Fun with QMA and non-unitary gates

Dan Browne

joint work with: **Naïri Usher**

QMA

- Kitaev (1999) - **QMA**: The quantum analogue of **NP**

QMA for beginners

NP

Arthur

Poly-time classical computer

Merlin

Unbounded computational power

NP

Arthur

Merlin

Decision problem:

*"Is there an answer to the **problem of gravity**?"*

Arthur can **verify** validity of proof in **poly-time**

← proof

"***Yes***"

*"And here's the **proof**"*

NP

Arthur

Merlin

Decision problem:

*"Is there an answer to the **problem of gravity**?"*

Arthur can **verify** validity of proof in **poly-time**

*"**No**"*

*"And there is **no fake proof** I could send to you to **trick you** into thinking the answer is yes"*

NP

Arthur

Merlin

Poly-time classical computer

Unbounded computational power

Decision problem

proof ←

*"**Yes = 1**"*
*"And here's the **proof**"*

*Arthur can **verify** validity of proof in **poly-time***

*"**No = 0**"*
*"And there is **no fake proof** …"*

~~NP~~ MA

Arthur

Poly-time classical computer

Decision problem

*Arthur can **verify** validity of proof in **poly-time***

Merlin

Unbounded computational power

*"**Yes = 1**"*
*"And here's the **proof**"*

proof ←

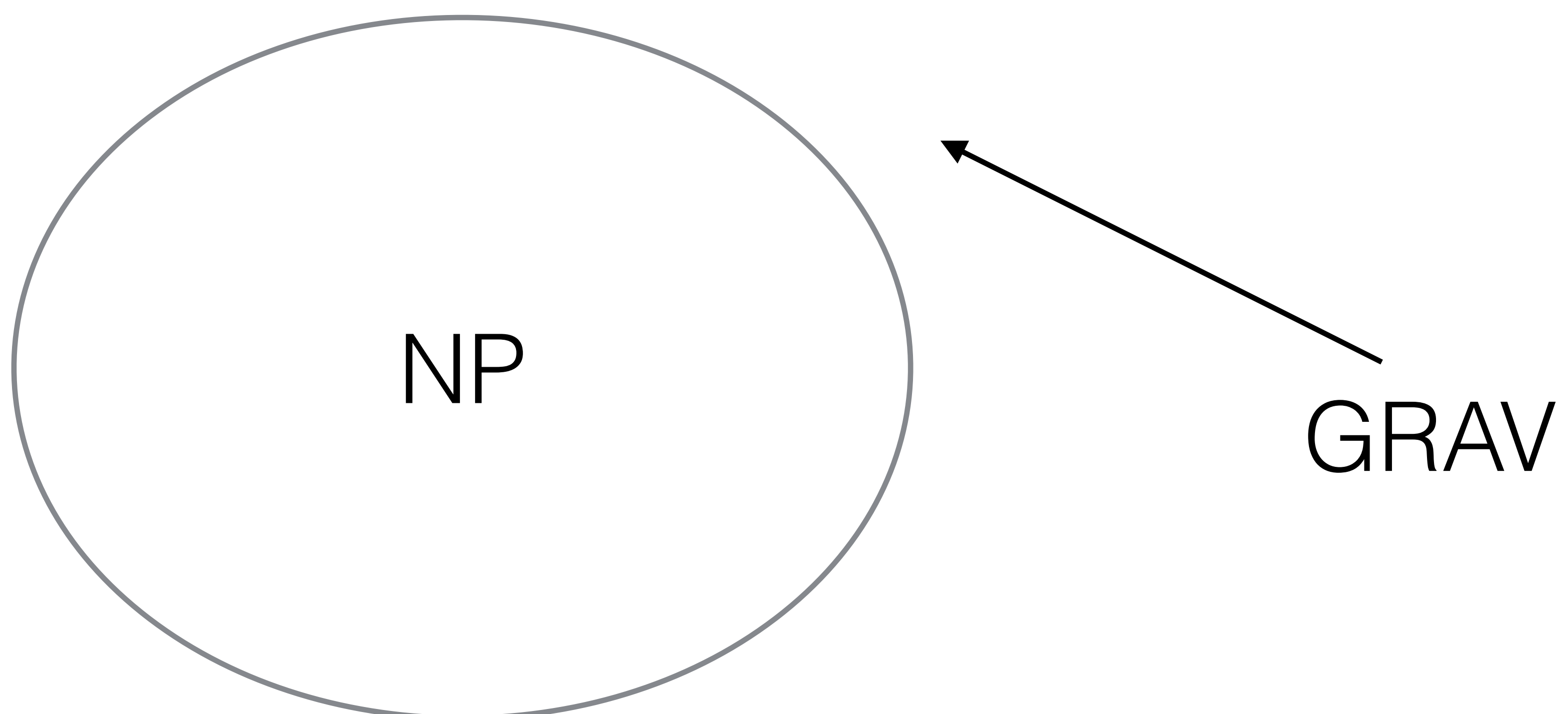
with **"high enough"** probability

*"**No = 0**"*
*"And there is **no fake proof** …"*

*"Is there an answer to the **problem of gravity**?"*

SPOILER: *"I lied. It is not in NP."*

*We need **quantum data!***



QMA

- Kitaev (1999) - **QMA**: The quantum analogue of **NP**

QMA

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Poly-time **quantum** computer

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Unbounded computational power

Decision problem

*Arthur can **verify** validity of proof in **poly-time***

quantum proof

$|\psi\rangle$

with **“high enough”** probability

*“**Yes = 1**”*

*“And here’s the **proof**”*

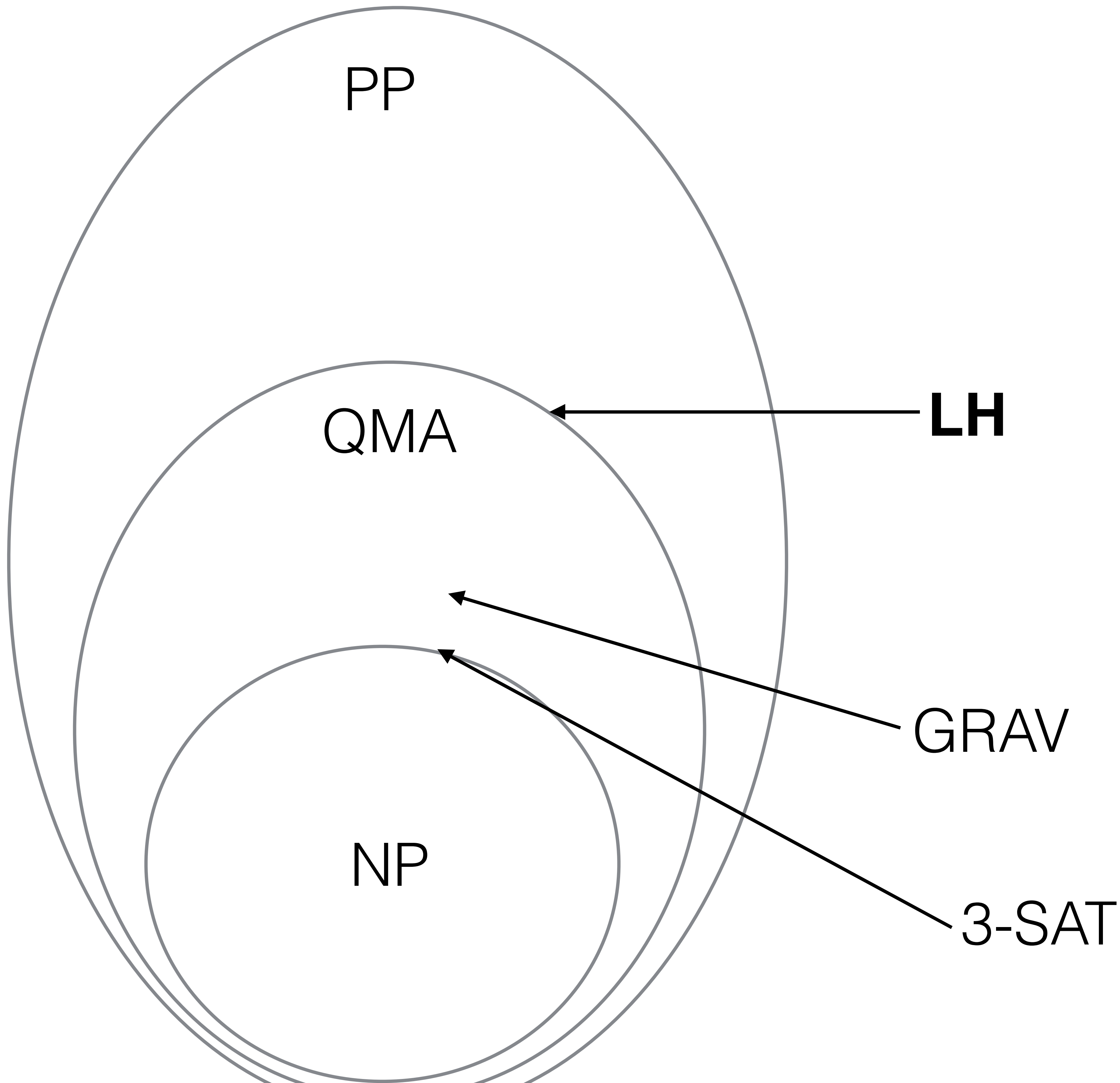
*“**No = 0**”*

*“And there is **no fake proof** …”*

Local Hamiltonian Problem

Kitaev (1999):

LH: k-local Hamiltonian Problem is QMA-complete



Local Hamiltonian Problem

Given: A k-local n-qubit Hamiltonian

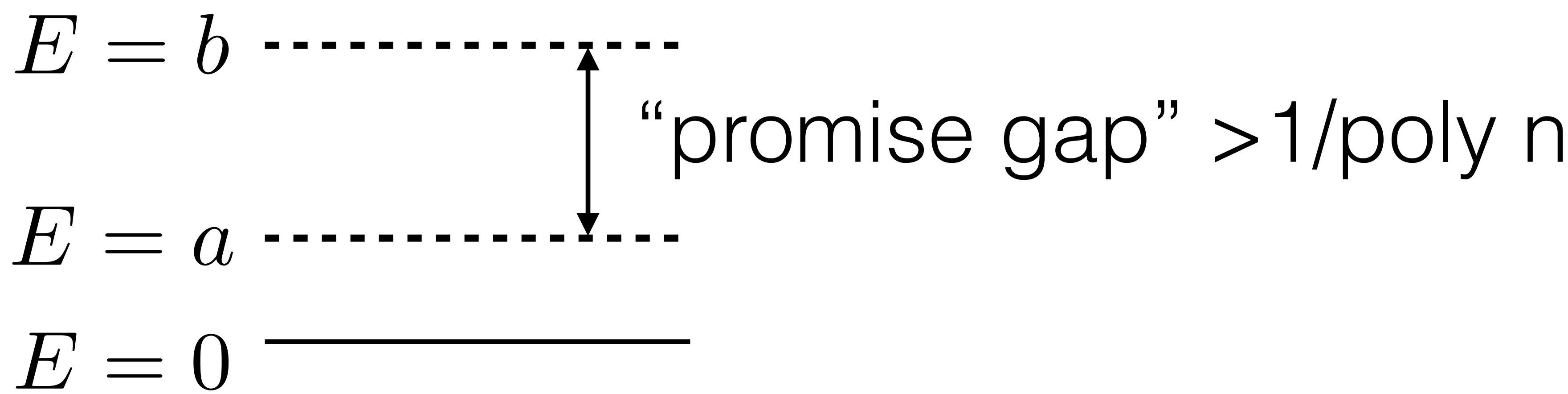
$$H = \sum_j H_j \qquad 0 \leq H_j \leq 1 \qquad H_j : k\text{-local}$$

with a promise that the **ground state energy** of H is:

$$E_0 > b$$

or

$$0 < E_0 < a$$



Problem: Determine whether $E_0 > b$ or $E_0 < a$?

LH is QMA-complete

LH is **in** QMA

ground state of **H**

←

$|\psi\rangle$

Can efficiently estimate the energy on a quantum computer.

LH is **hard** for QMA

Encode proof verification for any QMA problem
into a ground state of a LH

poly-time verification

quantum proof

$\leftarrow$ $|\psi\rangle$

ground state

Yes instances: **Low** energy
No instances: **Higher** energy

Encoding a computation in a ground state

Feynman's **History state** idea

History State

$$
\begin{aligned}
|\psi\rangle = &\ |\psi_{\text{in}}\rangle|0\rangle \\
&+U_1|\psi_{\text{in}}\rangle|1\rangle \\
&+U_2U_1|\psi_{\text{in}}\rangle|2\rangle \\
&+U_3U_2U_1|\psi_{\text{in}}\rangle|3\rangle \\
&+\cdots
\end{aligned}
$$

Encoding a computation in a ground state

Propagator Hamiltonian:

$$H_{\text{prop}} = \sum_j H_j$$

$$H_j = -\frac{1}{2}\left(U_j \otimes |j\rangle\langle j-1| + \text{h.c.}\right) + \frac{1}{2}\mathbb{1} \otimes \left(|j\rangle\langle j| + |j+1\rangle\langle j+1|\right)$$

Ground space is space of history states:

$$\begin{aligned} &|\psi_{\text{in}}\rangle|0\rangle \\ &+U_1|\psi_{\text{in}}\rangle|1\rangle \\ &+U_2U_1|\psi_{\text{in}}\rangle|2\rangle \\ &+U_3U_2U_1|\psi_{\text{in}}\rangle|3\rangle \\ &+\cdots \end{aligned}$$

Encoding a computation in a ground state

This works, because the **ground space** of

$$H_j = -\frac{1}{2}\left(U_j \otimes |j\rangle\langle j-1| + \text{h.c.}\right) + \frac{1}{2}\mathbb{1} \otimes \left(|j\rangle\langle j| + |j+1\rangle\langle j+1|\right)$$

is:

$$|\psi\rangle|j-1\rangle + U_j|\psi\rangle|j\rangle$$

Encoding a computation in a ground state

This works, because the **ground space** of

$$H_j = -\frac{1}{2}\left(U_j \otimes |j\rangle\langle j-1| + \text{h.c.}\right) + \frac{1}{2}\mathbb{1} \otimes \left(|j\rangle\langle j| + |j+1\rangle\langle j+1|\right)$$

is:

$$H_j(|\psi\rangle|j-1\rangle + U_j|\psi\rangle|j\rangle)$$

$$= (1/2)(-U_j|\psi\rangle|j\rangle + U_j|\psi\rangle|j\rangle) = 0$$

QMA

Arthur

Poly-time **quantum** computer

Decision problem

*Arthur can **verify** validity of proof in **poly-time***

quantum proof

$|\psi\rangle$

with **"high enough"** probability

Merlin

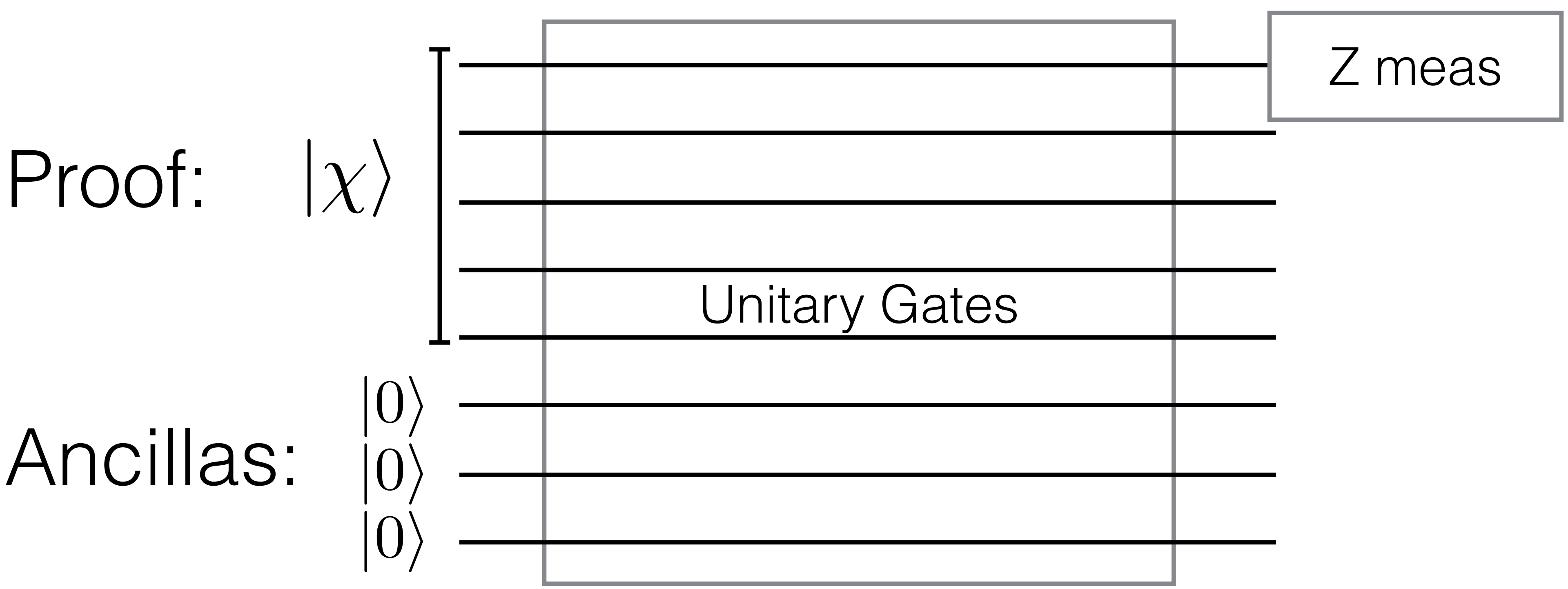
Unbounded computational power

"***Yes = 1***"
*"And here's the **proof**"*

"***No = 0***"
*"And there is **no fake proof** …"*

Encoding a computation in a ground state

The verification circuit we wish to encode:

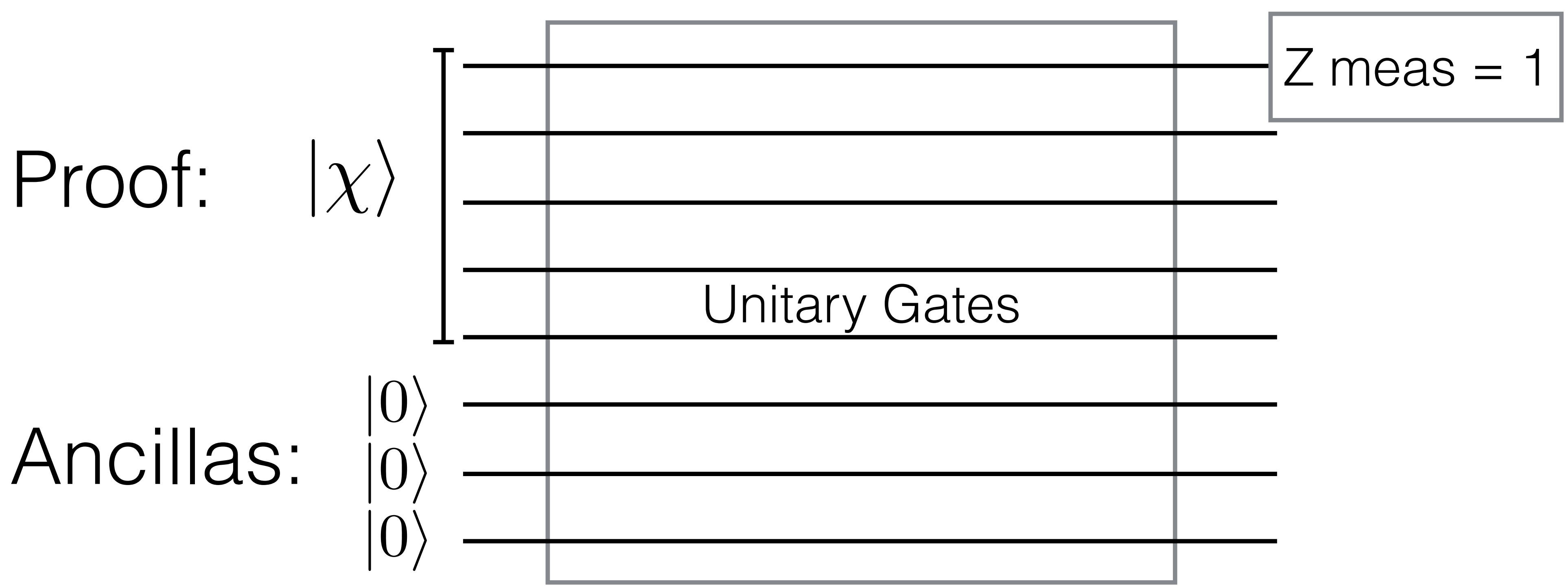


L = No. of unitary gates in circuit

Encoding a computation in a ground state

Maps into the **ground space** of

$$H = H_{\rm in} + H_{\rm prop} + H_{\rm out}$$



$$H_{\rm in} = \sum_{j \in \rm ancilla} |1\rangle_j\langle 1| \otimes |0\rangle_{\rm clock}\langle 0|$$

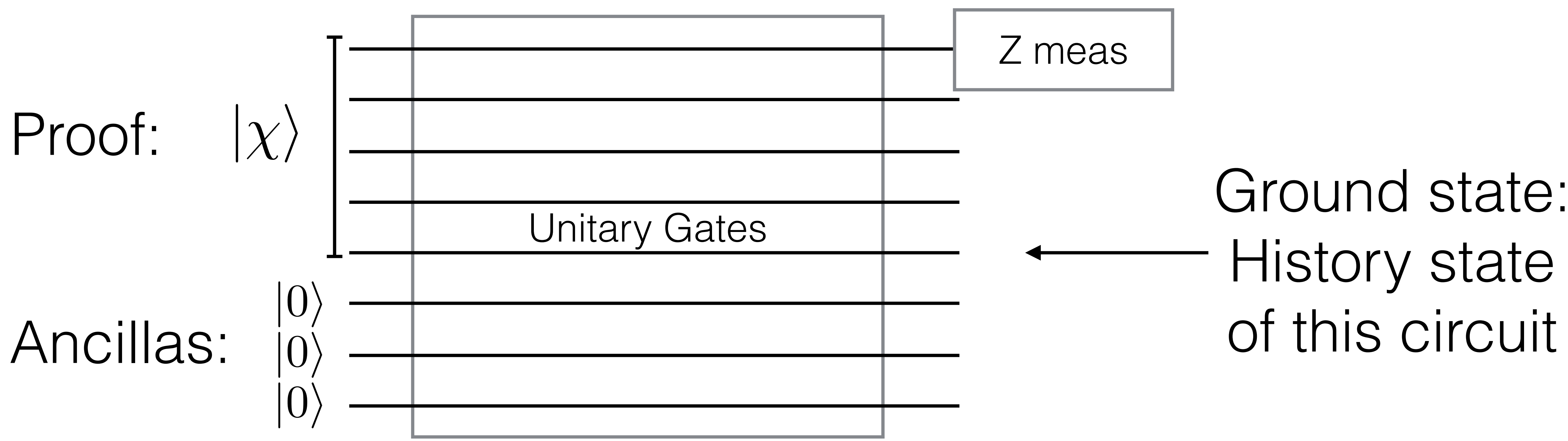
$$H_{\rm out} = |0\rangle_{\rm out}\langle 0| \otimes |L\rangle_{\rm clock}\langle L|$$

Encoding a computation in a ground state

Yes instance:

$$H = H_{\mathrm{in}} + H_{\mathrm{prop}} + H_{\mathrm{out}}$$

No frustration:
Each term of Hamiltonian can reach its **ground space:**

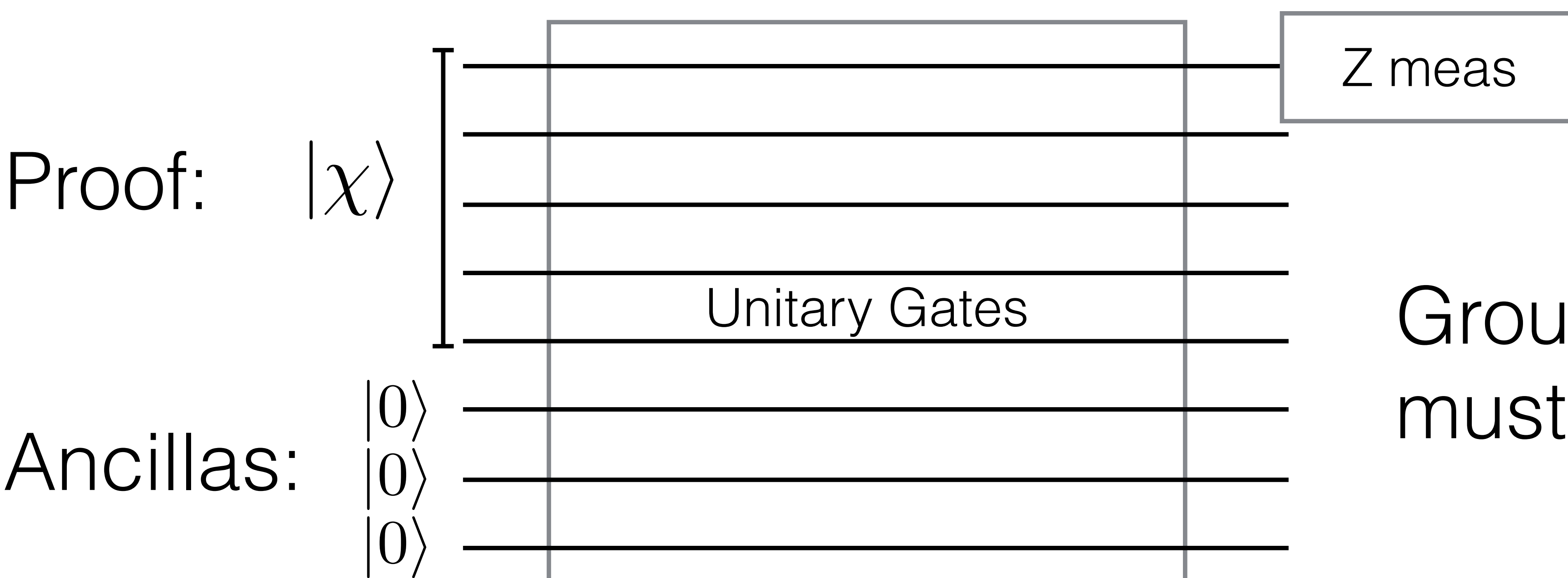


Encoding a computation in a ground state

No instance:

$$H = H_{\text{in}} + H_{\text{prop}} + H_{\text{out}}$$

Frustration:
H_{out} wants output qubit to be in state 1, but no proof state exists to allow this.



Ground state energy must be **increased**.

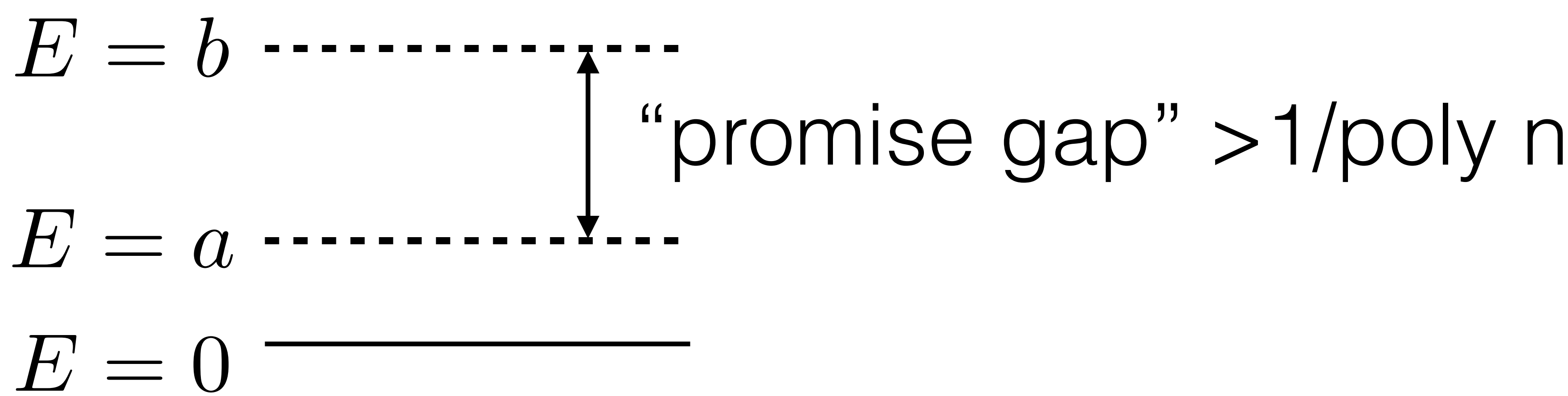
Promise gap scaling

LH Problem: Determine whether $E_0 > b$ or $E_0 < a$?

$$E_0 > b$$

or

$$0 < E_0 < a$$



We need the "promise gap" between **yes** and **no** to remain **inverse polynomial**.

Encoding a computation in a ground state

Kitaev:
Energy of **no** frustrated ground state instance scales with

$$E_0 \geq \lambda_1 2 \sin^2(\theta/2)$$

scales with 1/L

2nd lowest eigenvalue of H_{prop}

Kitaev's H_{prop}:
Lowest eigenvalue: $$1 - \cos(\pi/L+1) \approx \frac{\pi}{2(L+1)^2}$$

LH is QMA-complete

Non-unitary ground state computation?

Our work: (Usher/Browne, unfinished 2015)

Non-unitary ground state computation?

Our work: (Usher/Browne, unfinished 2015)

What if?

Unitary Gates
+ **non-Unitary projectors**

→

Ground State Computation

↓

Verifier circuits

Add **projectors** to the circuits

Non-unitary ground state computation?

Renormalised Projectors / Post-selection

E.g. quantum lottery ticket

$$\frac{1}{10000000000}|\mathrm{win}\rangle + |\mathrm{lose}\rangle$$

$$|\mathrm{win}\rangle\langle\mathrm{win}| \quad \downarrow$$

$$\frac{1}{10000000000}|\mathrm{win}\rangle$$

Renormalise $\downarrow$

$$|\mathrm{win}\rangle$$

Non-unitary ground state computation?

Why?

- Curiosity! - Feynman's construction pre-dates **all** quantum computing theory.

- Add **projectors** to unitary circuits and you get:

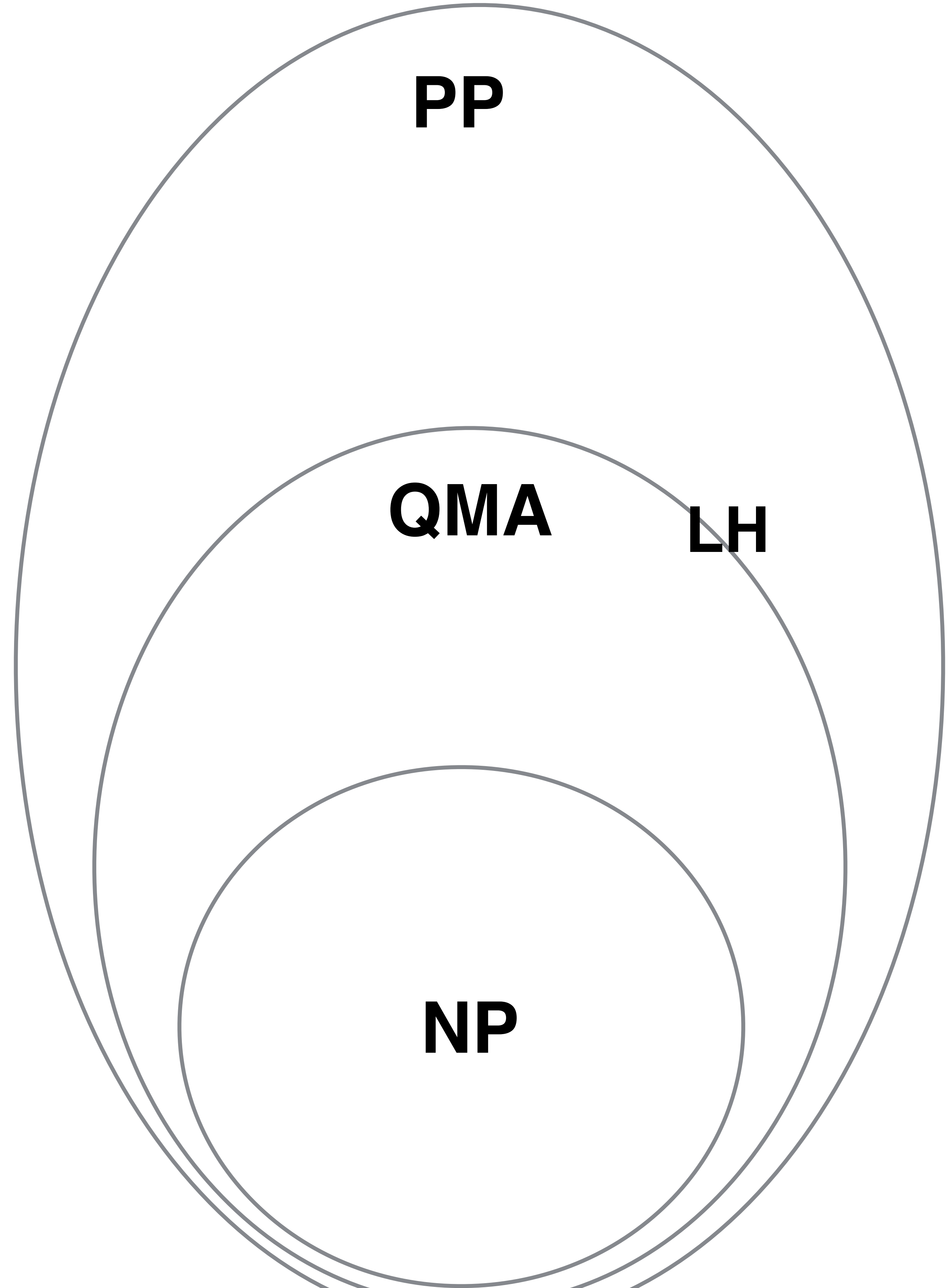
  Fault-tolerant Quantum Computation
  Measurement-based Quantum Computation
  postIQP = postBQP = PP

Non-unitary ground state computation?

Why it will never work!

postIQP = **postBQP = PP**

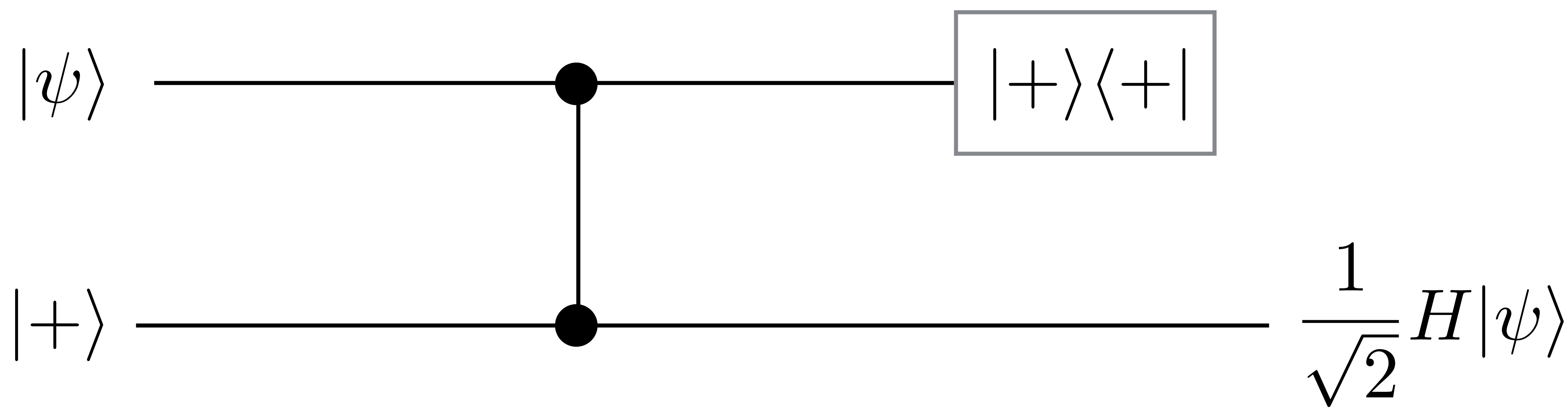
Aaronson:
If deterministic projectors (post-selection) are added to unitary gates, we can efficiently solve **PP-hard** problems.



Non-unitary ground state computation?

Why it might just work…..

postIQP = postBQP = PP

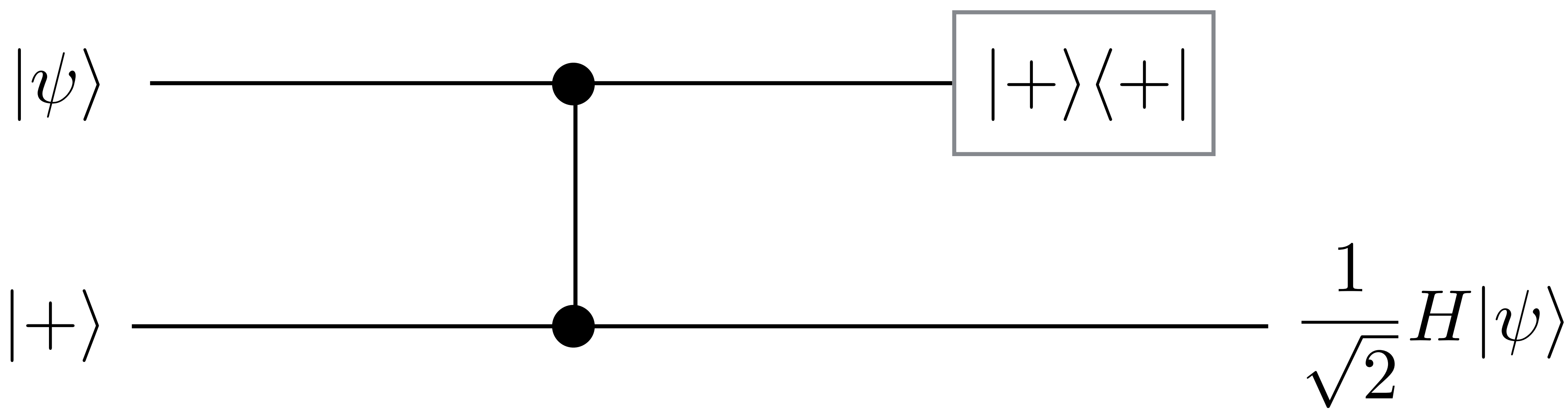


One-bit teleportation circuit (Zhou / Leung / Chuang 2000).

Non-unitary ground state computation?

Why it might just work…..

postIQP = postBQP = PP



~~**One-bit**~~ **Zero-bit teleportation** circuit.

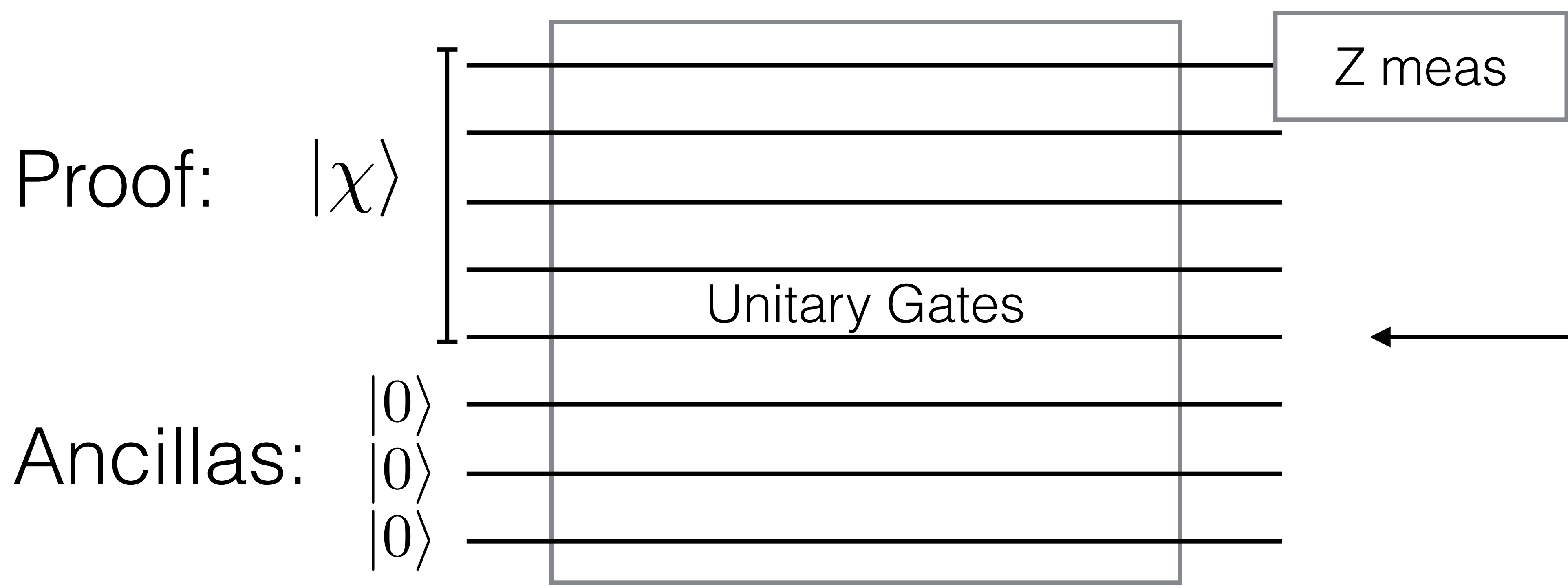
NB **Norm** of output is **independent** of input.

Encoding a computation in a ground state

Why it also might just work…..

In Kitaev construction, $\mathbf{H_{out}}$ **projects** the input to the correct proof state.

$$H = H_{\rm in} + H_{\rm prop} + H_{\rm out}$$



Accepting circuit **projects** input into **proof state**

The projector gadget

Recall Kitaev and Feynman's unitary gadget:

$$H_j = -\frac{1}{2}\left(U_j \otimes |j\rangle\langle j-1| + \text{h.c.}\right) + \frac{1}{2}\mathbb{1} \otimes \left(|j\rangle\langle j| + |j+1\rangle\langle j+1|\right)$$

that had ground space

$$|\psi\rangle|j-1\rangle + U_j|\psi\rangle|j\rangle$$

A projector gadget?

For projector P, want a gadget

$$H_p = ?$$

that has ground space

$$|\psi\rangle|j-1\rangle + \frac{P|\psi\rangle|j\rangle}{\beta}$$

where

$$\beta = \|P|\psi\rangle|j\rangle\|$$

To avoid PP-hardness, we **assume** β will be equal for **all** states |Ψ⟩.

The projector gadget

$$H_j = \left(\frac{\beta^2}{1+\beta^2}\right) P \otimes \left(-\frac{1}{\beta}|j-1\rangle\langle j| - \frac{1}{\beta}|j\rangle\langle j-1| + \frac{1}{\beta^2}|j-1\rangle\langle j-1| + |j\rangle\langle j|\right) + P^{\perp} \otimes |j\rangle\langle j|$$

has the ground space we want!

$$|\psi\rangle|j-1\rangle + \frac{P|\psi\rangle|j\rangle}{\beta}$$

NB In the limit $\beta \to 1$, $P \to \mathbb{1}$, we recover Kitaev/Feynman gadget.

The projector gadget

$$H_j = \left(\frac{\beta^2}{1+\beta^2}\right) P \otimes \left(-\frac{1}{\beta}|j-1\rangle\langle j| - \frac{1}{\beta}|j\rangle\langle j-1| + \frac{1}{\beta^2}|j-1\rangle\langle j-1| + |j\rangle\langle j|\right) + P^{\perp} \otimes |j\rangle\langle j|$$

$$H_j \left(|\psi\rangle|j-1\rangle + \frac{P|\psi\rangle|j\rangle}{\beta}\right)$$

$$\propto -\frac{P}{\beta}|\psi\rangle|j\rangle + \frac{P}{\beta}|\psi\rangle|j\rangle = 0$$

The projector gadget

$$H_j = \left(\frac{\beta^2}{1+\beta^2}\right) P \otimes \left(-\frac{1}{\beta}|j-1\rangle\langle j| - \frac{1}{\beta}|j\rangle\langle j-1| + \frac{1}{\beta^2}|j-1\rangle\langle j-1| + |j\rangle\langle j|\right) + P^{\perp} \otimes |j\rangle\langle j|$$

has the ground space we want!

$$|\psi\rangle|j-1\rangle + \frac{P|\psi\rangle|j\rangle}{\beta}$$

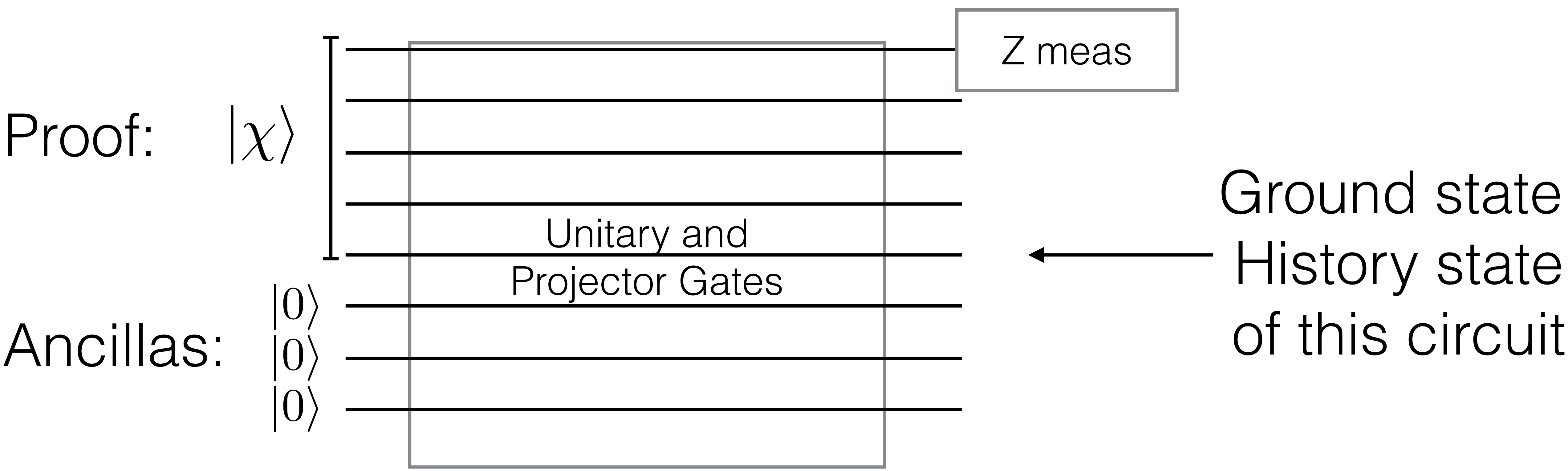
NB In the limit $\beta \to 1$, $P \to \mathbb{1}$, we recover Kitaev/Feynman gadget.

Encoding a computation in a ground state

With this gadget, we can construct Hamiltonians

$$H = H_{\text{in}} + H_{\text{prop}} + H_{\text{out}}$$

such that “**Yes**” instances have low energy…



Encoding a computation in a ground state

And **No** instances have frustration

$$H = H_{\text{in}} + H_{\text{prop}} + H_{\text{out}}$$

The “promise gap” satisfies Kitaev’s formula

$$E_0 \geq \lambda_1 2 \sin^2(\theta/2)$$

Need to check this! → should **scale** as **1 / poly(n)**

↑ 2nd lowest eigenvalue of H_{prop}

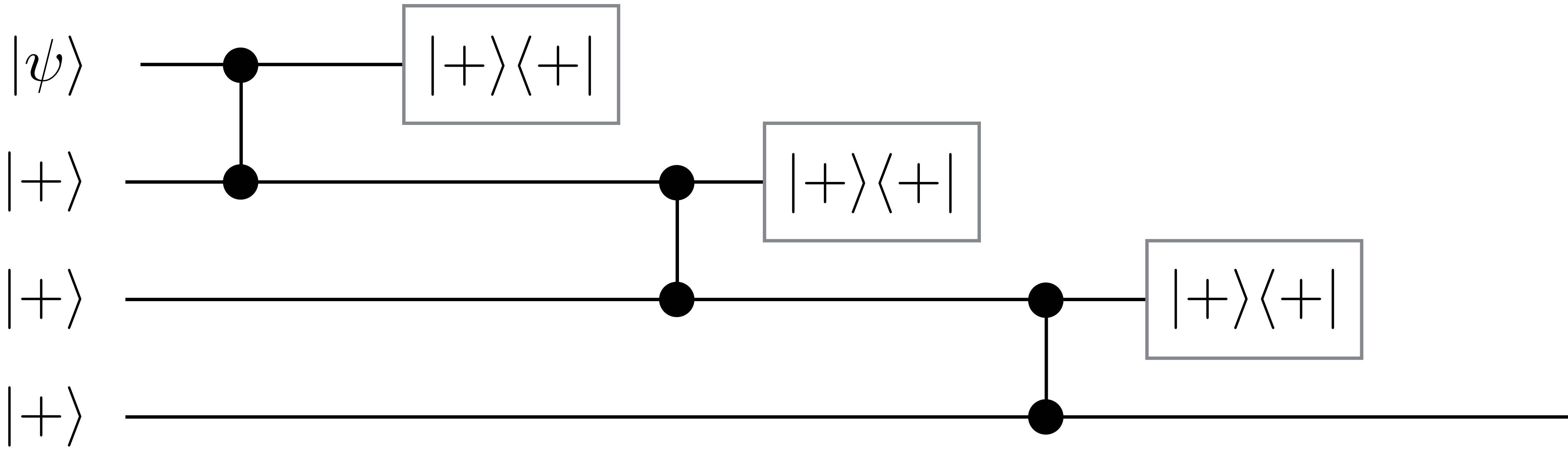
Characterising the promise gap

December 2014: We had a beautiful and elegant **analytic bound** on λ_1 for many circuits.

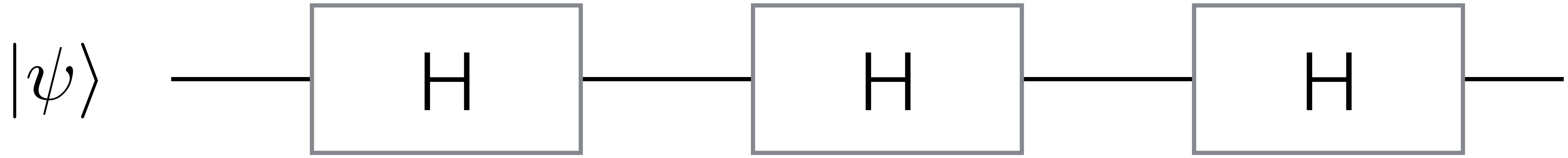
January 2015: We found one of **these** in the proof…

Characterising the promise gap

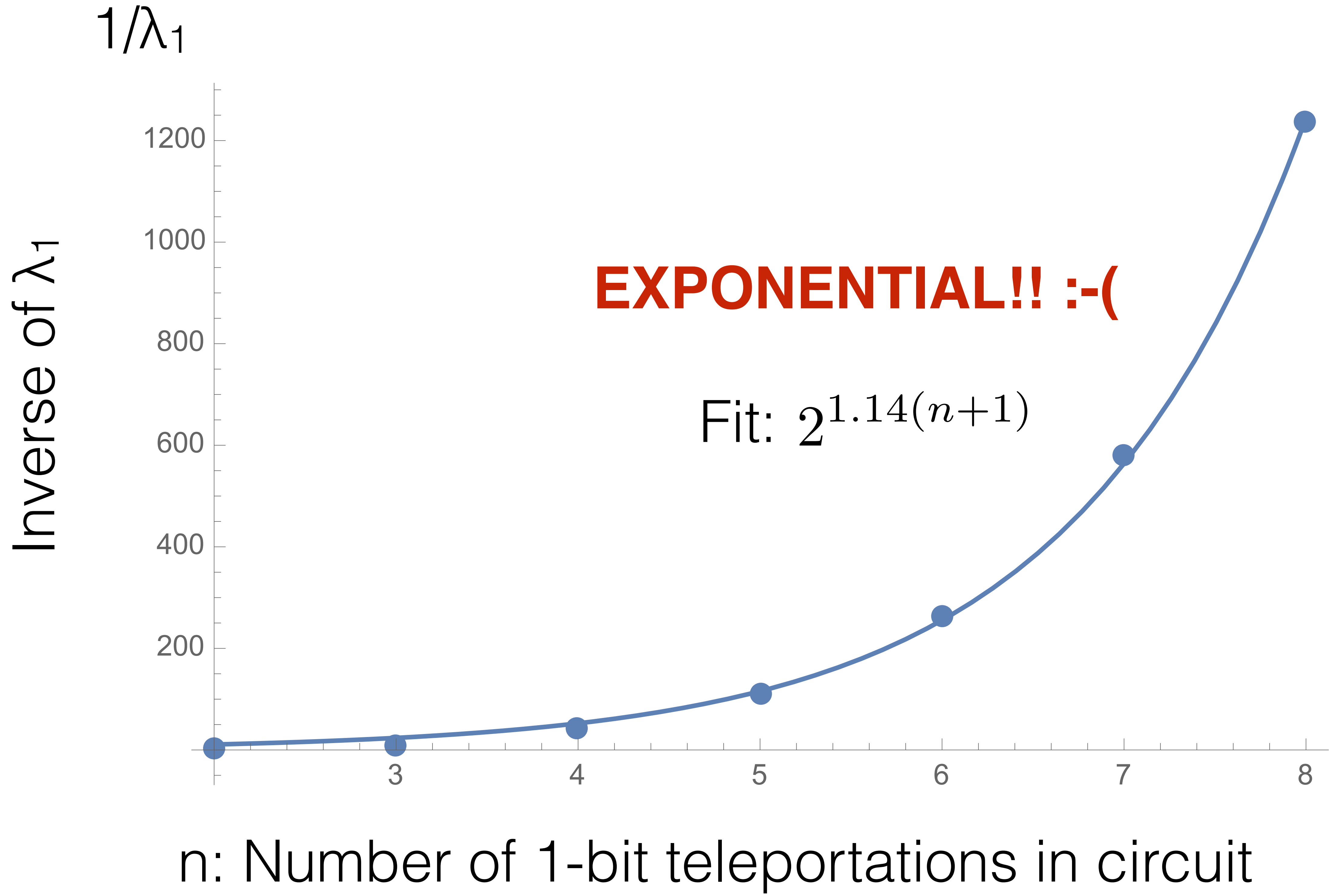
Instead - some quick and simple proof-of-principle numerics.



Equivalent to

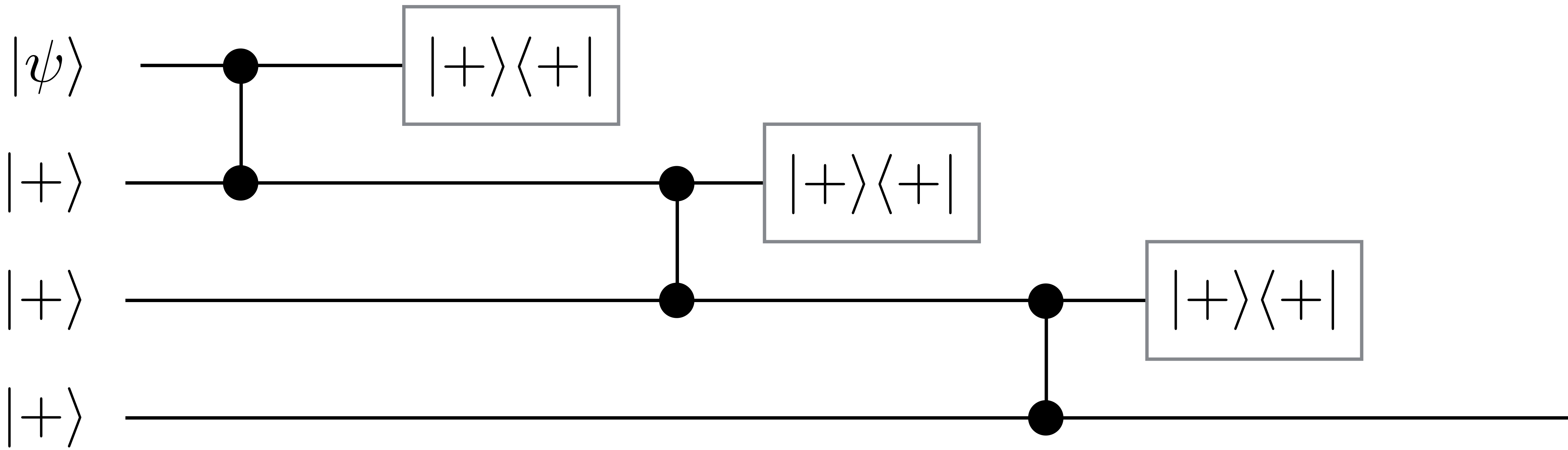


Characterising the promise gap



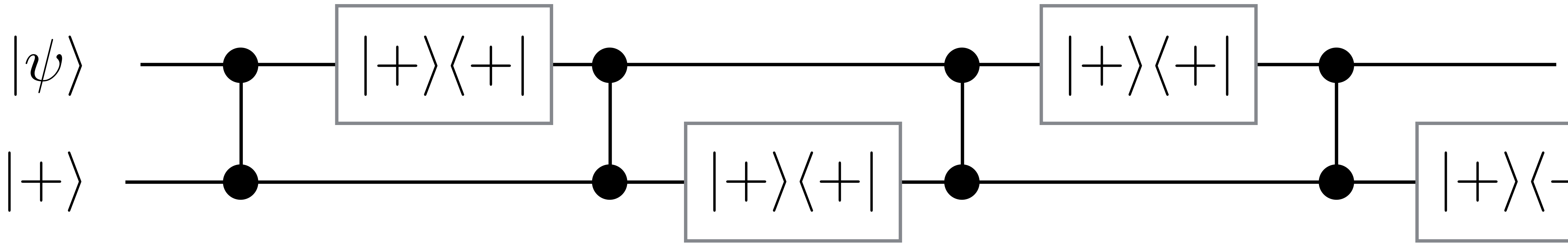
Characterising the promise gap

Extra ancillas giving **low energy** excited states. :-(

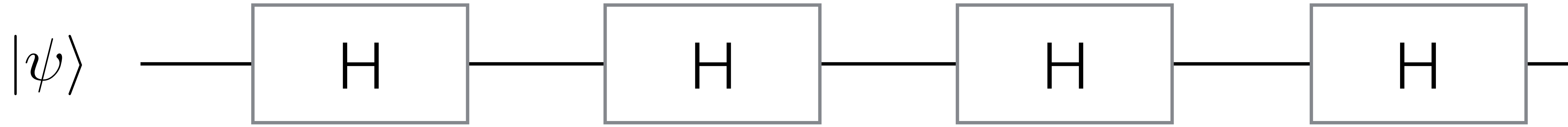


Characterising the promise gap

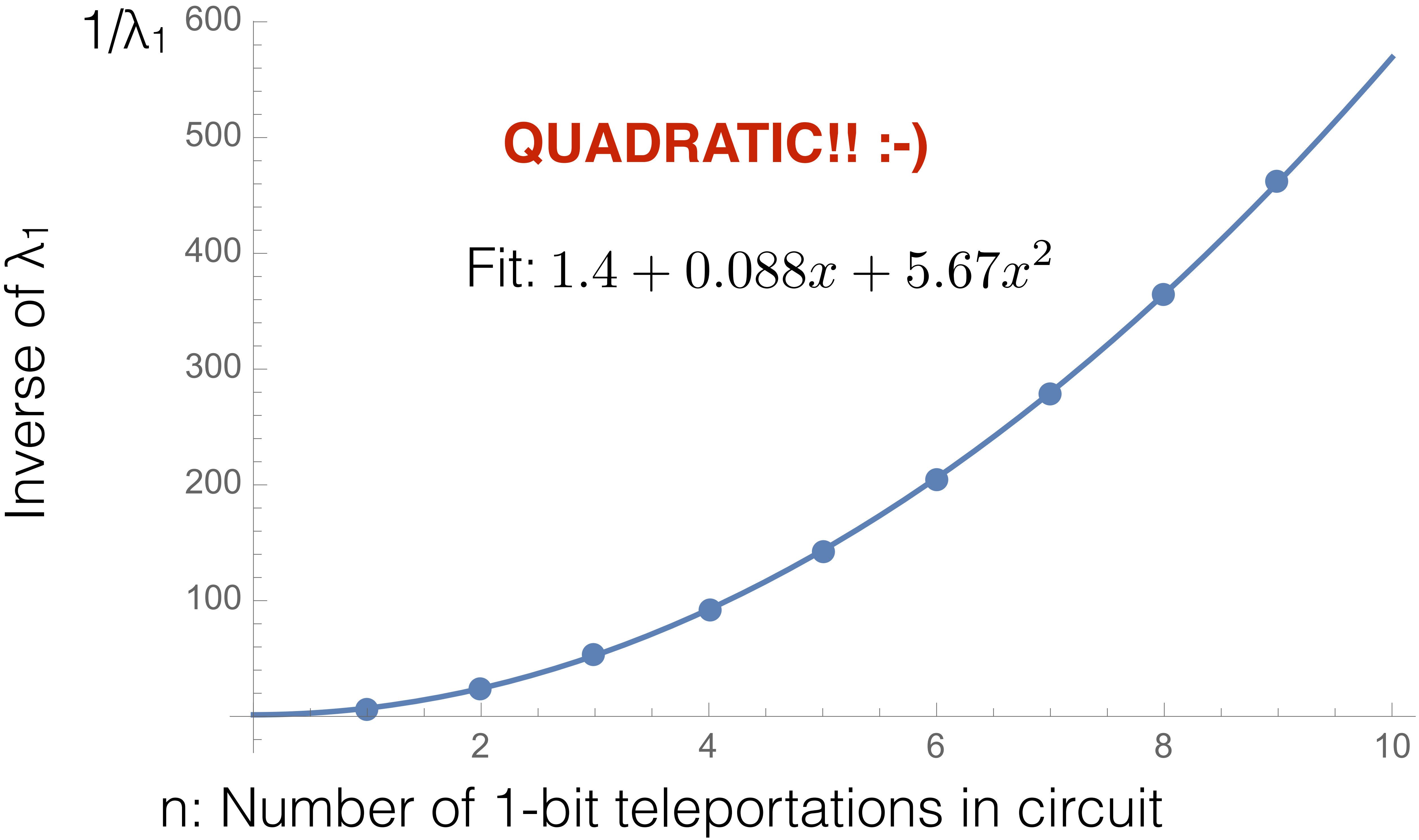
A second try



Equivalent to



Characterising the promise gap



Conclusion

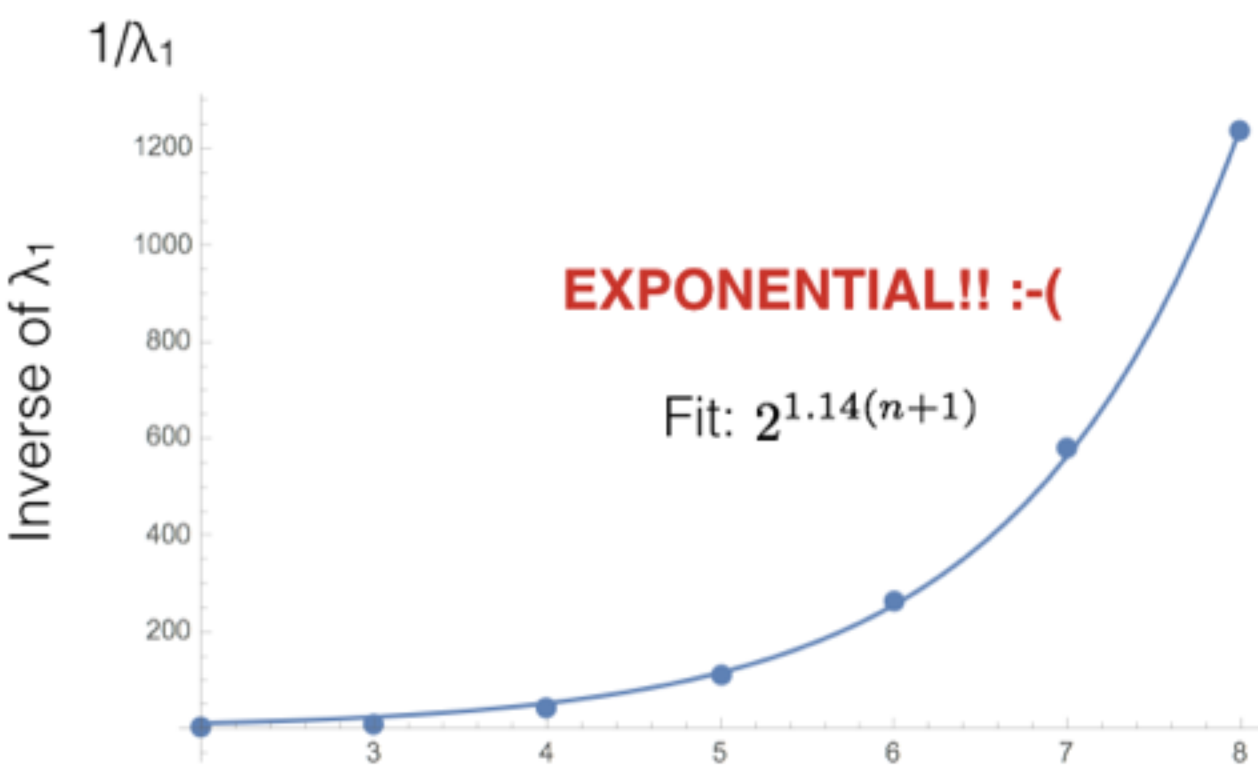
It **works**! Sometimes…..

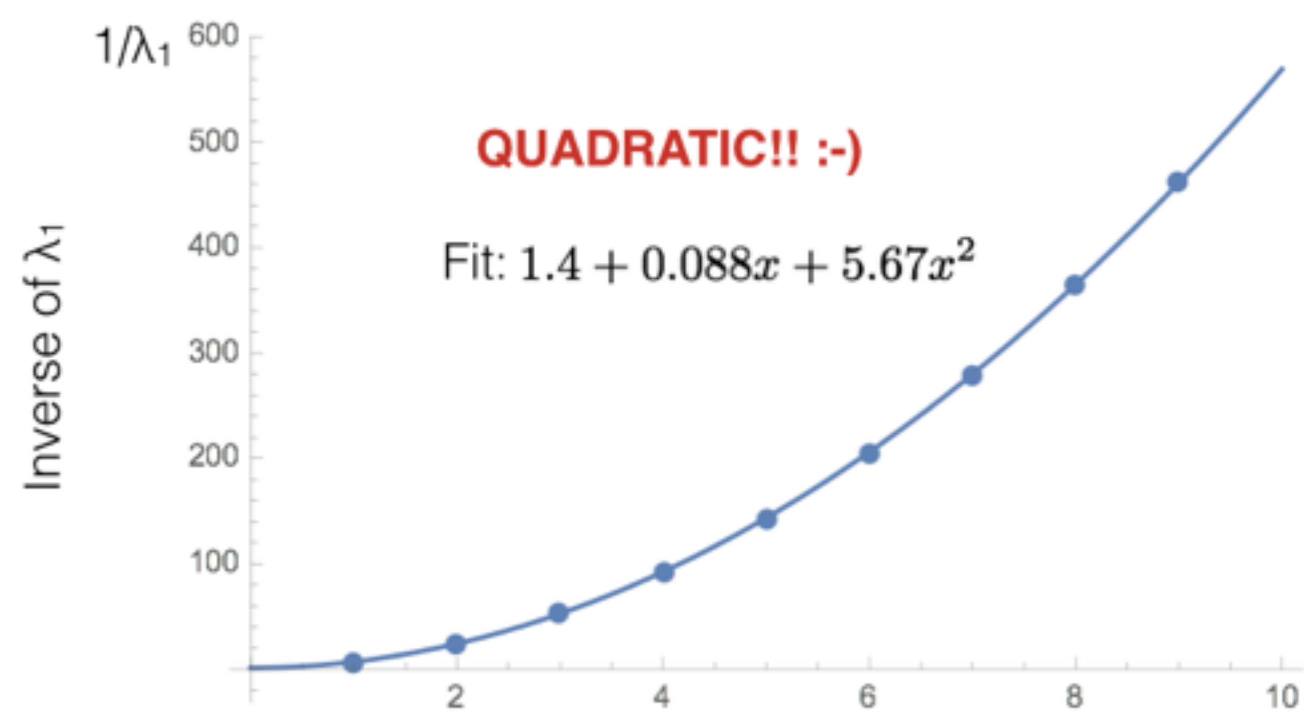
(probably)

Where do we go next?

Where do we go next?

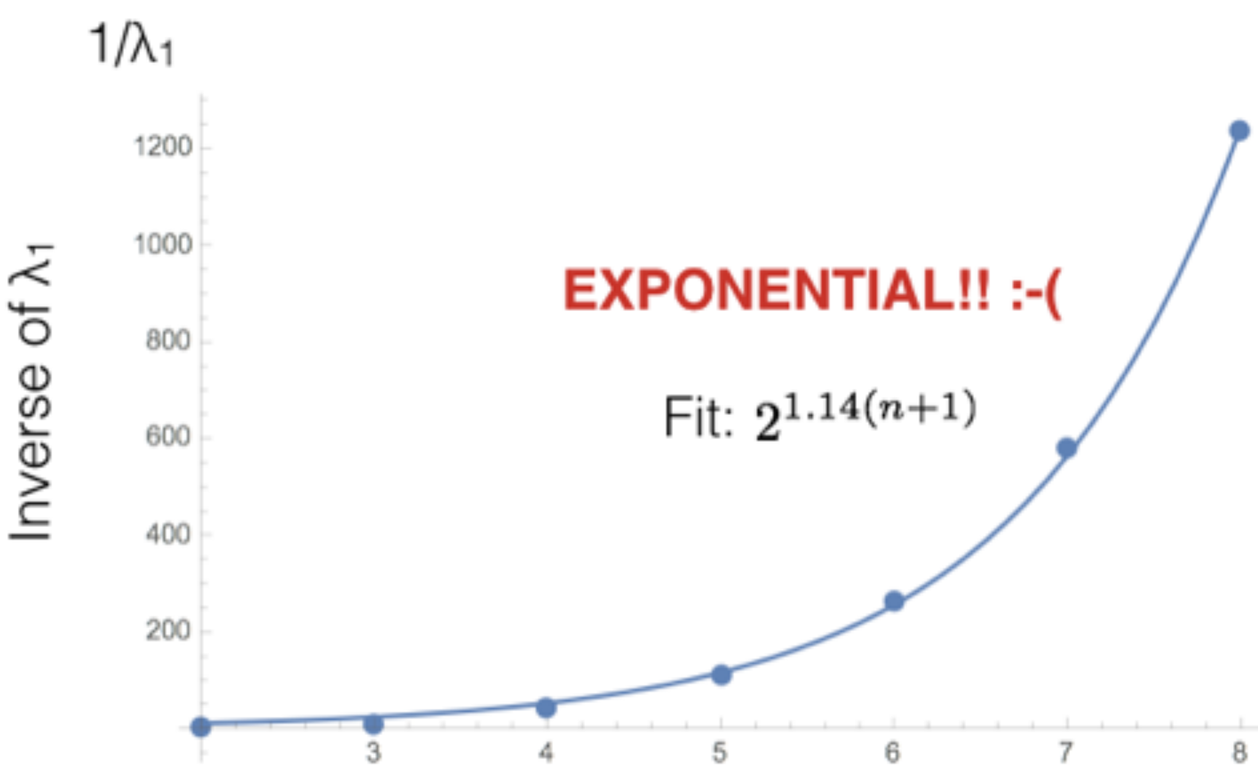
- Numerical evidence for 1/poly promise gap scaling:
  - But when does **exponential scaling** occur?
  - Analytic bounds?
- Aim: A QMA construction from a **non-universal** gate set with projectors (e.g. **IQP** circuit).

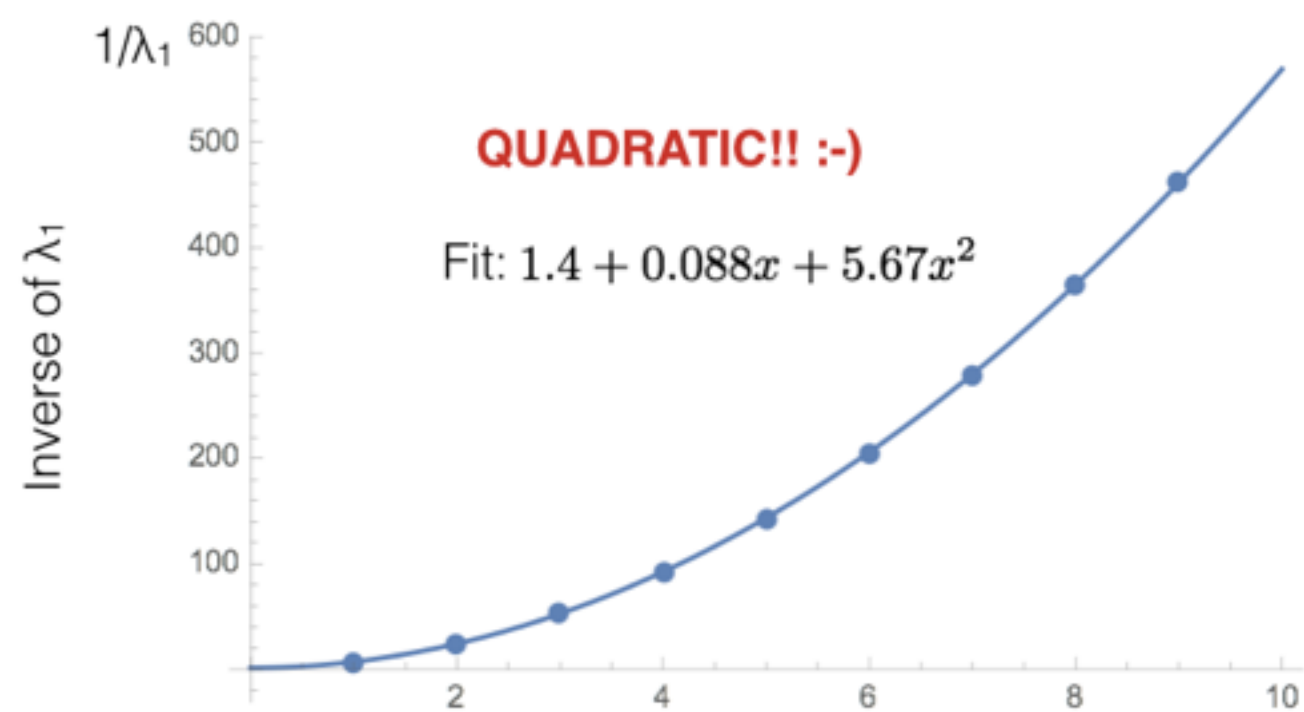




Where do we go next?

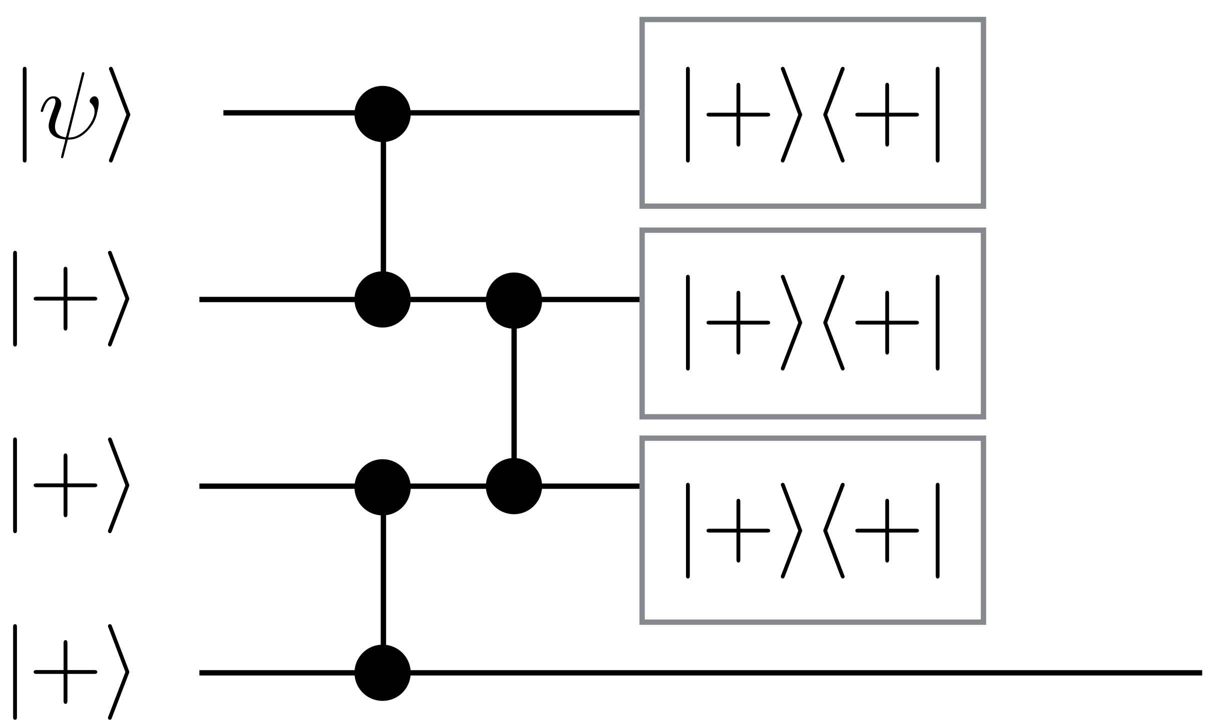
- Complexity
  - When does **BQP** + **projectors** = **BQP**
    - Is **constant probability** on **input states** sufficient?
  - Can we characterise **exponentially closing gap** with other complexity classes?





Where do we go next?

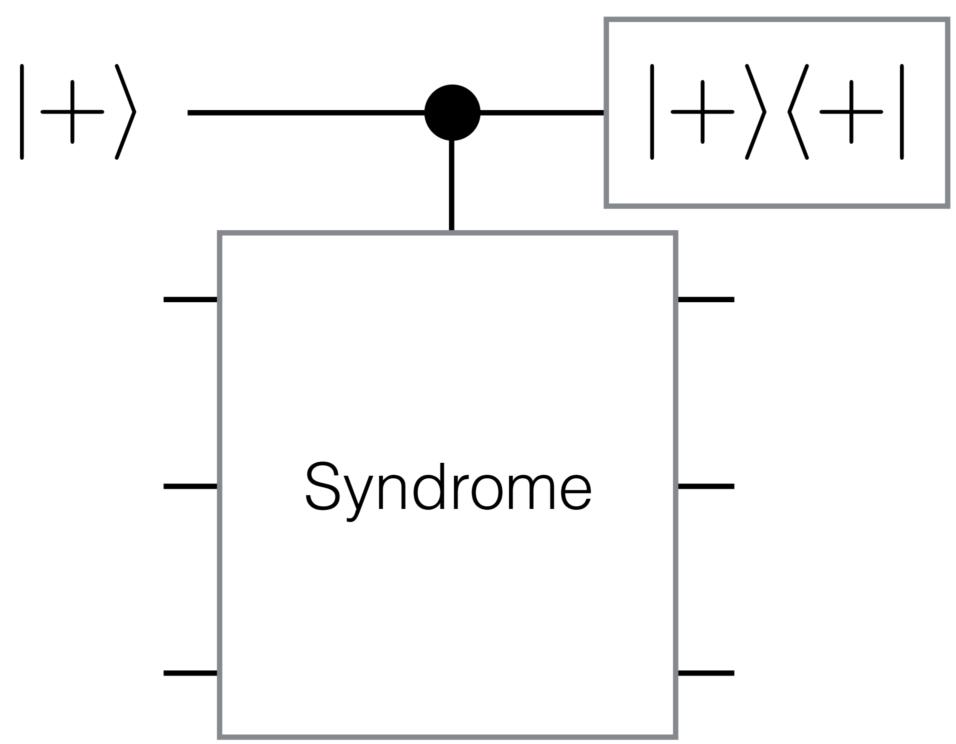
- Post-selection gives **IQP** / **MBQC** circuits trivial time complexity. QMA with **constant clock-steps**?



Need **many** clocks!

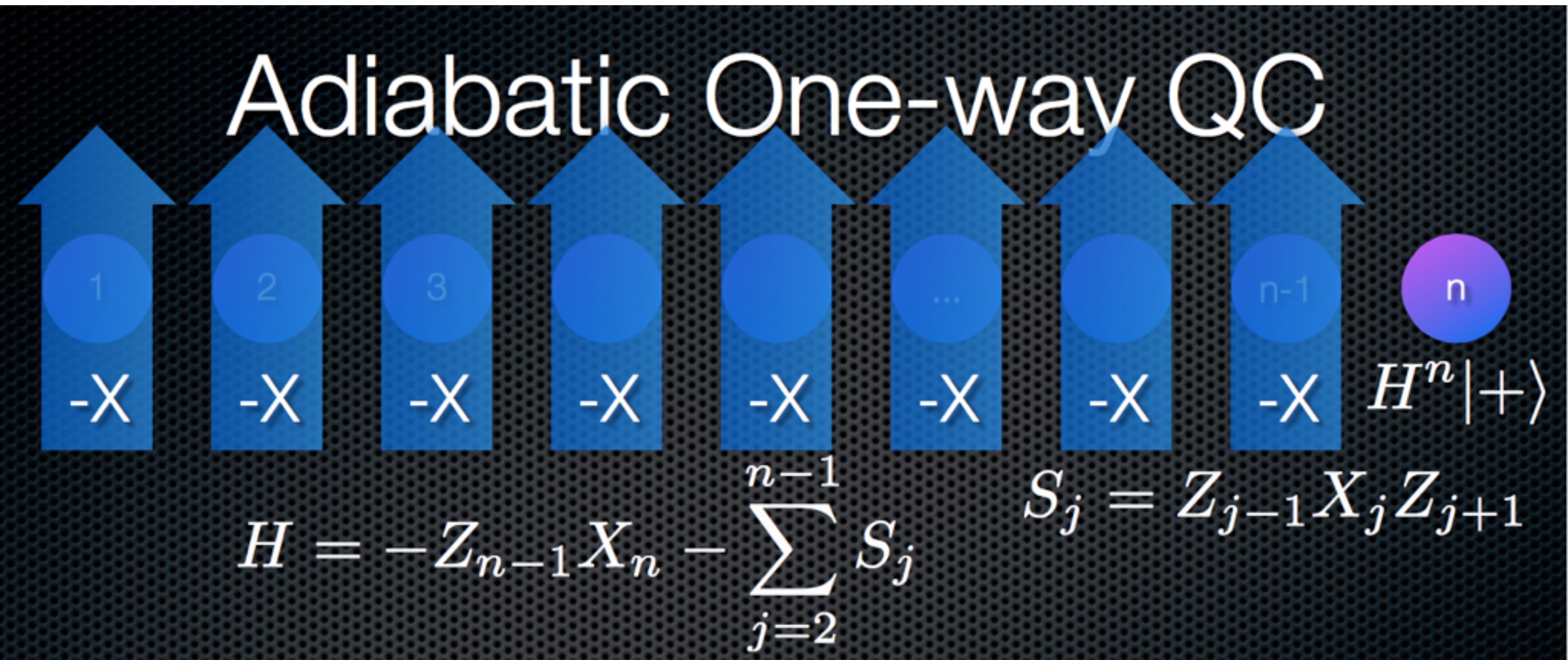
Where do we go next?

- Incorporate **fault tolerance**?
  - **Error detection** gadgets to project onto **error free** states?
  - Robustness of Hamiltonian to **perturbations**?
  - Norm of history state vectors problematic?



Where do we go next?

- **Other applications** for the projector gadget?
- **Adiabatic** Quantum Computation?
- Relationship with Bacon and Flammia's **adiabatic cluster state** model?



Thank you!