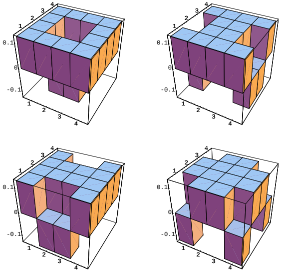
Positivity and sparsity in time-frequency distributions (with the benefit of hindsight)



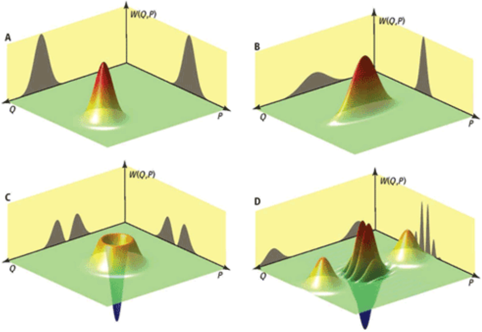
David Gross
Coogee (yeah!)
Jan '15

Outline

- Social science & math of phase spaces
- Why grown-ups should care
- Positivity & sparsity via uncertainty relations

The social science of phase spaces

The story as told by a quantum optician



- Maps density operators to pseudo-probability distribution on phase space (position-momentum plane).
- Displays most properties of a probability distribution
  - sums to one, marginal distributions, symplectic covariance,

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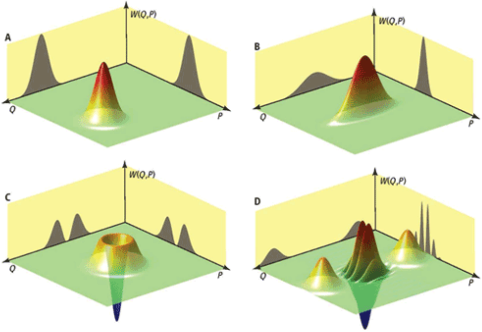


- Maps density operators to pseudo-probability distribution on phase space (position-momentum plane).
- Displays most properties of a probability distribution
  - sums to one, marginal distributions, symplectic covariance,

except...

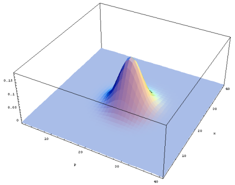
- ...it may take on negative values.

When does the analogy hold perfectly?

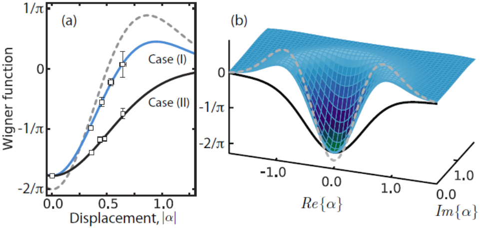
Natural question: which states give rise to non-negative Wigner distributions?

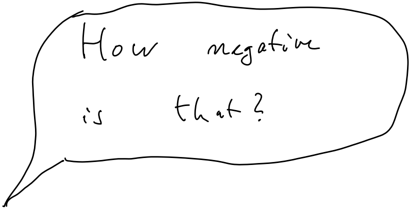
Theorem [Hudson, '74]
The only *pure states* to possess a non-negative Wigner functions are *Gaussian states*.

$$\psi(x) \propto e^{i(x\theta x + vx)}.$$



Common exchange at quantum optics conference





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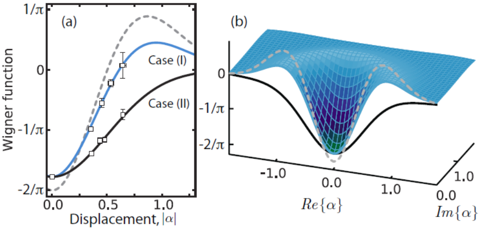

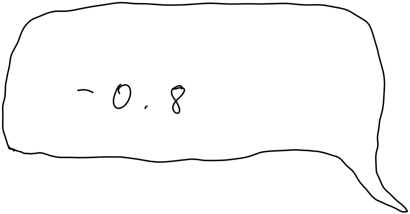

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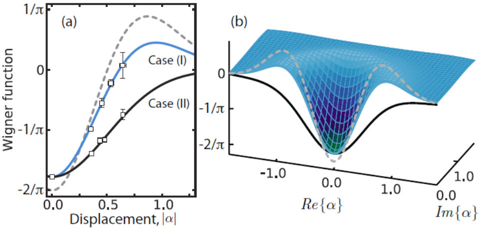

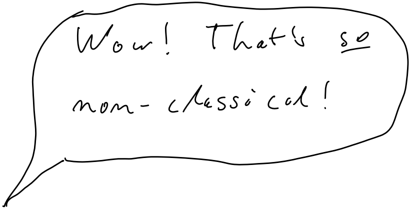

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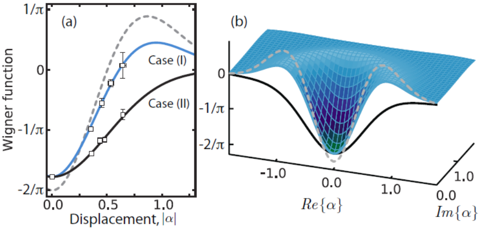

The quantum information lense

Goals of this program:

- "De-mystify" negativity,
- build a proper q'info resource theory of negativity,
- and pass to discrete systems along the way.

(Bonus: Connections to learnability of low-rank operators)

The math of quantum phase spaces.

(Bear with me).

CCR – Weyl – Heisenberg – characteristic function

- Canonical position / momentum operators:

$$[\hat{Q}, \hat{P}] = i\hbar\mathbb{1}.$$

That's a Lie algebra. Exponentiate...

CCR – Weyl – Heisenberg – characteristic function

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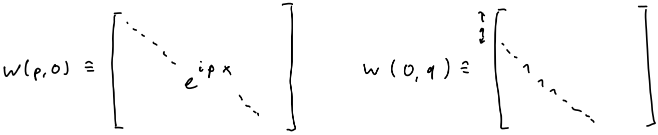
  That's a Lie algebra. Exponentiate...

- ...to get the *Weyl operators*:

$$w(p,q) \propto e^{ip\hat{Q}} e^{iq\hat{P}}$$

  for $(p,q) \in \mathbb{R}^2$.



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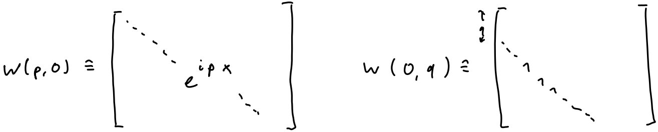

Weyl operators form a group (up to phases)

$$w(p_1, q_1)\, w(p_2, q_2) \;=\; w(p_1 + p_2, q_1 + q_2) \exp\{\pi i (p_1 q_2 - q_1 p_2)\}$$

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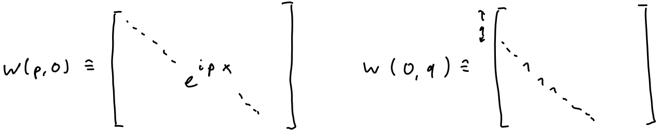

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Fun facts:

- The phase factor is *symplectic inner product* of parameters.
- The group is the *Heisenberg group* over $\mathbb{R}$.
- It acts irreducibly on $\mathcal{H} = L^2(\mathbb{R})$.

CCR – Weyl – Heisenberg – characteristic function

Fix a density operator ρ.

Def. The *characteristic function* of ρ

$$\chi_\rho(p, q) = \operatorname{tr} \rho w(p, q)$$

maps phase-space points (p, q) to the expectation value of associated Weyl operator.

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- Classically, the char. function is the Fourier transform of the probability density.
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Philosophical point:

- Classically, the char. function is the Fourier transform of the probability density.
- So name makes sense *if* "expanding in Weyl terms of Weyl ops" is some kind of FT...
- ...but it *is*. E.g. it's the non-commutative FT over the Heisenberg group.

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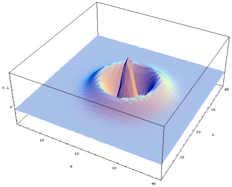
$$\chi_\rho(p, q) = \operatorname{tr} \rho w(p, q).$$

Def. The *Wigner function* of ρ

$$W_\rho(p, q) = \mathcal{F}_{(p', q') \to (p, q)} \, \chi_\rho(p', q')$$

is the (usual 2D) FT of the characteristic function.



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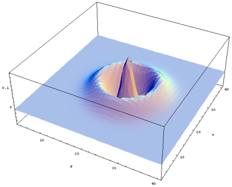
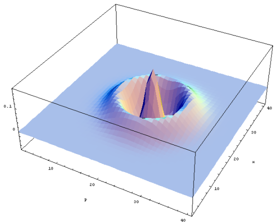
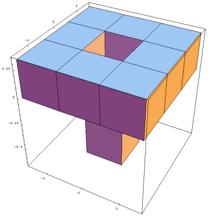


Philosophical point:

$$\rho \xrightarrow[\text{FT}]{\text{non-comm.}} \chi_\rho \xrightarrow[\text{FT}]{\text{comm.}} W_\rho$$

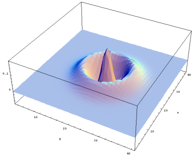
Let's go discrete.

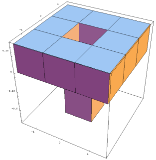




Dictionary 1

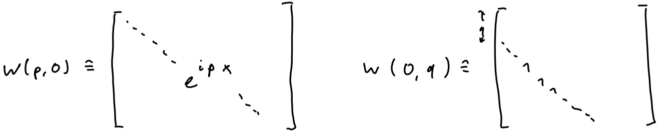
	Continuous	Discrete – d-dimensional
Configuration space	$\mathbb{R}^n$	$\mathbb{Z}_d^n = \{0, \ldots, d-1\}^n$ Arithmetic is modulo d
Hilbert space	$L^2(\mathbb{R}^n)$	$\mathbb{C}^d \simeq L^2(\mathbb{Z}_d^n)$
Phase space	$\mathbb{R}^{2n}$	$\mathbb{Z}_d^{2n}$
Weyl ops $w(p,q)$	$e^{ip\hat{Q}}e^{iq\hat{P}}$ $p, q \in \mathbb{R}^n$	??





Weyl operators

Continuous:



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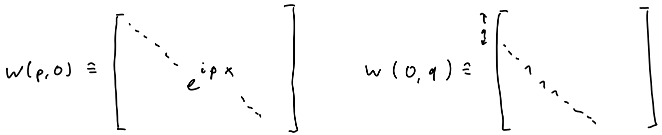

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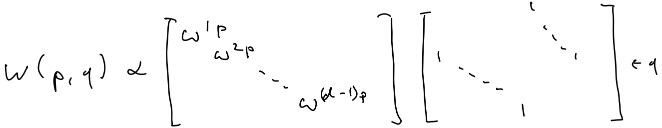
Discrete – for *odd d* – and with $\omega = e^{2\pi i/d}$:



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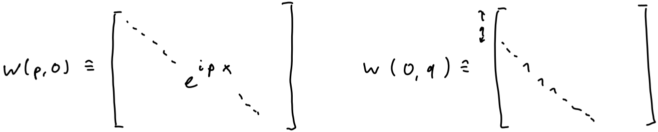

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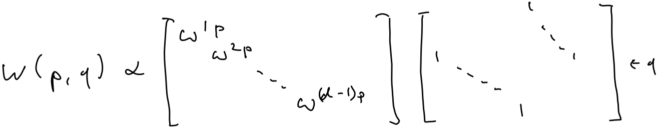

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Discrete Heisenberg group = generalized Paulis.

Dictionary 2

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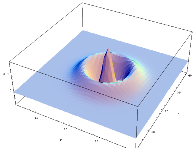
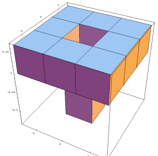
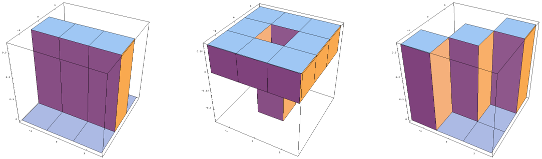
Charact. func.	$\mathrm{tr}(\rho\, w(p,q))$	$\mathrm{tr}(\rho\, w(p,q))$
Wigner func.	real FT of c.f.	DFT of c.f.





Shared properties



Approach very satisfactory. Some shared properties:

- Normalization

$$\sum_{p,q} W_\rho(p,q) = 1,$$

- Inner products

$$\operatorname{tr} \rho A = \sum_{p,q} W_\rho(p,q) W_A(p,q)$$

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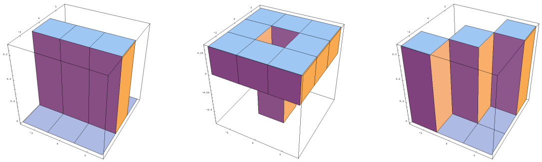

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...and also (next slides)...

- symplectic covariance,
- positivity exactly for "Gaussians",
- described by "displaced parity operators".

Positivity

Recall continuous case:

Thm. [Hudson, '74] If $\rho = |\psi\rangle\langle\psi|$, then W_ρ non-negative iff ψ is a Gaussian state:

$$\psi(x) \propto e^{i(x\theta x + vx)} \qquad (x \in \mathbb{R}^n).$$

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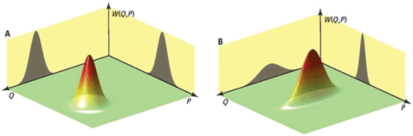
My source of early pride:

Thm. ("Discrete Hudson") [DG, '06] If $\rho = |\psi\rangle\langle\psi|$, then W_ρ non-negative iff ψ is a stabilizer state.
What is more, stabilizer states are those of the form

$$\psi(x) \propto e^{i2\pi/d(x\theta x + vx)} \qquad (x \in \mathbb{Z}_d^n)$$

(at least when restricted to their support).

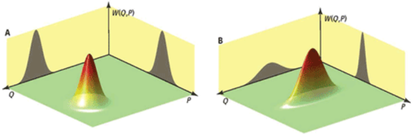
Symplectic Covariance



Let S be a symplectic phase space transformation. (I.e. det-1 matrix for one system).
Then there is a unitary U such that

$$W_{U\rho U^\dagger}(p,q) = W_\rho(S(p,q)).$$

Symplectic Covariance



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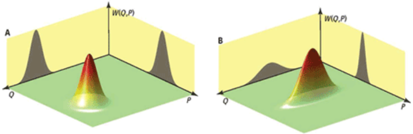

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Remarks:

- In quantum optics, these are the ops of *linear optics*
- In math-phys U is the *metaplectic representation* of S
- In q'info, these Us are the *Clifford group*
- The ops preserve positivity $\Rightarrow$ map Gaussians to Gaussians and stabs to stabs.

Parity operators

For every p, q, the map $\rho \mapsto W_\rho(p, q)$ is linear in ρ, i.e. there is a *phase space point operator* $A(p, q)$ such that

$$W_\rho(p, q) = \operatorname{tr} \rho A(p, q).$$

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Short calculation:

$$A(p, q) = w(p, q) A(0, 0) w(p, q)^\dagger,$$

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the *parity operator*.

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In particular, the $A(p, q)$'s are *unitary* (and hermitian).

Summary

Configuration space	$\mathbb{R}^n$	$\mathbb{Z}_d^n = \{0, \dots, d-1\}^n$
Hilbert space	$L^2(\mathbb{R}^n)$	$\mathrm{L}^2(\mathbb{Z}_d^n)$
Phase space	$\mathbb{R}^{2n}$	$\mathbb{Z}_d^{2n}$
Weyl operators	$e^{i(p\hat{Q}-q\hat{P})}$	$\hat{z}(p)\hat{x}(q)$
Characteristic function	$\mathrm{tr}(\rho\, w(p,q))$	$\mathrm{tr}(\rho\, w(p,q))$
Wigner function	FT of char. function $=$ exp. of disp. parity	FT of char. function $=$ exp. of disp. parity
Non-negative	$\psi(x) = e^{2\pi i(x\theta x + vx)}$	$\psi(x) = e^{\frac{2\pi}{d} i(x\theta x + vx)}$
Symmetries	$Sp(\mathbb{R}^{2n})$	$Sp(\mathbb{Z}_d^{2n})$

- Nice'ish. But looks like a kiddo-project ending up undercited in J. Phys. A. Which grown-up problems does it solve?

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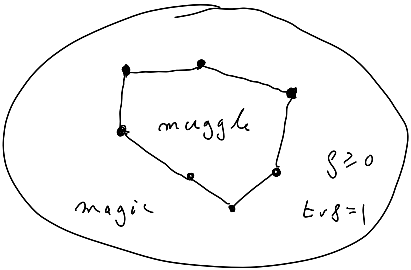
A few:

- Shows that Spekken's *episdemic toy theory* is actually stabilizer QM represented as Wigner functions
- Lead to some *simulability* results for mixed many-body dynamics [U Sydney, ongoing]
- Featured in construction of certain *quantum expanders* [DG, Eisert '07]
- But the real deal is. . .

The Resource Theory of Stabilizer Computation

[Veitch, Housavin, Gottesman, Emerson '13
Some of the above + Ferrie, DG '12]



Magic State Model

Recall that Clifford operations on stabilizer states

- Are efficiently simulable
- Cheap to implement fault-tolerantly.

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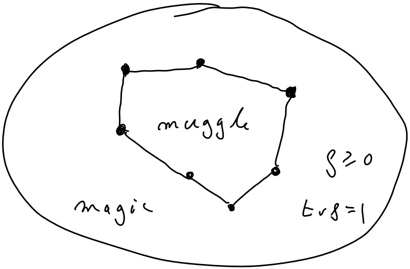
Of interest

- Practically: Error-correction thresholds
- Conceptually: "What drives putative QC speedup?" – in part. for mixed states?

Which states qualify as magic resources?

- Call ρ *muggle* if it is the convex combination of stabs
- Otherwise, ρ is *magic*.



Resource Theory 1

	Stabilizer comp	entanglement
free operations	Clifford	LOCC
free states	muggle	separable
non-free states	magic	entangled
tractable approx.	???	pos. partial transp. (tight for pure states)
bound states	???	PPT
quantitative meas.	???	log negativity

Re-Visit magic state circuit

Looking at computation in Wigner rep. . .

. . . it's plain that

- Inputs are positively represented,
- Cliffords preserve that (symplectic covariance),
- Measurements are contractions with positive functions. . .

Re-Visit magic state circuit

Looking at computation in Wigner rep. . .

. . . it's plain that

- Inputs are positively represented,
- Cliffords preserve that (symplectic covariance),
- Measurements are contractions with positive functions. . .

Hence. . .

- . . . entire scheme *efficiently simulable* unless resource states introduce negativity!

Negativity in mixed states

For mixed states: positive Wigner $\not\Rightarrow$ muggle

- Continuous case [Brocker, Werner '95]
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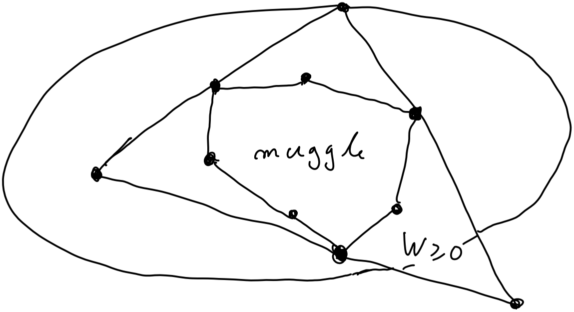

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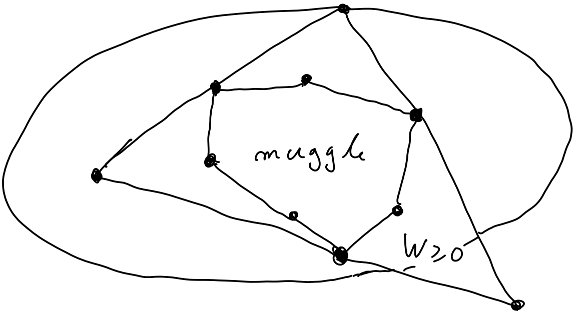


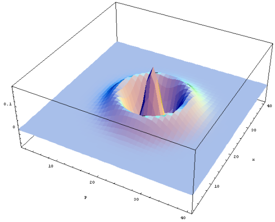
Pos-Wig is *simplicial outer approx.* of muggle

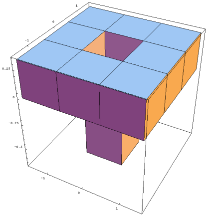
Resource Theory 2

	Stabilizer comp	entanglement
free operations	Clifford	LOCC
free states	muggle	separable
non-free states	magic	entangled
tractable approx.	pos. Wigner (tight for pure states)	pos. partial transp. (tight for pure states)
bound states	poswig	PPT
distillable	negwig?	NPT?
quantitative meas.	log negativity	log negativity

Proof sketch of discrete Hudson

. . . via phase-space uncertainty relations





Step 1: Parseval

Ingredient 1: Re-scaled $A(p,q)$'s are ONB matrix space:

$$\mathrm{tr}\left(\frac{1}{\sqrt{d}}A(p,q)\right)\left(\frac{1}{\sqrt{d}}A(p',q')\right) = \delta_{p,p'}\delta_{q,q'}.$$

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So ρ and $\frac{1}{\sqrt{d}}W_\rho =: W'_\rho$ have "same energy".

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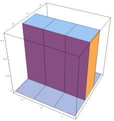

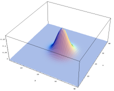

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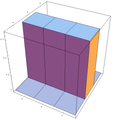

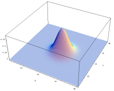

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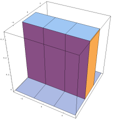

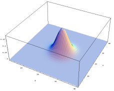

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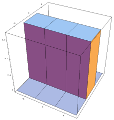

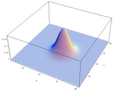

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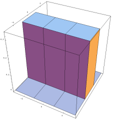

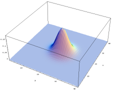

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- Fact: This characterizes stabilizer states.

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Simple and general fact:

A low-rank matrix cannot have a sparse representation in a matrix basis with small operator norm.
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Final step: Non-negativity implies minimal uncertainty

$$\sqrt{d} = \sum_{p,q} W'_\rho(p,q) = \sum_{p,q} |W'_\rho(p,q)| = \|W'_\rho\|_1 = \min .$$

and we are done.

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Advantages:

- Non-trivial also for mixed states,
- works for qubits, too.

Q: Measures of magic based on char. function uncertainties?

Thanks for your attention.