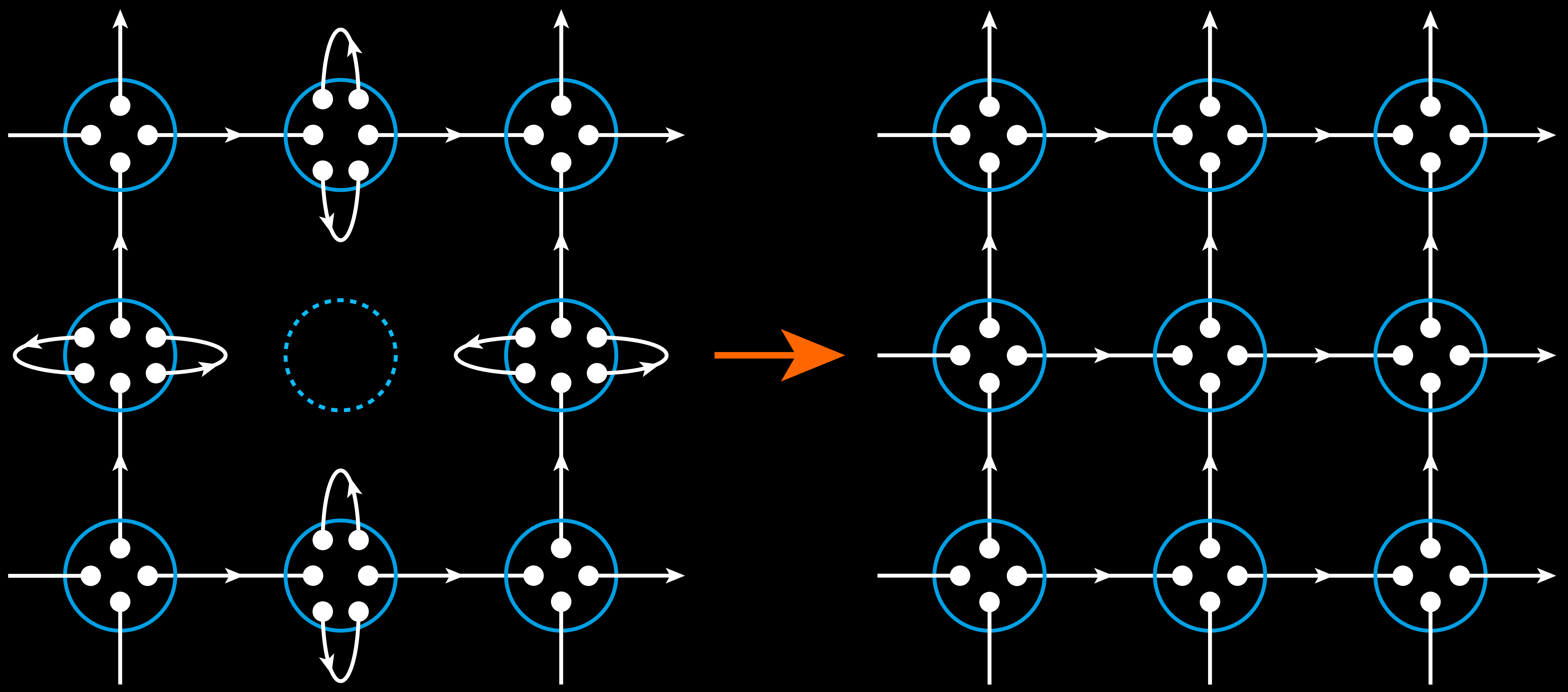












This process is an **isometry** $\mathcal{U}$

Ground-state ansatz

$$|\Phi(m)\rangle \equiv \mathcal{U}_m \mathcal{U}_{m-1} \cdots \mathcal{U}_1 |\Omega(\infty)\rangle$$

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$$m \Rightarrow \xi(m) \Rightarrow a(m)$$

$|\Phi(m)\rangle$ is a tree tensor network state (MERA)

E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. **43**, 4452 (2002)
M. Aguado and G. Vidal, Phys. Rev. Lett. **100**, 070404 (2008)
O. Buerschaper, M. Aguado, and G. Vidal, Phys. Rev. B **79**, 085119 (2009)
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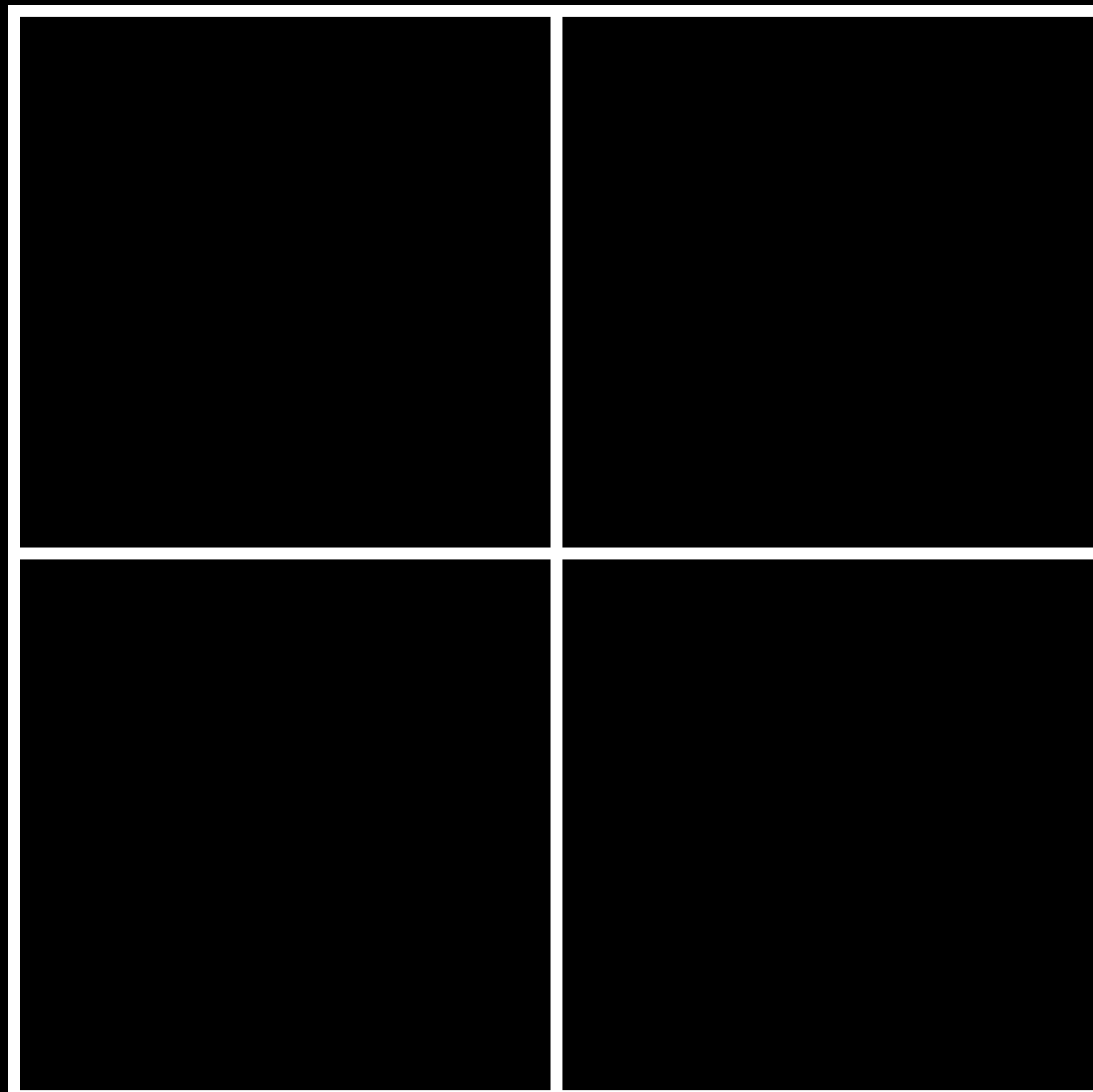
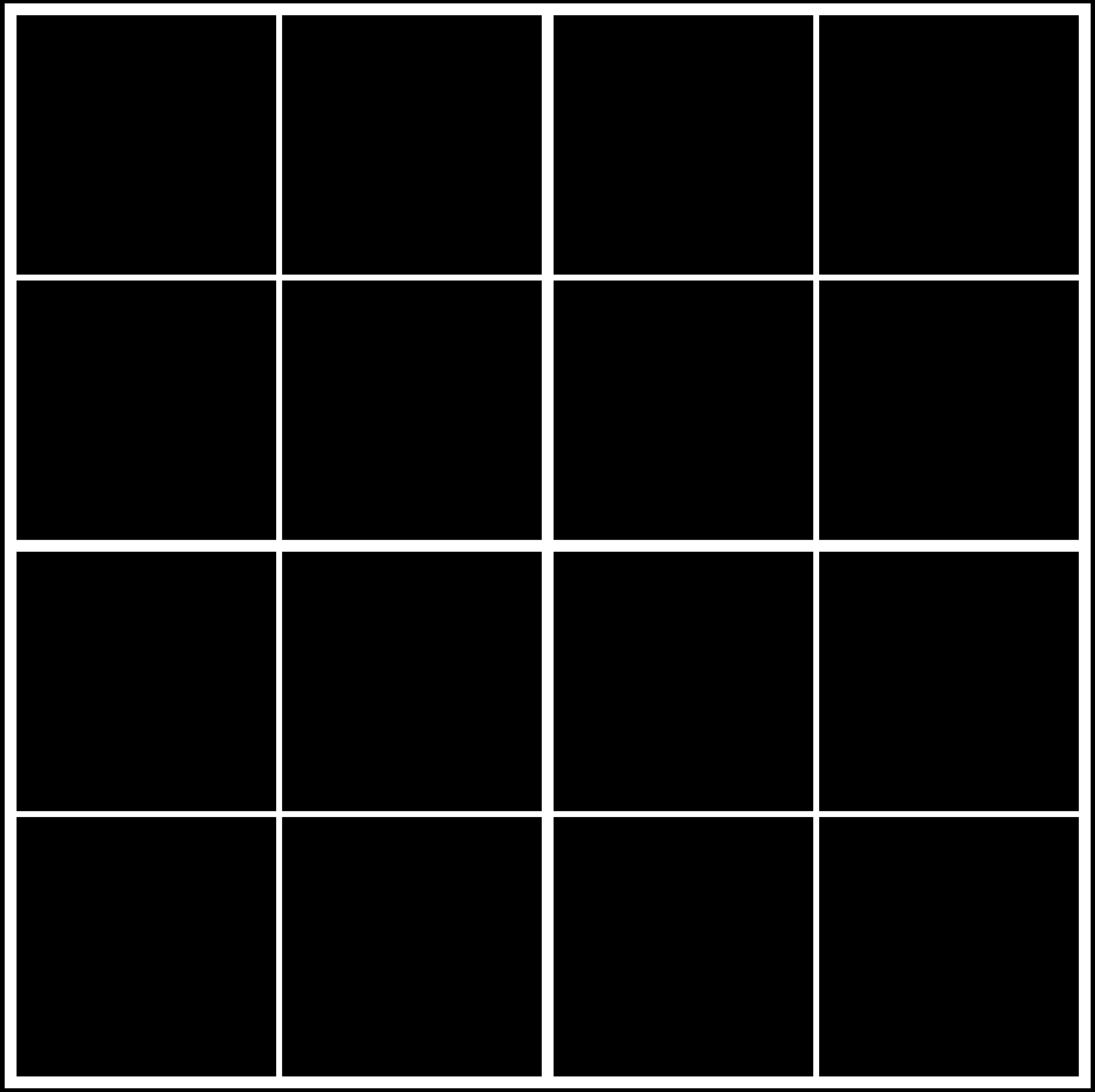
How to match g to m?

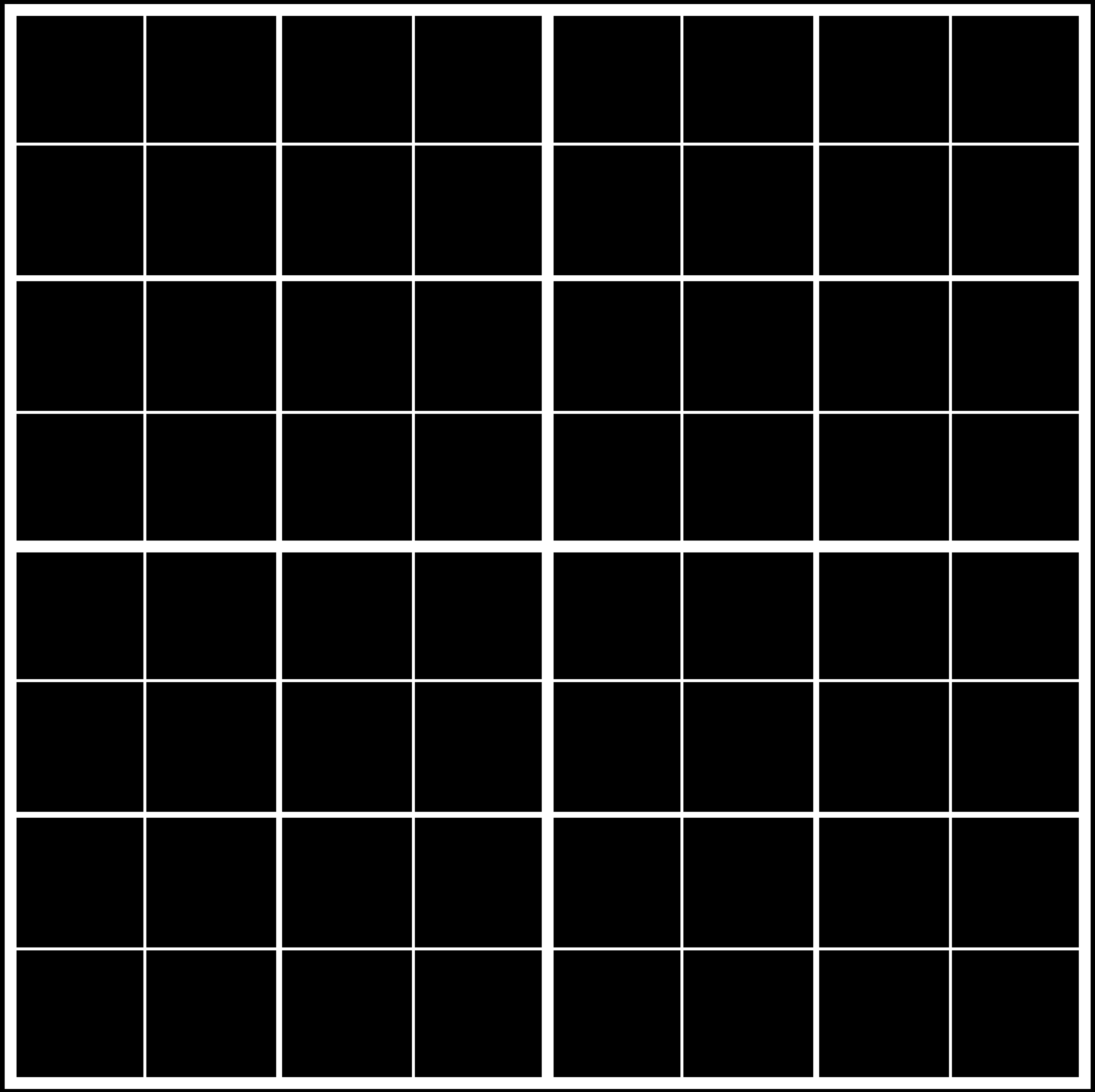
4. Renormalisation

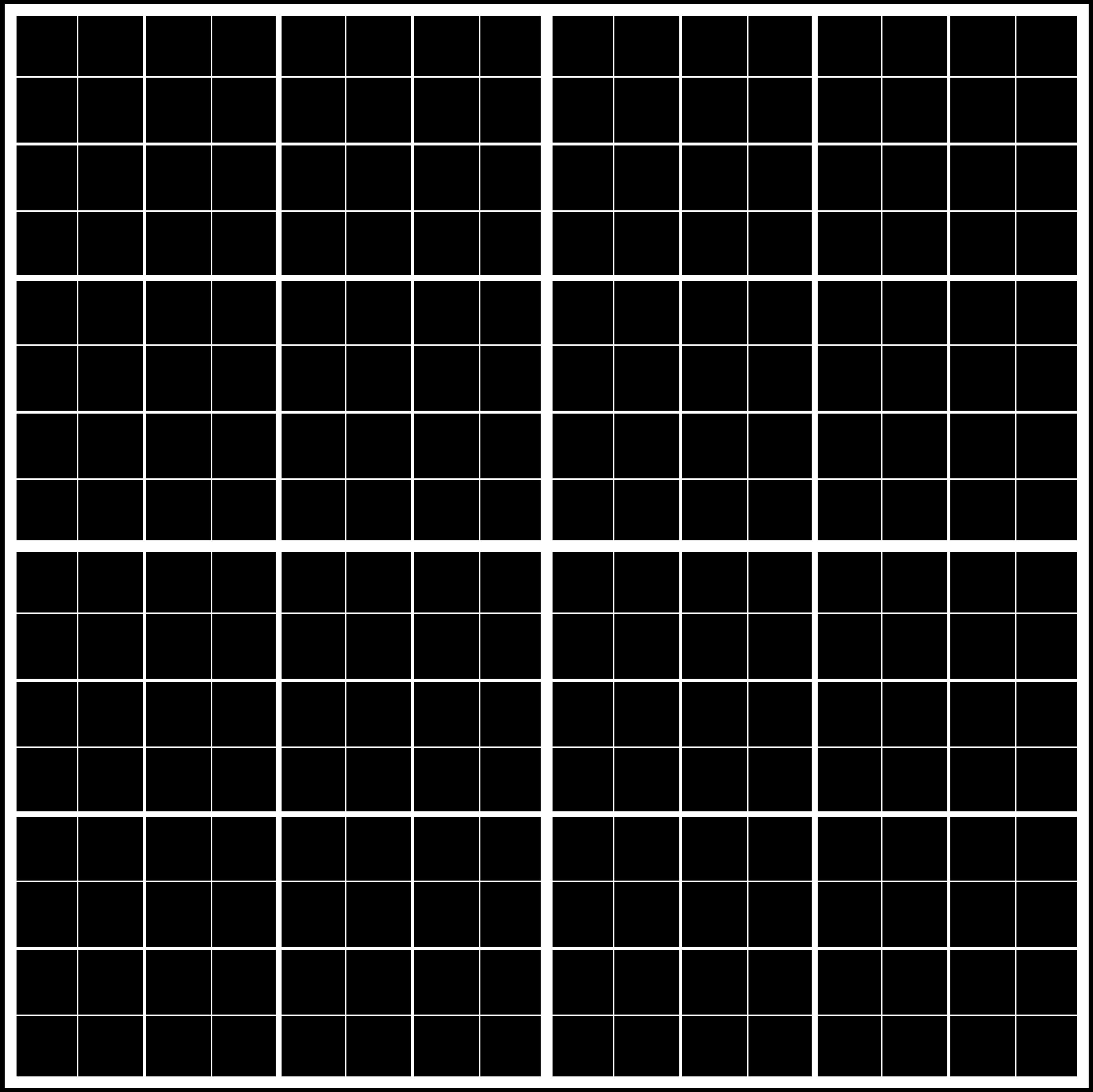
(see C. Bény and T. J. Osborne, arXiv:1310.3188)

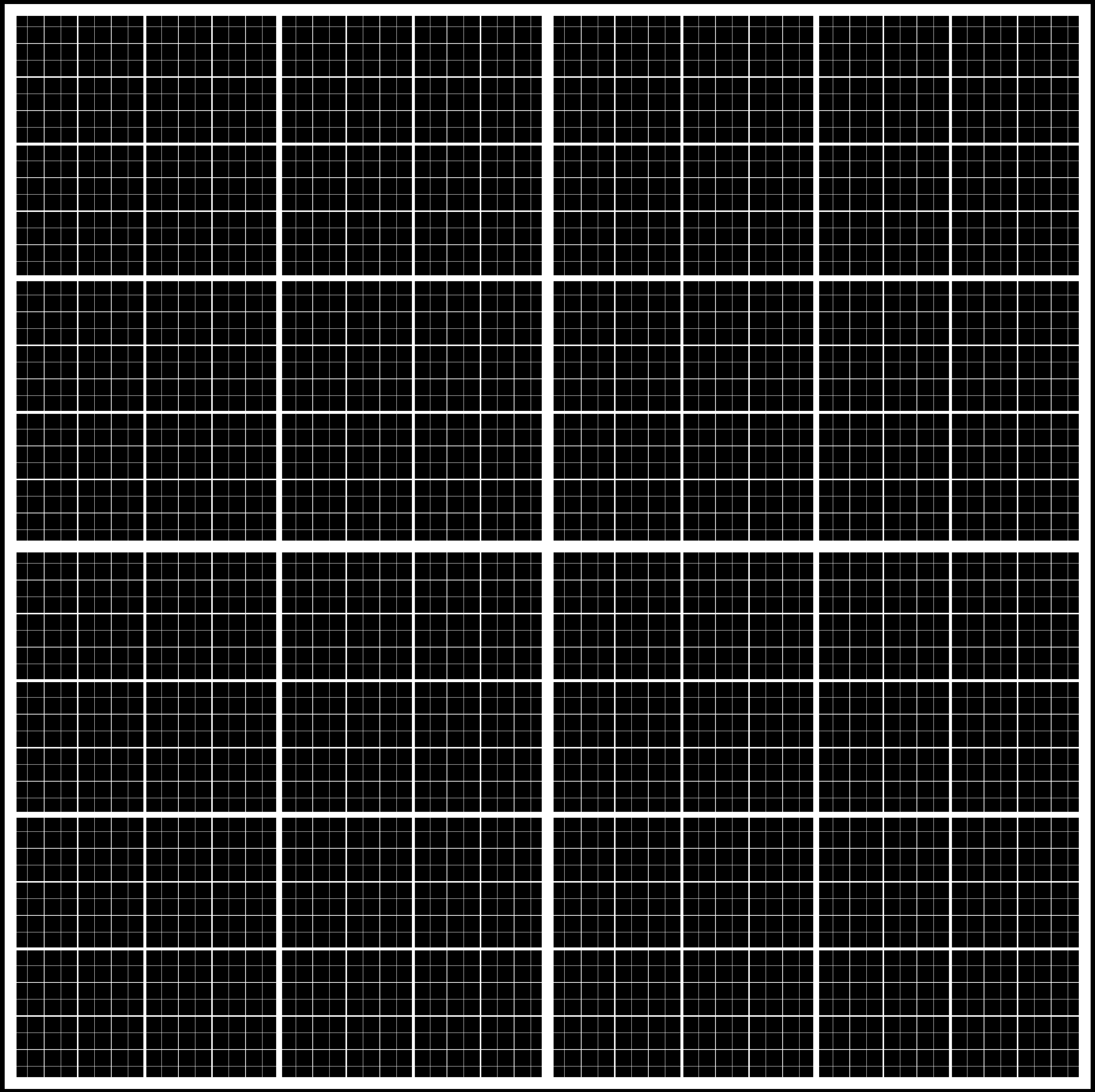
5. Continuum limit











Where are the quantum fields?

Fluctuation operators

$$\widehat{F}_{\mu\nu}(f) \equiv \lim_{a\to 0} Z(a) \sum_j f(aj)(\mathrm{tr}(\widehat{u}_{\Box_{\mu\nu}}) - \langle \mathrm{tr}(\widehat{u}_{\Box_{\mu\nu}})\rangle)$$

$$\widehat{E}_{\mu}(f) \equiv \lim_{a\to 0} Z(a) \sum_j f(aj)(\widehat{\ell}_{\mu} - \langle \widehat{\ell}_{\mu}\rangle)$$

K. Hepp and E. H. Lieb, Helv. Phys. Acta **46**, 573 (1973).
A. F. Verbeure, Many-Body Boson Systems (Springer, London, 2011)

Lorentz invariance?

$$|E_k\rangle = \lim_{a\to 0} \sum_j \int_{-\infty}^{\infty} e^{iajk} e^{i\frac{t}{a}H} \widetilde{F}_{\mu\nu}^{(a)}(j)|\Omega_a\rangle \, dt$$

J. Haegeman, S. Michalakis, B. Nachtergaele, T. J. Osborne, N. Schuch, F. Verstraete, Phys. Rev. Lett. **111**, 080401 (2013)

Lorentz invariance:

$$\omega(k) = \sqrt{k^2 + m^2}$$

Quantum-information inspired ground state ansatz for Yang-Mills