Fawzi-Renner Inequality by State Redistribution

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based on arXiv:1411.4921 with

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Strong Subadditivity

(Lieb and Ruskai '73) For ρ_{CRB}

$$S(CB) + S(RB) \geq S(CRB) + S(B)$$

Strong Subadditivity and CMI

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Conditional Mutual Information (CMI)

$$I(C:R|B) := S(CB) + S(RB) - S(CRB) - S(B)$$

By SSA: $I(C:R|B) \geq 0$

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By SSA: $I(C:R|B) \geq 0$

Can we improve SSA?

I.e. Is there a positive non-identically-zero function f *s.t.*

$$I(C:R|B) \geq f(\rho_{CRB})$$

CMI for Probability Distributions

For a probability distribution p_{XYZ}:

$$I(X:Y|Z) = \mathop{\mathbb{E}}_{z' \sim p(z)} I(X:Y)_{p(x,y|z=z')}$$

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and

$$I(X:Y|Z) = \min_{q \in \mathrm{MC}} S(p_{XYZ} || q_{XYZ})$$

MC := {q : x-z-y form a *Markov chain*}

Relative entropy: $$S(p||q) := \sum_i p_i \log(p_i/q_i)$$

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Relative entropy: $S(p||q) := \sum_i p_i \log(p_i / q_i)$

Since $S(p||q) \geq \|p-q\|_1^2 / \ln(4)$ (Pinsker's inequality),

$I(X{:}Y|Z) \leq \varepsilon$ implies p is $O(\varepsilon^{1/2})$ close to a Markov chain

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and

$$I(X:Y|Z) = \min_{q \in \mathrm{MC}} S(p_{XYZ} \| q_{XYZ})$$???

How about for quantum states?

We don’t know how to condition on quantum information…

The second equality could still work. But what is the set of “quantum Markov chains”?

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Classical:

i) *x*-*z*-*y* Markov if *x* and *y* are independent conditioned on *z*

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Quantum:

i) ρ_{CRB} Markov quantum state if *C* and *R* are independent conditioned on *B*, i.e. $H_B \simeq \bigoplus_k H_{B_{L,k}} \otimes H_{B_{R,k}}$ and

$$\rho_{CRB} = \bigoplus_k p_k \rho_{CB_{L,k}} \otimes \rho_{B_{R,k}R}$$

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ii) ρ_{CRB} Markov if there is channel $\Lambda : B \to RB$ s.t. $\Lambda(\rho_{CB}) = \rho_{CRB}$

Quantum Markov States vs CMI

(Hayden, Jozsa, Petz, Winter '03) *I(C:R|B)=0* iff ρ_{CRB} is quantum Markov

$$I(C:R|B)=0$$

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$$I(C:R|B) = 0$$

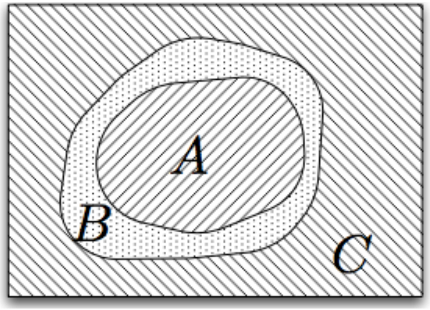
$$\exists \ \Lambda : B \to BR \ \text{ s.t.}$$
$$I_C \otimes \Lambda(\rho_{CB}) = \rho_{CRB}$$

obs:
$$\Lambda(X) = \rho_{RB}^{1/2} \rho_B^{-1/2} X \rho_B^{-1/2} \rho_{RB}^{1/2}$$

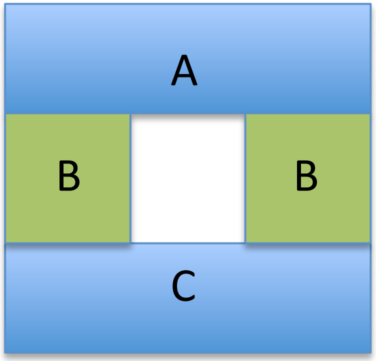
$$\rho_{CRB} = \bigoplus_k p_k \rho_{CB_{L,k}} \otimes \rho_{B_{R,k}R}$$

Applications

(Brown and Poulin '12)
(Quantum Hammersley-Clifford thm) Every many body state with vanishing CMI is the Gibbs state of a commuting model (on triangle-free graphs)



(Lewin and Wen '06)
CMI is equal to twice topological entanglement entropy. If $S(X) = a|\partial X| - \gamma$, I(A:C|B) = 2γ



(Kitaev, unpublished)
If γ=0, the state (+ ancilas) can be created by a constant depth circuit

What if merely CMI ≈ 0? (Ex. $S(X) = a|\partial X| - \gamma + 2^{-O(|\partial X|)}$)

CMI vs Quantum Markov States

Does $I(C:R|B)_\rho \geq \Omega\left(\min_{\sigma_{CRB} \in QMC} S(\rho||\sigma)\right)$?

or just $I(C:R|B) \geq f\left(\min_{\sigma_{CRB} \in \mathrm{QMC}} \|\rho - \sigma\|_1\right)$?

QMC := {σ_{CRB} : $\sigma_{CRB} = \bigoplus_k p_k \rho_{CB_{L,k}} \otimes \rho_{B_{R,k}R}$ }

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NO!!!

QMC := {σ_{CRB} : $\sigma_{CRB} = \bigoplus_k p_k \rho_{CB_{L,k}} \otimes \rho_{B_{R,k}R}$ }

(Ibinson, Linden, Winter '06; Christandl, Schuch, Winter '11)

If ρ_{CRB} is QMC, ρ_{CR} is *separable* ($\sigma_{CR} = \sum_k p_k \rho_{C,k} \otimes \rho_{R,k}$)

For an extension ρ_{CRB} of the *d x d* anti-symmetric state ρ_{CR},

$$I(C:R|B)_\rho \leq 2/d \qquad \min_{\sigma_{CR} \in \text{SEP}} \|\rho_{CR} - \sigma_{CR}\|_1 \geq 1/4$$

Partial Progress

(B., Christandl, Yard '10; Li, Winter '12; B., Harrow, Lee, Peres '13)

$$I(C:R|B) \geq \min_{\sigma_{CR} \in \mathrm{SEP}} \max_{M_R \in \mathcal{M}} S(I_C \otimes M_R(\rho_{CR}) || I_C \otimes M_R(\sigma_{CR}))$$

$$\mathcal{M} := \{M(X) \sum_i \mathrm{tr}(L_i X)|i\rangle\langle i|, \quad L_i \geq 0, \sum_i L_i = I\}$$

Applications: quasipolynomial-time algorithm for testing separability, faithfulness squashed entanglement, SoS hierarchy, etc…

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How about the distance to QMC? Open question:

$$I(C:R|B) \overset{?}{\geq} \min_{\sigma \in \mathrm{QMC}} \max_{M_C, M_R, M_B} S(M_C \otimes M_R \otimes M_B(\rho) || M_C \otimes M_R \otimes M_B(\sigma))$$

Approximate Reconstruction

"Small CMI" and "being close to QMC state" are *not* equivalent
(in a dimensional independent way for trace norm or fidelity)

Classically:

$$I(X:Y|Z) = \min_{q \in \mathrm{MC}} S(p||q)$$
$$= \min_{\Lambda : Z \to YZ} S(p_{XYZ}||I_X \otimes \Lambda(p_{XZ}))$$

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Conjecture (I. Kim; A. Kitaev; L. Zhang; Berta, Seshadreesan, Wilde, …)

$$I(C:R|B) \geq \min_{\Lambda : B \to RB} S(\rho_{CRB}||I_C \otimes \Lambda(\rho_{CB}))$$

or $$I(C:R|B) \geq f\left(\min_{\Lambda : B \to RB} \|\rho_{CRB} - I_C \otimes \Lambda(\rho_{CB})\|_1\right)$$

Fawzi-Renner Inequality

thm (Fawzi, Renner '14)

$$I(C:R|B) \geq \min_{\Lambda: B \to RB} -2\log(F(\rho_{CRB}, I_C \otimes \Lambda_B(\rho_{CB})))$$

Fidelity: $F(\rho, \sigma) := \mathrm{tr}(\sqrt{\sigma^{1/2}\rho\sigma^{1/2}})$

½ Renyi Relative Entropy: $S_{1/2}(\rho||\sigma) := -2\log(F(\rho,\sigma))$

We have: $S(\rho||\sigma) \geq S_{1/2}(\rho||\sigma)$

Weaker than classical inequality

Strengthening of Fawzi-Renner

thm (B., Harrow, Oppenheim, Strelchuk '14)

$$I(C:R|B)_\rho$$

$$\geq \lim_{n\to\infty} \min_{\Lambda_n : B^n \to B^n C^n} \frac{1}{n} S(\rho_{BCR}^{\otimes n} || \Lambda_n \otimes I_{R^n}(\rho_{BR}^{\otimes n}))$$

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$$\geq \min_{\Lambda : B \to BC} \mathbb{M}S(\rho_{BCR} || \Lambda \otimes I_R(\rho_{BR}))$$

$$\mathbb{M}S(\rho||\sigma) := \max_{M \in \mathcal{M}} S(M(\rho)||M(\sigma))$$

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From: $S(\rho||\sigma) \geq S_{1/2}(\rho||\sigma)$ and $\min_{M \in \mathcal{M}} F(M(\rho), M(\sigma)) = F(\rho, \sigma)$

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- From: Properties relative entropy (Piani '09)
- From: State Redistribution Protocol (Devetak, Yard '04)

Comparison Lower Bounds on CMI

$$I(C:R|B) \geq \min_{\sigma_{CR} \in \text{SEP}} \max_{M_R \in \mathcal{M}} S(I_C \otimes M_R(\rho_{CR}) || I_C \otimes M_R(\sigma_{CR}))$$

1. State Redistribution:

$$I(C:R|B) \quad \geq \quad E_R^\infty(\rho_{C:BR}) - E_R^\infty(\rho_{C:B})$$

Regularized relative entropy of entanglement:

$$E_R^\infty(\rho_{A:B}) := \lim_{n \to \infty} \frac{1}{n} \min_{\sigma_{A^n B^n} \in \text{SEP}} S(\rho_{AB}^{\otimes n} || \sigma_{A^n B^n})$$

2. Hypothesis Testing:

$$E_R^\infty(\rho_{C:BR}) - E_R^\infty(\rho_{C:B}) \geq \min_{\sigma_{CR} \in \text{SEP}} \max_{M_R \in \mathcal{M}} S(I_C \otimes M_R(\rho_{CR}) || I_C \otimes M_R(\sigma_{CR}))$$

$$I(C:R|B) \geq \min_{\Lambda: B \to BC} \mathbb{M}S(\rho_{BCR} || \Lambda \otimes I_R(\rho_{BR}))$$

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$$I(C:R|B) \geq \lim_{n \to \infty} \min_{\Lambda_n : B^n \to B^n C^n} \frac{1}{n} S(\rho_{BCR}^{\otimes n} || \Lambda_n \otimes I_{R^n}(\rho_{BR}^{\otimes n}))$$

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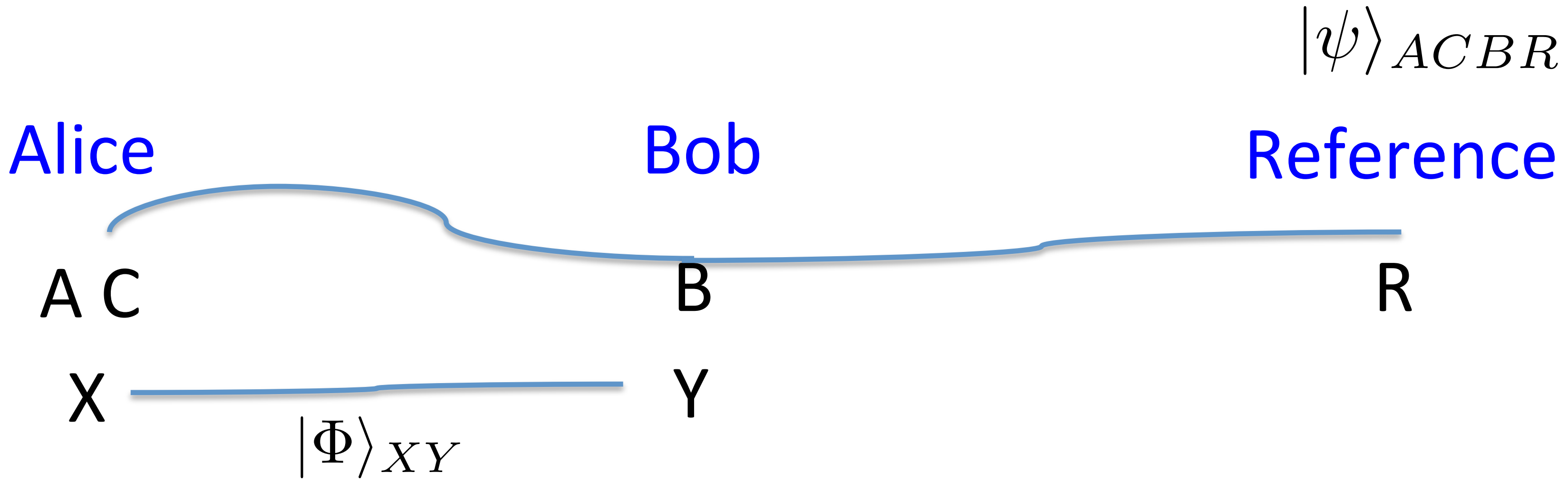
State Redistribution



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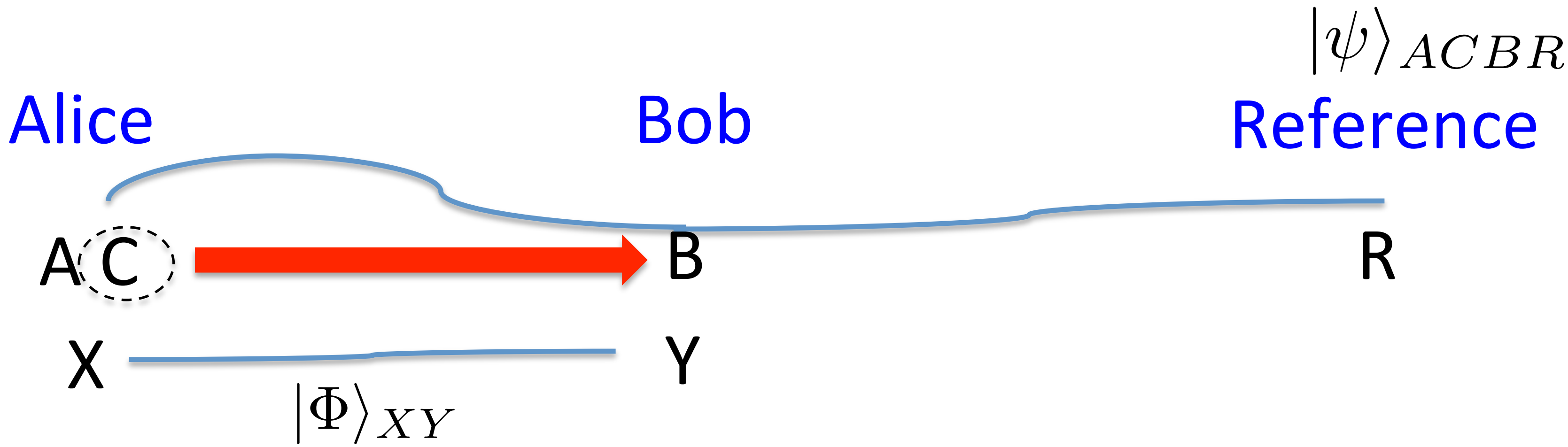

(Devetak, Yard '06)

Optimal quantum communication rate: *½ I(C:R|B)*

I.e. there exist encodings E_n : $A^nC^nX_n$ -> A^nG_n and
decodings D_n : $B^nG_nY_n$ -> B^nC^n s.t.

$$\lim_{n\to\infty} \| D_n \circ E_n(|\psi\rangle\langle\psi|_{ABCR}^{\otimes n} \otimes \Phi_{X_nY_n}) - |\psi\rangle\langle\psi|_{ABCR}^{\otimes n}\|_1 = 0$$

$$\lim_{n\to\infty} \frac{\log \dim(G_n)}{n} = \frac{1}{2} I(C:R|B)$$

Proof of …

… $I(C:R|B) \geq \lim_{n\to\infty} \min_{\Lambda_n : B^n \to B^n C^n} \frac{1}{n} S(\rho_{BCR}^{\otimes n} || \Lambda \otimes I_{R^n}(\rho_{BR}^{\otimes n}))$

Idea: Consider the optimal state redistribution protocol and replace the quantum communication by white noise (maximally mixed state)

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Idea: Consider the optimal state redistribution protocol and replace the quantum communication by white noise (maximally mixed state)

After encoding: $\phi_{G_n Y_n A^n B^n R^n} := E_n \otimes I_{B^n R^n Y_n}(|\psi\rangle\langle\psi|_{ABCR}^{\otimes n} \otimes \Phi_{X_n Y_n})$

As $\phi_{G_n Y_n B^n R^n} \leq \dim(G_n) I_{G_n} \otimes \phi_{Y_n B^n R^n}$ and $\phi_{Y_n B^n R^n} = \tau_{Y_n} \otimes \rho_{BR}^{\otimes n}$

$$(D_n \otimes I_{R^n})(\tau_{G_n} \otimes \tau_{Y_n} \otimes \rho_{BR}^{\otimes n}) \geq \dim(G_n)^{-2} (D_n \otimes I_{R^n})(\phi_{G_n Y_n B^n R^n})$$

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By op monotonicity of the log:

$$\begin{aligned} & S(\rho_{BCR}^{\otimes n} || (D_n \otimes I_{R^n})(\tau_{G_n} \otimes \tau_{Y_n} \otimes \rho_{BR}^{\otimes n})) \\ \leq \quad & S(\rho_{BCR}^{\otimes n} || (D_n \otimes I_{R^n})(\phi_{G_n Y_n B^n R^n})) + 2\log(\dim(G_n)) \end{aligned}$$

0 $\qquad\qquad\qquad\qquad$ $nI(C:R|B)$

Open questions:

- Improve bound to relative entropy (like the classical)

- Prove the bound for the transpose channel

- Prove Li-Winter conjecture:
  For every states *ρ, σ* and channel *Λ* there is a channel *Γ* s.t.
  Γ(Λ(σ)) = σ and

$$S(\rho||\sigma) \geq S(\Lambda(\rho)||\Lambda(\sigma)) + S(\rho||\Gamma \circ \Lambda(\rho))$$

  see (Berta, Lemm, Wilde '14) for partial progress

- Find (more) applications of the FR inequality!
- Find more improvements of SSA; see (Kim '12) for another

- (dis)Prove: $I(C:R|B) \geq$

$$\min_{\sigma \in \text{QMC}} \max_{M_C, M_R, M_B} S(M_C \otimes M_R \otimes M_B(\rho)||M_C \otimes M_R \otimes M_B(\sigma))$$