Digging in the mud: New perspectives on many-body localisation*

Jens Eisert, FU Berlin

Coogee, January 2015

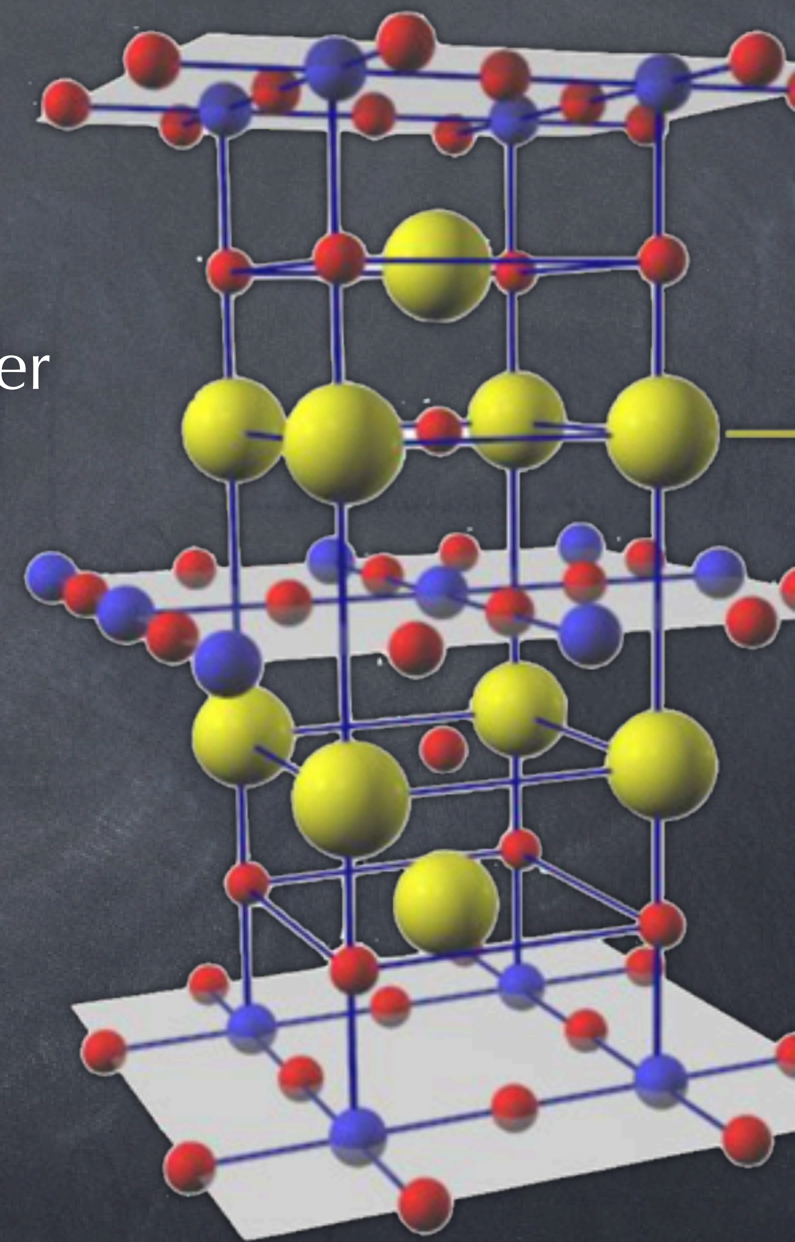
Joint work with M. Friesdorf, A. Werner, V. Scholz, W. Brown, mentions work of I. H. Kim, G. Vidal, J. Carrasquilla, D. A. Abanin, A. Chandran, T. J. Osborne and others

* Possible bonus on area laws

Quantum many-body systems

$$H = \sum_j h_j$$

- **Local Hamiltonians** as models for strongly correlated matter



This talk

Quantum information

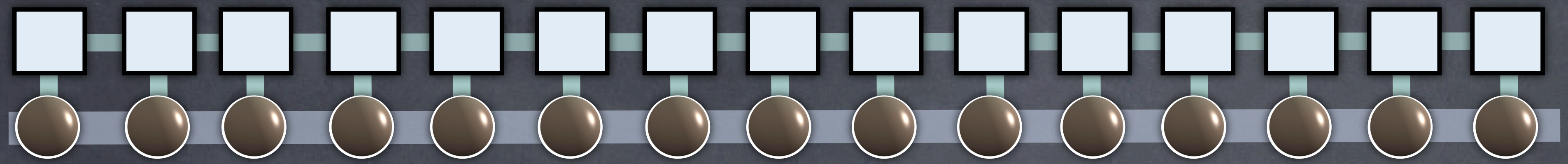
Condensed matter

Ground state properties of gapped models

- **Ground state properties** of gapped local Hamiltonians
  - Clustering of correlations
  - Satisfy in 1D area laws for entanglement entropies $S_\alpha(\rho_A) = O(|\partial A|)$

Nachtergaele, Sims, Commun Math Phys 265, 119 (2006)
Hastings, Koma, Commun Math Phys 265, 781 (2006)
Eisert, Cramer, Plenio, Rev Mod Phys (2010)
Hastings, J Stat Mech P08024 (2007)
Brandao, Horodecki, Nature Physics 9, 721 (2013)

Ground state properties of gapped models



- **Ground state properties** of gapped local Hamiltonians
  - Can in 1D be approximated by matrix-product states
  - Can exhibit topological order, used for quantum computing, etc

Fannes, Nachtergaele, Werner, Lett Math Phys 25, 249 (1992)
Verstraete, Cirac, Murg, Adv Phys 57, 143 (2008)

Equilibration

- **Dynamical properties of local Hamiltonians** $\rho(t) = e^{-itH}\rho(0)e^{itH}$
  - Quenches: Do systems equilibrate?
  - Local expectation values $\mathrm{tr}(A\rho(t)) = \mathrm{tr}(A\omega)$ take in expectation (or time intervals) the expectation values of time average
    $$\omega = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt \rho(t)$$

Eisert, Friesdorf, Gogolin, Nature Physics (2015)
Polkovnikov, Sengupta, Silva, Vengalattore, Rev Mod Phys 83, 863 (2011)
Cramer, Dawson, Eisert, Osborne, Phys Rev Lett 100, 030602 (2008)
Linden, Popescu, Short, Winter, Phys Rev E 79, 061103 (2009)

Thermalisation

- **Dynamical properties of local Hamiltonians**
  - Do systems thermalise? $\omega_A \sim \mathrm{tr}_{A^c}(e^{-\beta H})$
  - Common expectation in non-integrable models: Yes

Eisert, Friesdorf, Gogolin, Nature Physics (2015)
Polkovnikov, Sengupta, Silva, Vengalattore, Rev Mod Phys 83, 863 (2011)

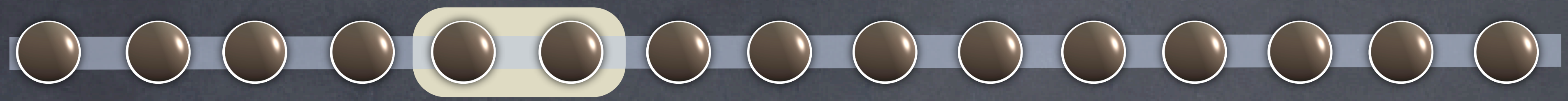
Eigenstate thermalisation hypothesis

- Thermalisation follows, e.g., from **eigenstate thermalisation hypothesis**
  - (Most) individual eigenstates locally 'appear thermal'

$$\mathrm{tr}_{A^c}(|k\rangle\langle k|) \approx \mathrm{tr}_{A^c}(e^{-\beta H})$$

Deutsch, Phys Rev A 43, 2046 (1991)
Srednicki, Phys Rev E 50, 888 (1994)

Absence of thermalisation

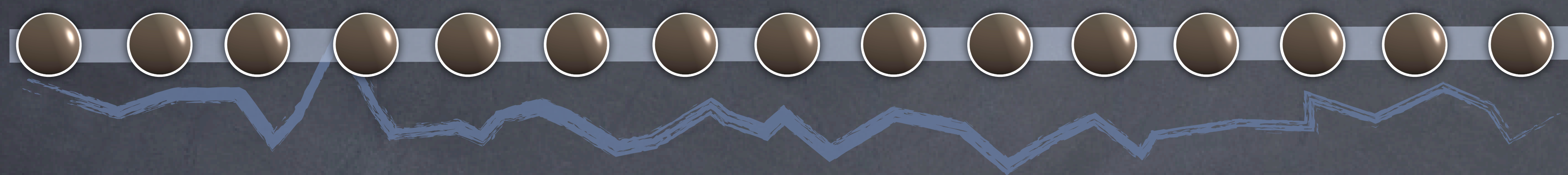


$$H = \sum_j h_j$$

- Some systems seem to **fail to thermalise**
  - **Many-body localisation** is one incarnation

How about disordered models?

Anderson localisation



- One particle hopping on a line in i.i.d. random potential

$$H = \sum_j (|j\rangle\langle j+1| + |j+1\rangle\langle j| + f_j |j\rangle\langle j|)$$

(or non-interacting particles)

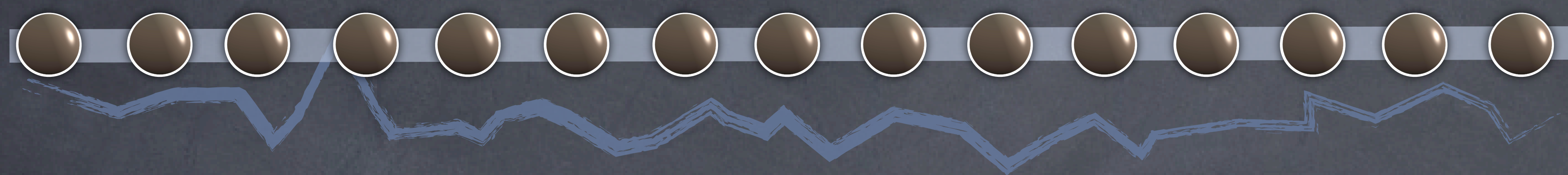
- **Static reading:** With prob increasing with lattice, all eigenfunctions exponentially clustering correlations

- **Dynamical reading:**

$$\mathbb{E}(\sup_t |\langle n|e^{-itH}|m\rangle|) \le c_1 e^{-c_2 \mathrm{dist}(n,m)}$$

Anderson, Phys Rev 109, 1492 (1958)

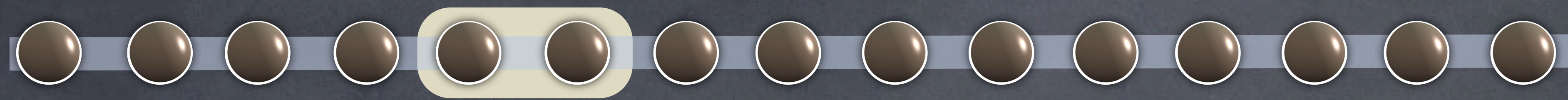
Many-body localisation



- Does localisation survive finite interactions?
- Yes: **Many-body localisation** (MBL)
- Far from well-understood (but explosion of interest)

Anderson, Phys Rev 109, 1492 (1958)
Nandkishore, Huse, arXiv:1404.0686

Breakdown of eigenstate thermalisation hypothesis



- **Eigenstate thermalisation hypothesis (ETH)**:

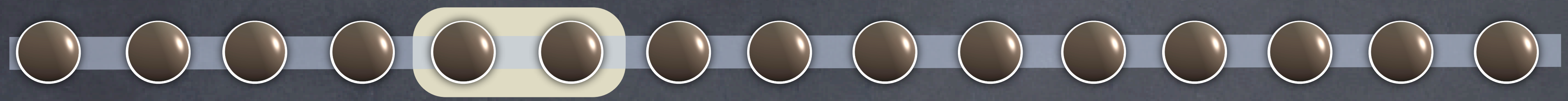
$$\mathrm{tr}_{A^c}(|k\rangle\langle k|) \sim \mathrm{tr}_{A^c}(e^{-\beta H})$$

Srednicki, Phys Rev E 50, 888 (1994)
Deutsch, Phys Rev A 43, 2046 (1991)

- System exhibits MBL if ETH breaks down

Pal, Huse, Phys Rev B 82, 174411 (2010)
Ogenesyan, Huse, Phys Rev B 75, 155111 (2007)
Gogolin, Mueller, Eisert, Phys Rev Lett 106, 040401 (2011)

Localisation in Fock space



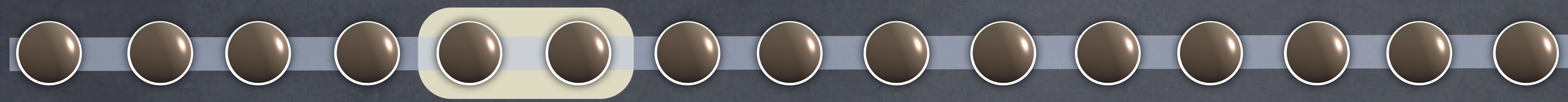
- Anderson insulators with **perturbative interactions**

$$H = H_0 + \lambda H_{\text{int}}$$

- Solve single-particle problem, build Fock space of Slater dets
- Consider "localisation in Fock space"

Basko, Aleiner, Altshuler, Ann Phys 321, 1126 (2006)

Static definition of having MPS eigenstates



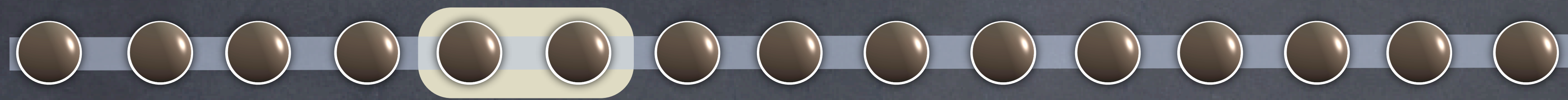
- More quantum information inspired
- MBL eigenstate ⟷ Finite depth circuit ⟷ • Slater determinants
- Most eigenstates are **matrix-product states** of low bond dimension

Bauer, Nayak, J Stat Mech P09005 (2013)

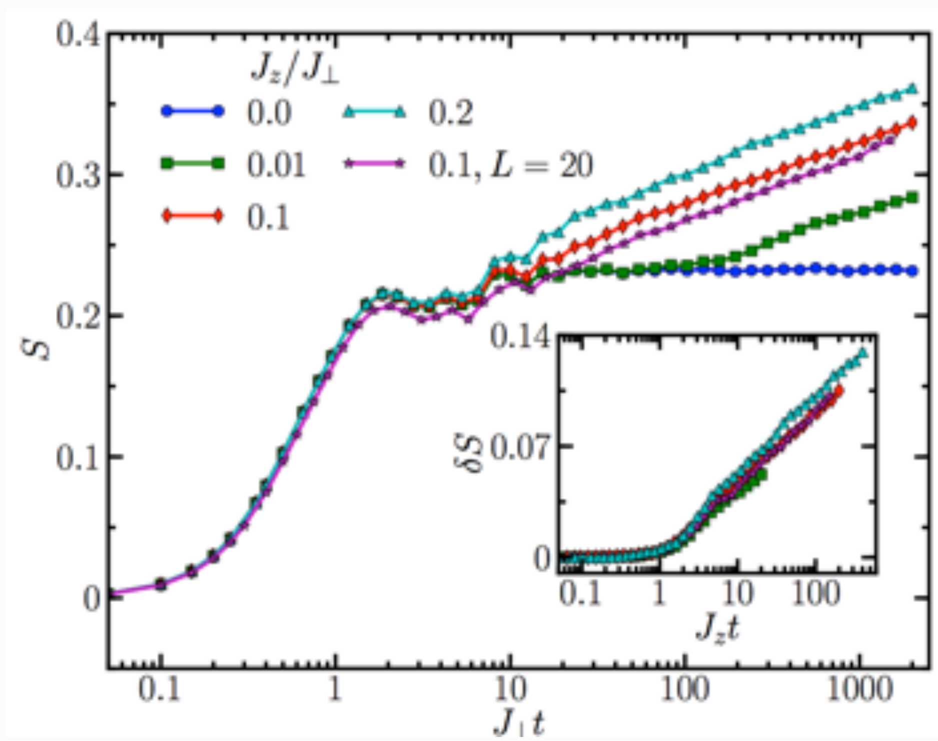
- Most comprehensive numerical study to date

Luitz, Laflorencie, Alex, arXiv:1411.0660

Slow entanglement growth



- Slow (logarithmic) entanglement entropy growth following quenches

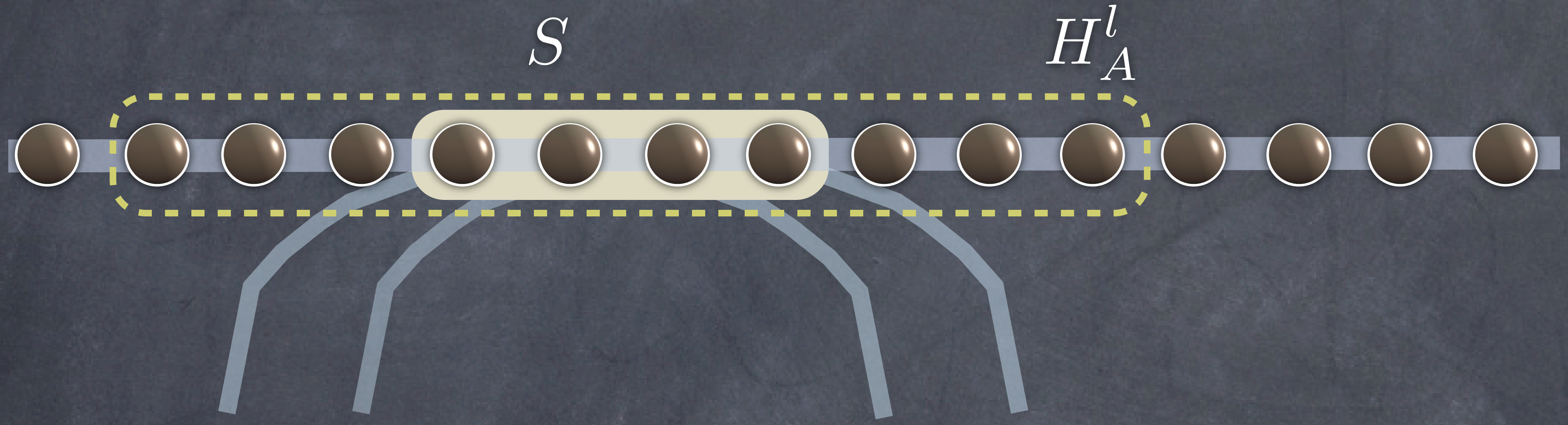


Badarson, Pollmann, Moore, Phys Rev Lett 109, 017202 (2012)

Dynamical localisation

- **Strong dynamical localisation:** All transport is blocked for arbitrary states

$$\|A(t) - e^{itH_A^l} A e^{-itH_A^l}\| \leq c_{\mathrm{loc}} e^{-\mu l}$$



Dynamical localisation

- **Strong dynamical localisation:** All transport is blocked for arbitrary states

$$\|A(t) - e^{itH_A^l} A e^{-itH_A^l}\| \leq c_{\mathrm{loc}} e^{-\mu l}$$

- **Mobility edge:** Transport suppressed on the low-energy sector below E_{mob}

$$\forall \rho \in \{|l\rangle\langle k| : E_l, E_k \leq E_{\mathrm{mob}}\} :$$

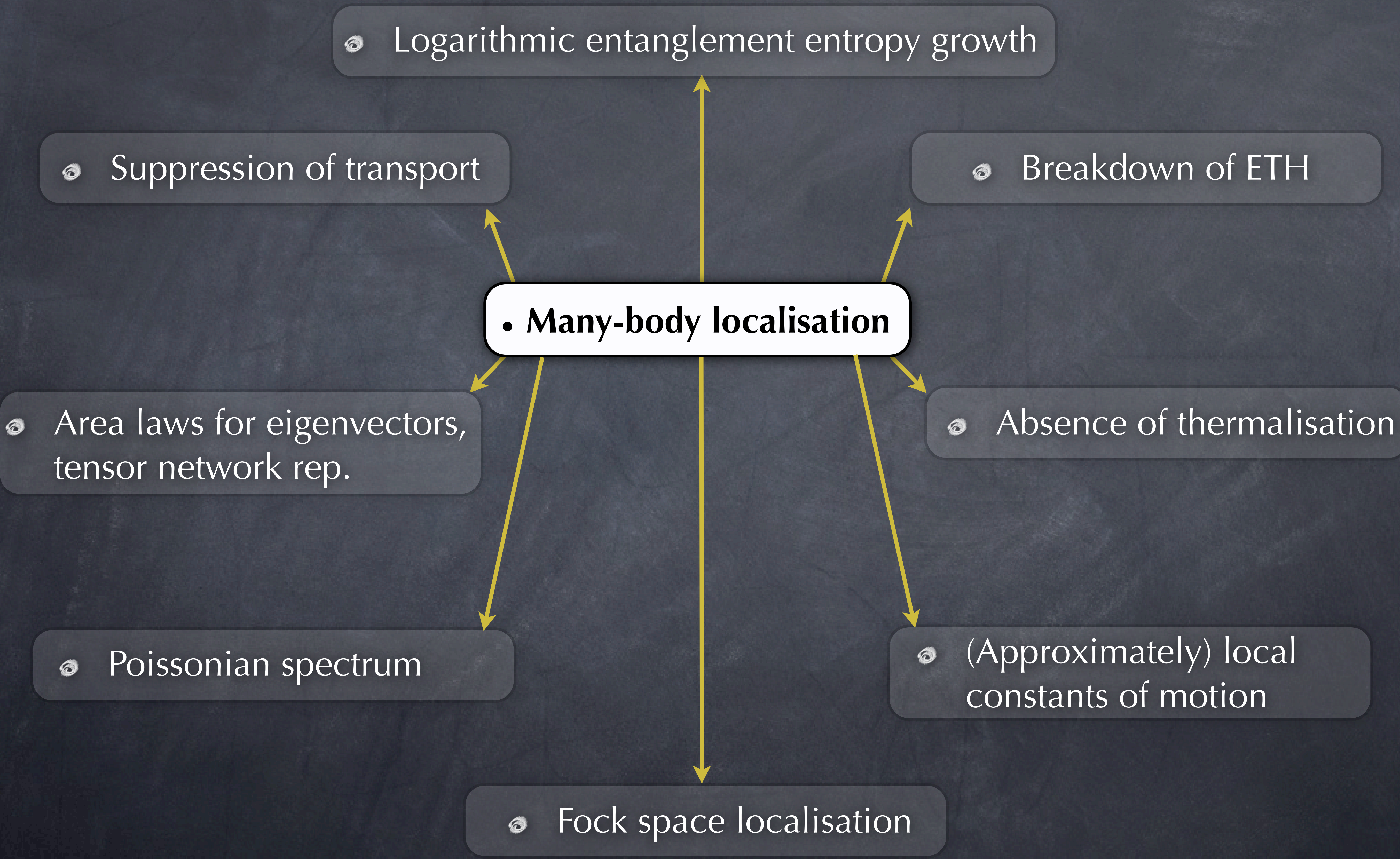
$$|\mathrm{tr}(\rho[A(t), B])| \leq \min(t, 1) c_{\mathrm{mob}} e^{-\mu d(A,B)}$$

- **Excitations get stuck:** Action of local unitaries not detectable far away $|\psi\rangle = U|0\rangle = e^{-iG}|0\rangle$

$$|\langle 0|A(t)|0\rangle - \langle 0|e^{iG} A(t) e^{-iG}|0\rangle| \leq \min(t, 1)\|G\| C e^{-\mu d(A,U)}$$

How are all these pictures related?

Many-body localisation



Many-body localisation

- Suppression of transport

- Area laws for eigenvectors, tensor network rep.

Can the dynamical picture and the static be related?

Theorem: Clustering of correlations from dynamics

- **Theorem** (clustering of correlations of eigenvectors)

a) If the Hamiltonian shows strong dynamical localisation then *all its eigenvectors* have exponentially clustering correlations

$$|\langle k|AB|k\rangle - \langle k|A|k\rangle\langle k|B|k\rangle| \leq 4c_{\mathrm{loc}} e^{-\mu d(A,B)/2}$$

b) If the Hamiltonian has a mobility edge at energy, eigenstates below mob edge

$$|\langle k|AB|k\rangle - \langle k|A|k\rangle\langle k|B|k\rangle|$$
$$\leq \left(12\pi 2^N \mathcal{N}(E_k+\kappa)c_{\mathrm{mob}} + \ln\frac{\pi\mu d(A,B)e^{4+2\pi}}{\kappa^2}\right)\frac{e^{-\mu d(A,B)/2}}{2\pi}$$

where $\mathcal{N}(E)$ is normalized integrated density of states at energy E and $\kappa > 0$ can be chosen arbitrarily

Friesdorf, Werner, Scholz, Brown, Eisert, arXiv:1409.1252

A glimpse at the proof

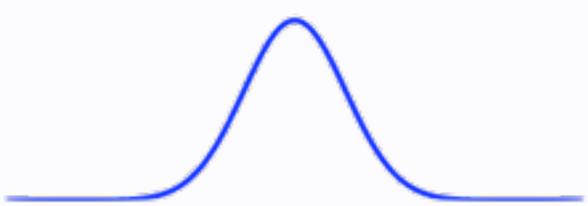
Use of filter functions $I_f^H(A) = \int_{-\infty}^{\infty} dt f(t) A(t)$

Partly diagonalizes observable in Hamiltonian basis

$$\langle k | I_f^H(A) | l \rangle = \hat{f}(E_k - E_l) \langle k | A | l \rangle$$

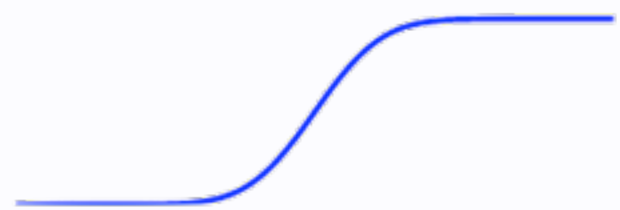
while still keeping some locality $\| I_f^H(A) - A_l \| \approx \int_{l/(2v)}^{\infty} dt |f(t)|$

Gaussian filter



$$I_\alpha^H(A) = \frac{\alpha^{1/2}}{\pi^{1/2}} \int_{-\infty}^{\infty} dt e^{-\alpha t^2} A(t)$$

High-pass filter



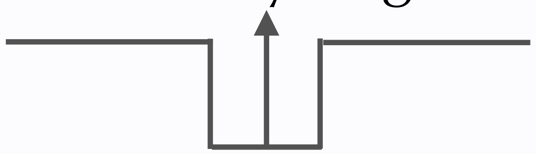
$$I_\alpha^H(A) = \lim_{\epsilon \to 0} \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{e^{-\alpha t^2}}{t + i\epsilon} A(t)$$

Gaussia

A glimpse at the proof

$\langle k|AB|k\rangle$

Mobility edge



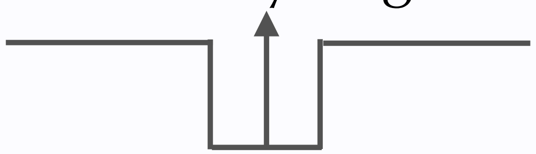
High pass filter

$\approx \langle k|AP_{\text{high}}B|k\rangle\ ,\qquad \mathbb{I} = P_{\text{high}} + P_{\text{low}}$

$\approx \langle k|I_f(A)B|k\rangle$

$\approx \langle k|[B, I_f(A)]|k\rangle$

Mobility edge



$\approx \mathcal{N} c_{\text{mob}} e^{-\mu d(A,B)}$

Goal: Bound $|\langle k|AB|k\rangle| - \langle k|A|k\rangle\langle k|B|k\rangle|$

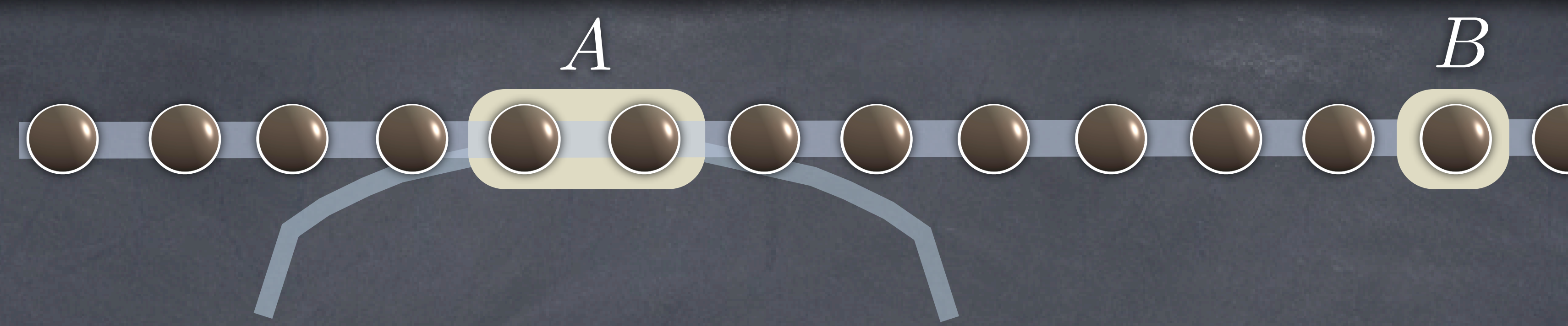
W.l.o.g. $\langle k|AB|k\rangle = 0$

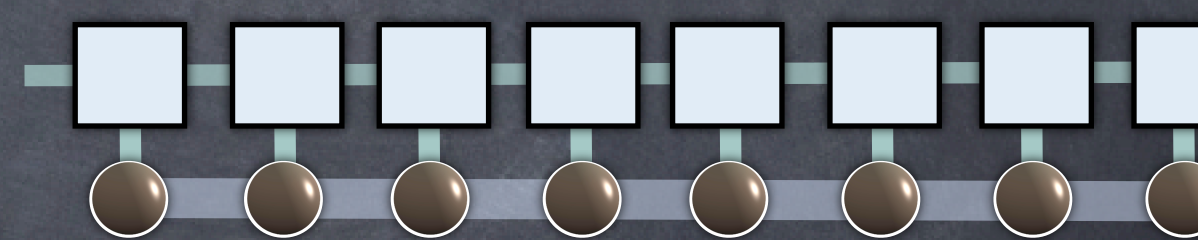
Clustering of correlations

…rf, Werner, Scholz, Brown, Eisert, arXiv:1409.1252
Compare Hastings, Koma, Commun Math Phys 265, 119 (2006)
Hamsa, Sims, Stolz, Commun Math Phys 315, 215 (2012)

Area laws and matrix-product states

Dynamical localisation with mobility edge





Goal: Bound $|\langle k|AB|k\rangle| - \langle k|A|k\rangle\langle k|B|k\rangle|$

W.l.o.g. $\langle k|AB|k\rangle = 0$

Clustering of correlations

Brandao, Horodecki, Nat Phys 9, 721 (2013
Friesdorf, Werner, Scholz, Brown, Eisert, arXiv:1409.1252
Verstraete

Many-body localisation

- Suppression of transport

[1]

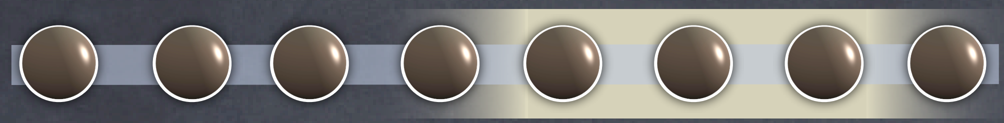
- Area laws for eigenvectors, tensor network rep.

[1] Friesdorf, Werner, Scholz, Brown, Eisert, arXiv:1409.1252

Local and approximately local constants of motion

- Local **constants of motion** (sIOM)**:** Local $\mathcal{Z}$ with $[\mathcal{Z}, H] = 0$

Local and approximately local constants of motion



- Local **constants of motion** (sIOM): Local $\mathcal{Z}$ with $[\mathcal{Z}, H] = 0$
- Local restriction of observables

$$\Gamma_S(A) := d^{-|S^c|} \mathbb{I}_{S^c} \otimes \mathrm{tr}_{S^c} A$$

- Approximately local constants of motion (qIOM) : ex $c_1, c_2 \geq 0$ s.t.

$$\|\mathcal{Z} - \Gamma_{X_l}(\mathcal{Z})\| \leq c_1 \|\mathcal{Z}\| e^{-c_2 l}$$

Local constants of motion (sIOM)

- Assume local constants of motion $\{\mathcal{Z}^{(i)} : i = 1, \ldots, N_{\mathcal{Z}}\}$ to commute with each other $[\mathcal{Z}^{(i)}, \mathcal{Z}^{(j)}] = 0$
- Algebraically independent

Local constants of motion (sIOM)

- Each sIOM $\mathcal{Z}^{(j)} = \sum_{\mu_j=1}^{q} \lambda_{\mu_j}^{(j)} \mathcal{P}_{\mu_n}^{(j)}$ can be written as matrix-product operator

Chandran, Carresquilla, Kim, Abanin, Vidal, arXiv:1410.0687

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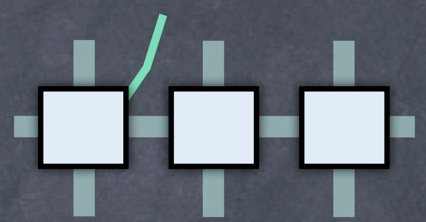

Chandran, Carresquilla, Kim, Abanin, Vidal, arXiv:1410.0687

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- Finite number of sIOM at each site: Number $N(j)$ of sIOP overlapping at site j bounded by $N(j) \leq (2l_{\mathcal{Z}} - 1)^D$

Chandran, Carresquilla, Kim, Abanin, Vidal, arXiv:1410.0687

Spectral tensor networks

$$\mathcal{P}_\mu = \mathcal{P}^{(1)}_{\mu_1} \dots \mathcal{P}^{(N)}_{\mu_N}$$

Chandran, Carresquilla, Kim, Abanin, Vidal, arXiv:1410.0687

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Spectral tensor networks

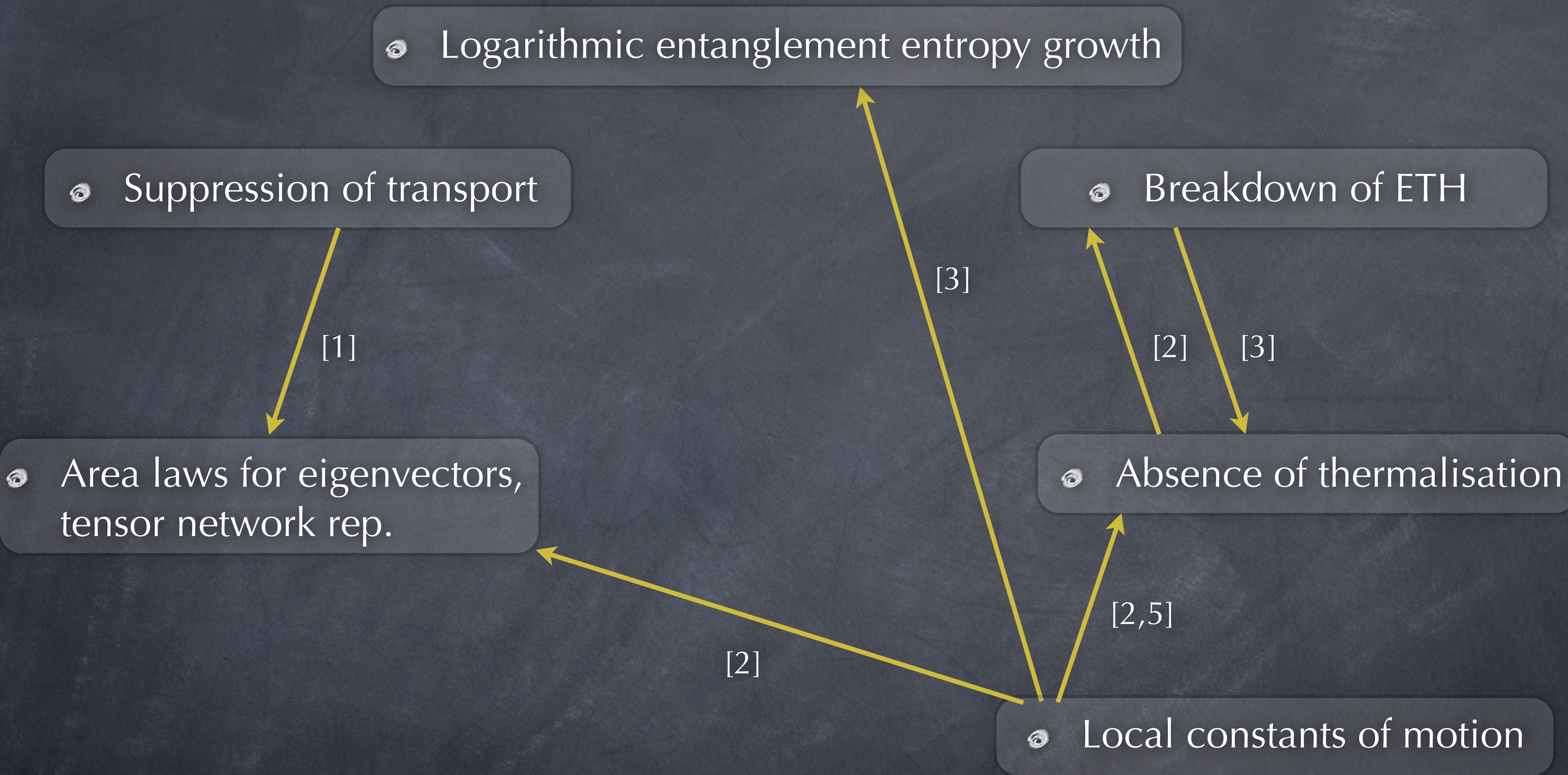
$$\mathcal{P}_\mu = \mathcal{P}^{(1)}_{\mu_1} \ldots \mathcal{P}^{(N)}_{\mu_N}$$

- **Theorem:** Gives rise to efficient **spectral tensor network**

- Can in 1D compute all local expectation values efficiently with one tensor network

Chandran, Carresquilla, Kim, Abanin, Vidal, arXiv:1410.0687

Many-body localisation



[1] Friesdorf, Werner, Scholz, Brown, Eisert, arXiv:1409.1252
[2] Chandran, Carresquilla, Kim, Abanin, Vidal, arXiv:1410.0687
[3] Kim, Chandran, Abanin, arXiv:1412.3073
[4] Gogolin, Mueller, Eisert, Phys Rev Lett 106, 040401 (2011)
[5] Chandran, Kim, Vidal, Abanin, arXiv:1407.8480

Eigenvector entanglement and dynamics?

What is the influence of eigenvector entanglement for the dynamics?

Transport using local constants of motion

- **Transport:** Ex observable A such that, for each finite region S

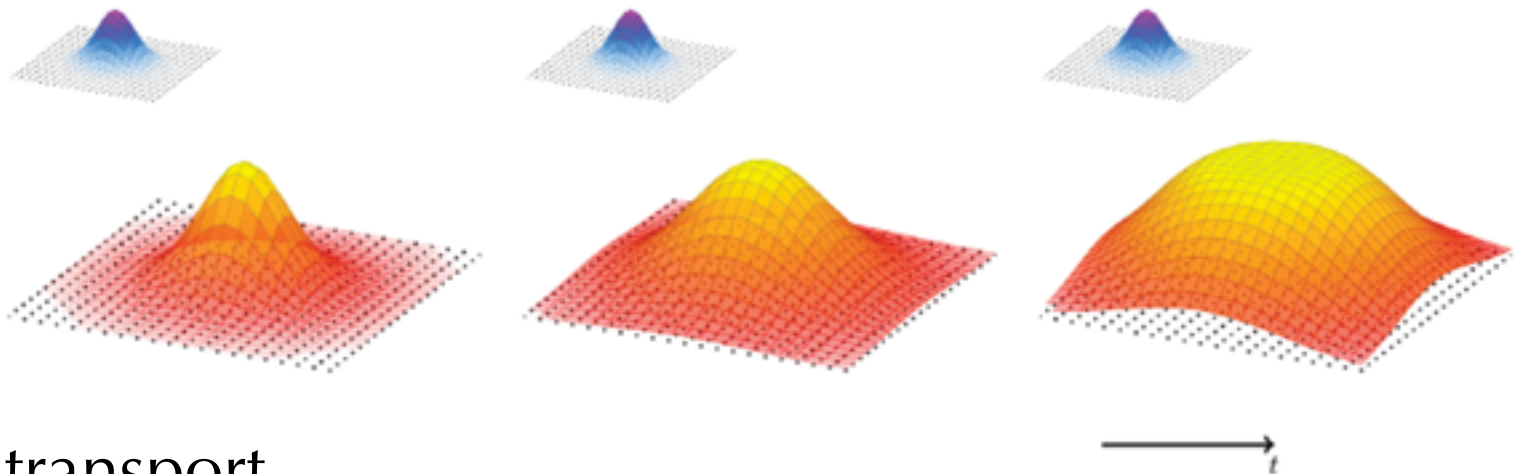
$$\mathbb{E}\|A(t) - \Gamma_S(A(t))\| \geq 1 - \epsilon$$

- Is quantum information propagation
- Implies that system can be used for signalling
- **Generic spectrum:** Energies and gaps are non-degenerate

Friesdorf, Werner, Goihl, Eisert, Brown, arXiv:1412.5605

Transport using local constants of motion

- **Theorem:**
- Let H have generic spectrum, then each of the following
    - The Hamiltonian has product eigenstates
    - The Hamiltonian has f-local eigenstates
    - There exists an exactly local constant of motion (sIOM)
    - There exists an approximately local constant of motion (qIOM)



imply transport

- Compatible with mobility edge

Friesdorf, Werner, Goihl, Eisert, Brown, arXiv:1412.5605

Survey statics versus dynamics

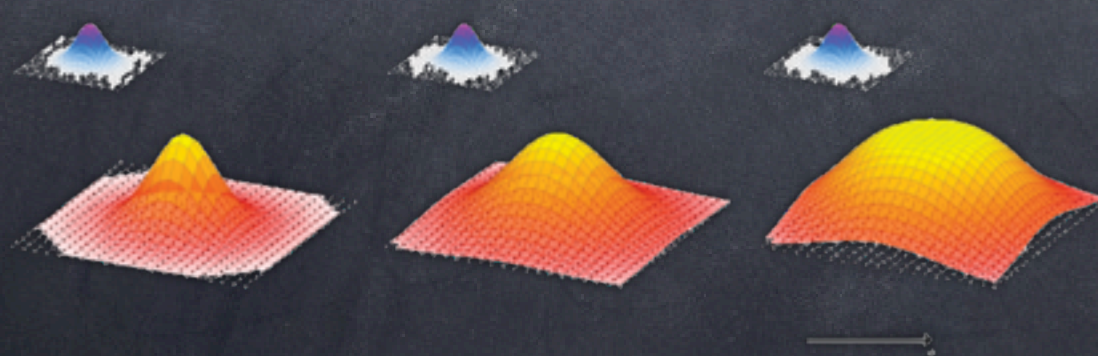
- Complete suppression of transport for low energies

  $\Rightarrow$ efficient MPS representation of low-energy eigenstates

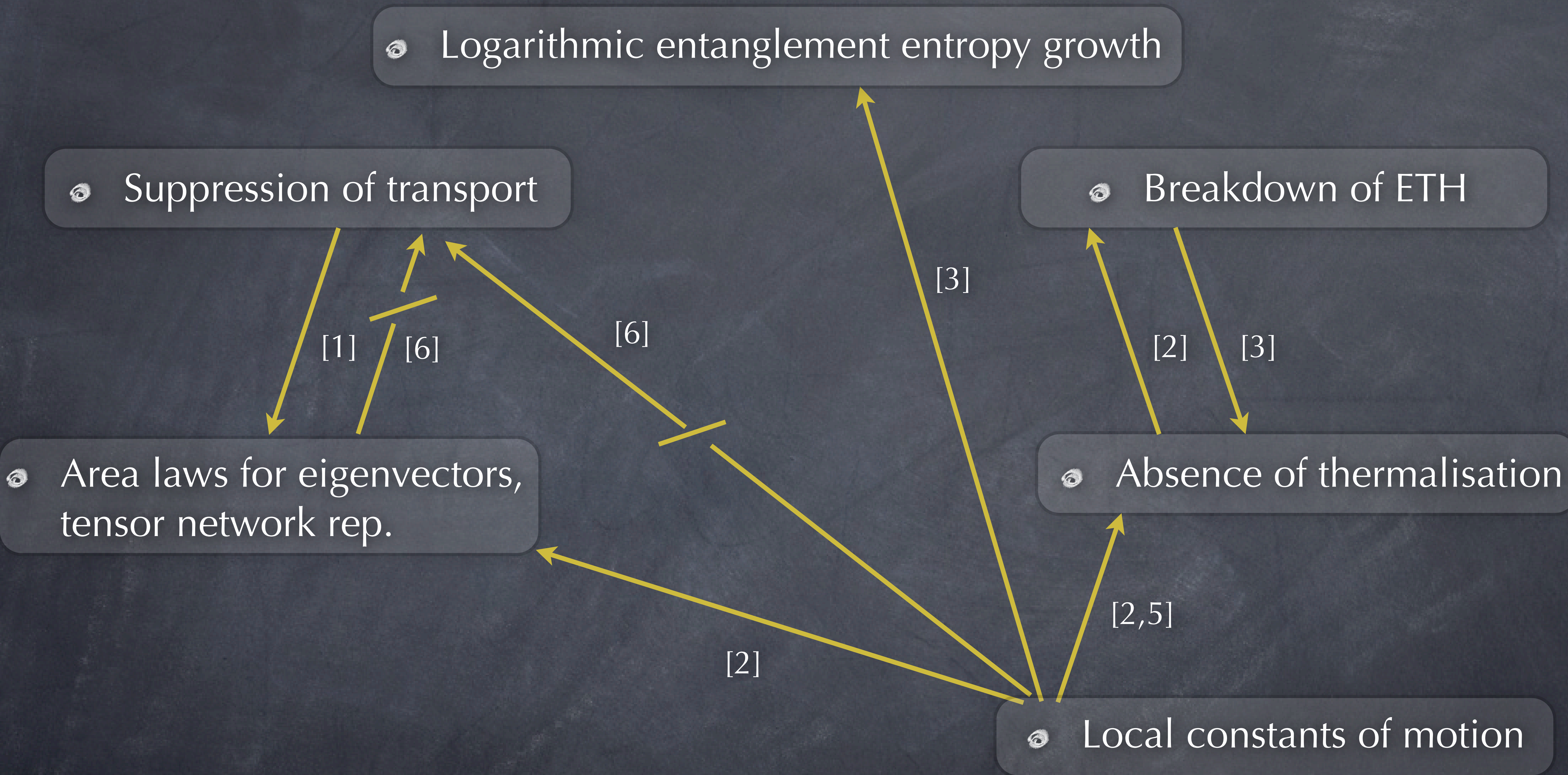
Eigenvector $|k\rangle \approx$ 

- Constants of motion + generic spectrum

  $\Rightarrow$ Information propagation possible



Many-body localisation



[1] Friesdorf, Werner, Scholz, Brown, Eisert, arXiv:1409.1252
[2] Chandran, Carresquilla, Kim, Abanin, Vidal, arXiv:1410.0687
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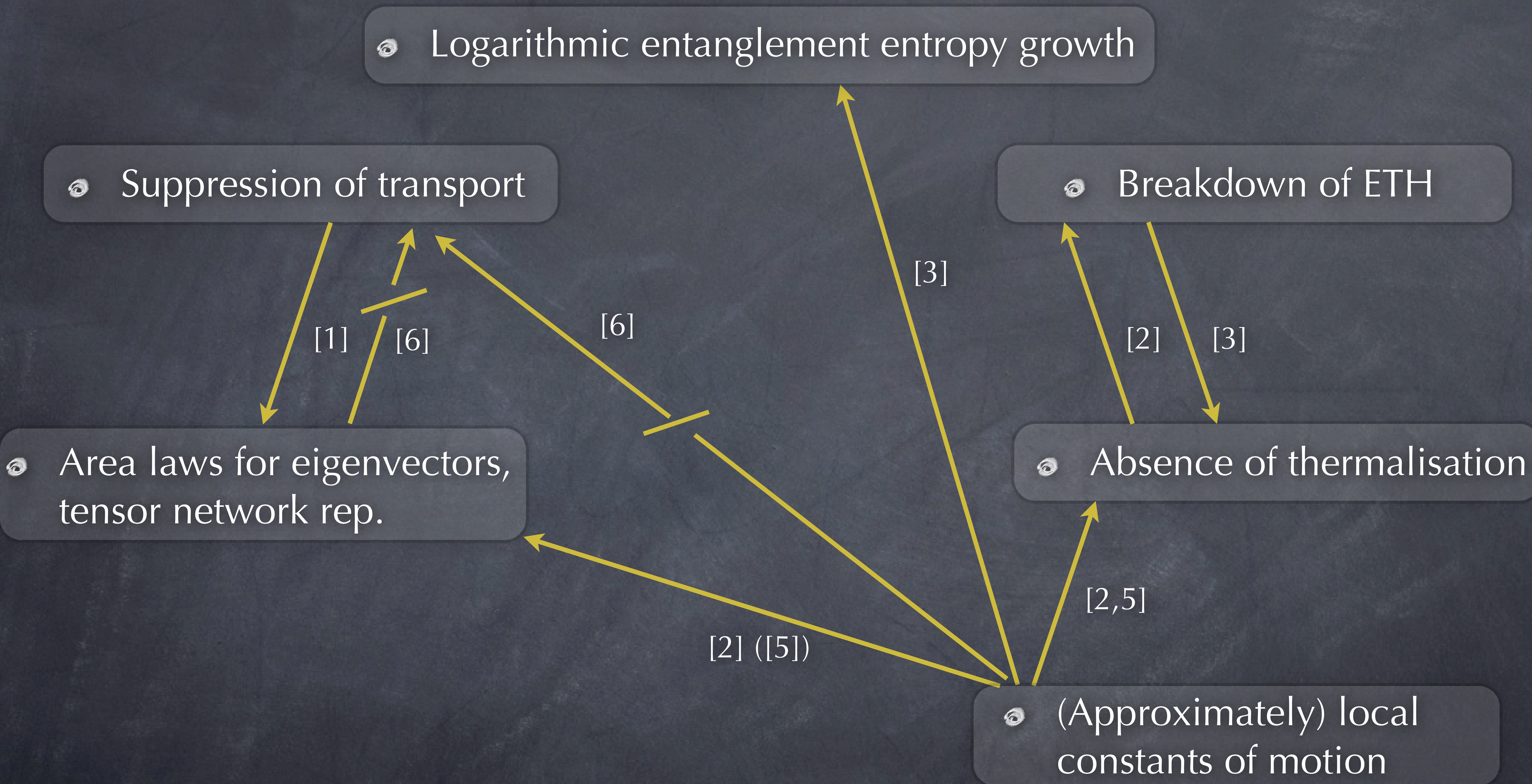
[4] Gogolin, Mueller, Eisert, Phys Rev Lett 106, 040401 (2011)
[5] Chandran, Kim, Vidal, Abanin, arXiv:1407.8480
[6] Friesdorf, Werner, Goihl, Eisert, Brown, arXiv:1412.5605

Many-body localisation



[1] Friesdorf, Werner, Scholz, Brown, Eisert, arXiv:1409.1252
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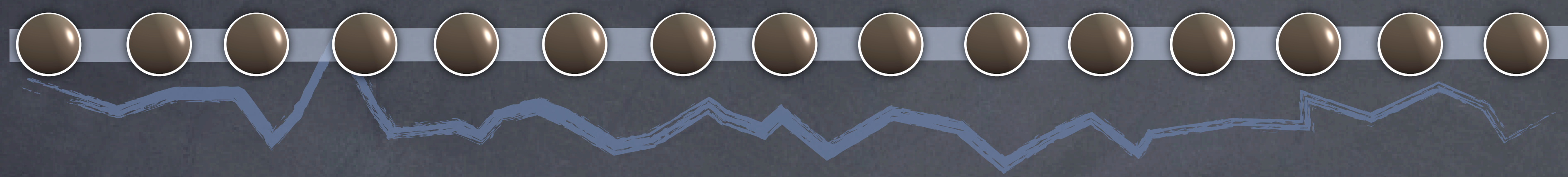
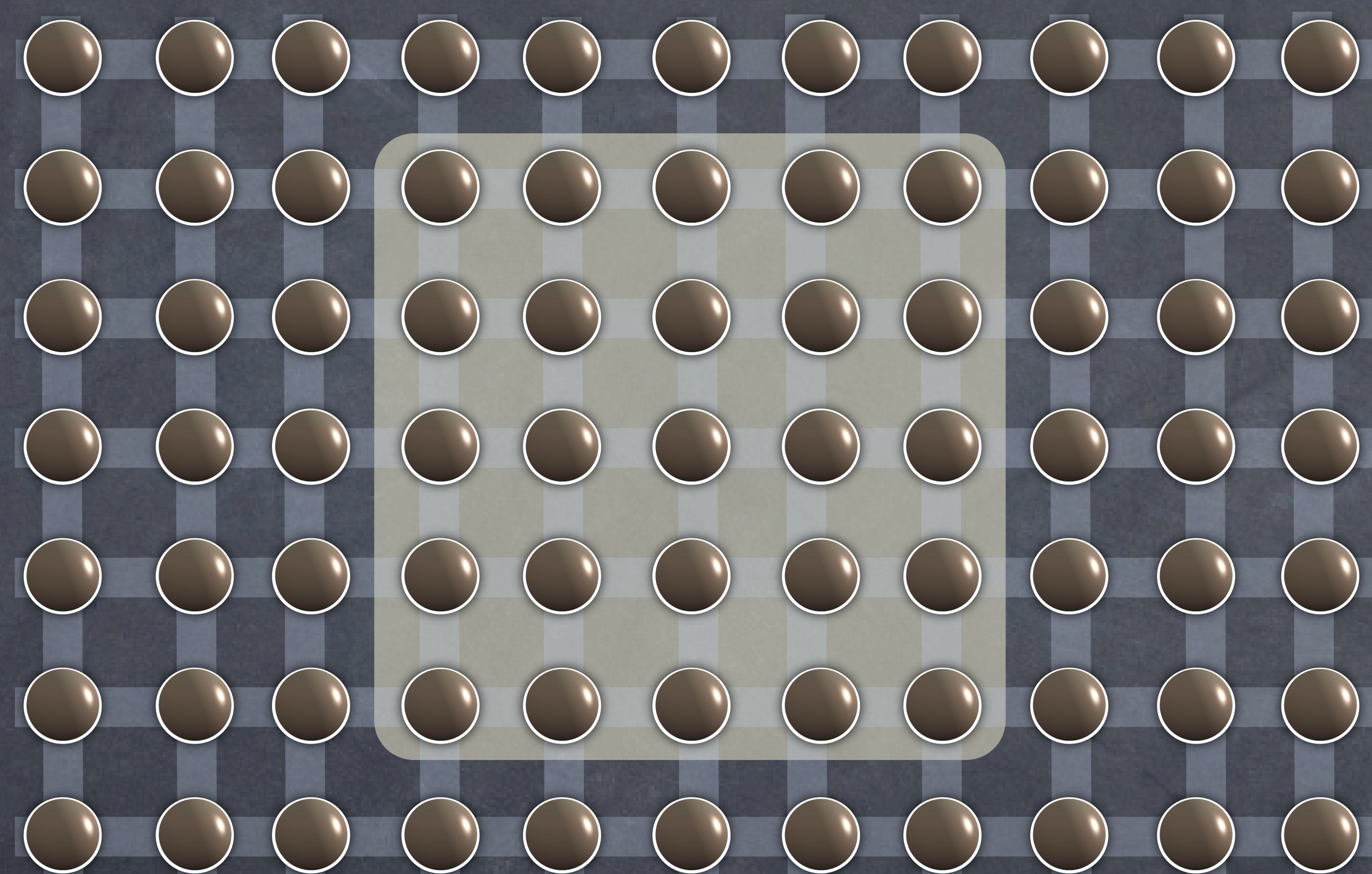
Many-body localisation



- **Lesson**:
- Even though many **aspects of MBL** are far from well-understood...
- ... **quantum information ideas** seem to bring in fresh insights beyond what can be obtained from numerics from very small systems
- **Invitation to a programme**

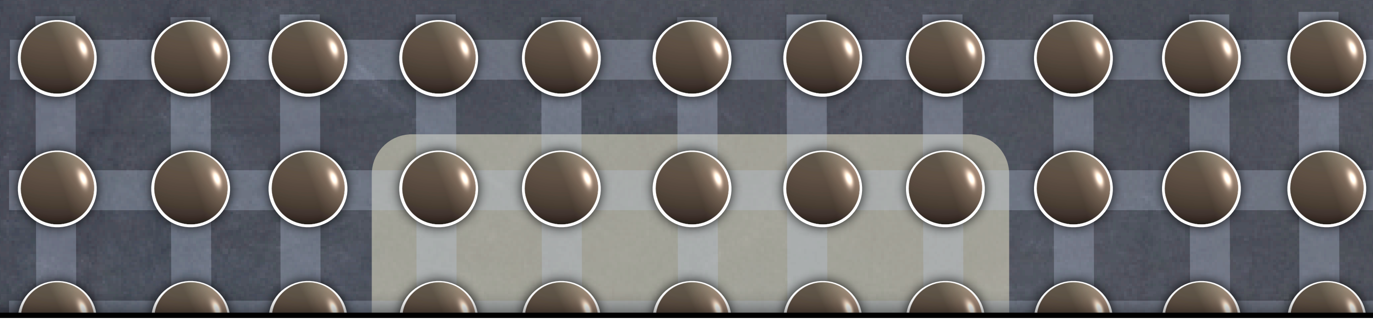
Bonus track

Commonly stated: Entanglement area laws imply efficient simulatability in terms of tensor networks



- Indeed, in 1D, (Renyi) entanglement area laws $S_\alpha(\rho_A) = O(|\partial A|)$ for $\alpha < 1$ imply an efficient approximation in terms of MPS

- Should be the same in other dimensions, or not?

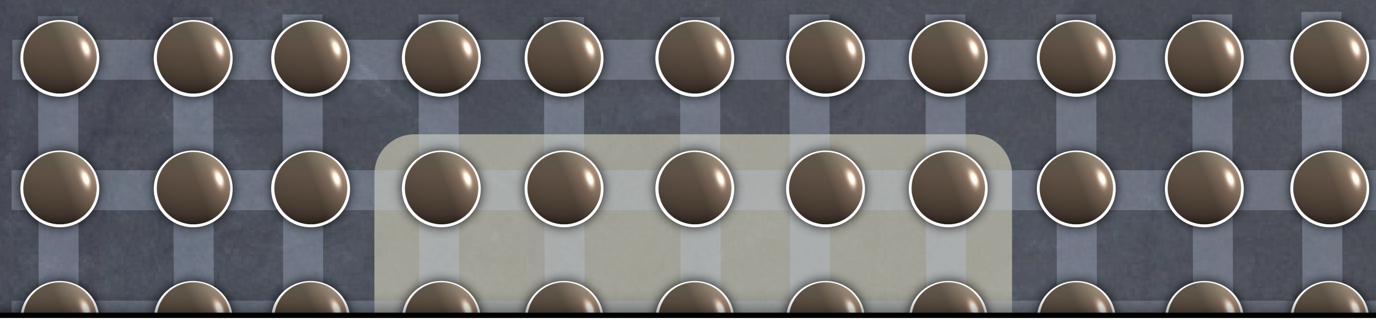


- **Theorem:** For all cubic lattices with $D > 1$, there ex states that satisfy an entanglement area law for all Renyi entropies, $S_\alpha(\rho_A) = O(|\partial A|)$ for all $\alpha \geq 0$, yet they cannot be efficiently approximated to constant error in trace-norm with

  - PEPS or MERA

Ge, Eisert, arXiv:1411.2995
Compare Poulin, Qarry, Somma, Verstraete, Phys Rev Lett 106, 170501 (2011)

Bonus track



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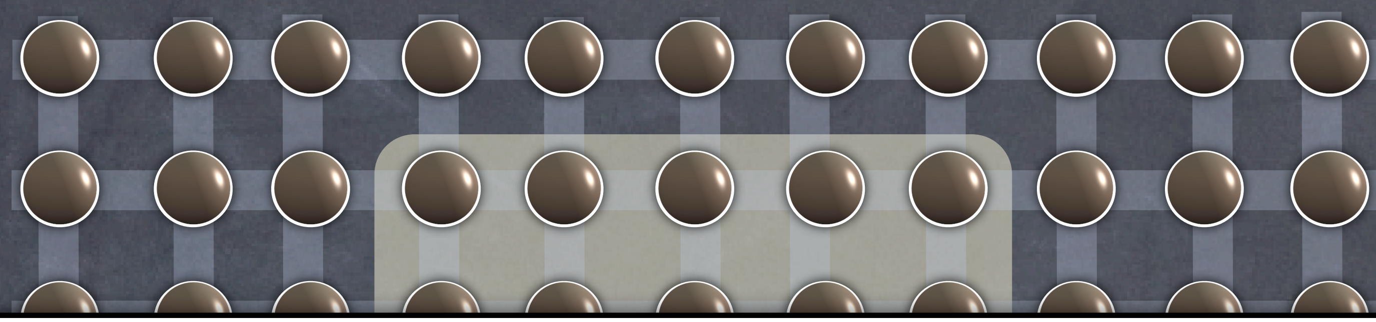
  - PEPS or MERA
  - any state with efficient classical description (Kolmogorov-complexity)

Ge, Eisert, arXiv:1411.2995
Compare Poulin, Qarry, Somma, Verstraete, Phys Rev Lett 106, 170501 (2011)

Bonus track



- **Theorem:** For all cubic lattices with $D > 1$, there ex states that satisfy an entanglement area law for all Renyi entropies, $S_\alpha(\rho_A) = O(|\partial A|)$ for all $\alpha \geq 0$, yet they cannot be efficiently approximated to constant error in trace-norm with

  - PEPS or MERA
  - any state with efficient classical description (Kolmogorov-complexity)

- Can be taken isotropic, translationally invariant, algebraic correl. decay

What else is needed?

Ge, Eisert, arXiv:1411.2995
Compare Poulin, Qarry, Somma, Verstraete, Phys Rev Lett 106, 170501 (2011)

Summary

This talk

Thanks for your attention!