Positivity and sparsity in time-frequency distributions
(with the benefit of hindsight)

David Gross
Coogee (yeah!)
Jan '15
Outline

- Social science & math of phase spaces
- Why grown-ups should care
- Positivity & sparsity via uncertainty relations
The social science of phase spaces
The story as told by a quantum optician

- Maps density operators to pseudo-probability distribution on phase space (position-momentum plane).
- Displays most properties of a probability distribution
  - sums to one, marginal distributions, symplectic covariance, except...
The story as told by a quantum optician

- Maps density operators to pseudo-probability distribution on phase space (position-momentum plane).
- Displays most properties of a probability distribution
  - sums to one, marginal distributions, symplectic covariance, except...
- ...it may take on negative values.
When does the analogy hold perfectly?

Natural question: which states give rise to non-negative Wigner distributions?

**Theorem [Hudson, ’74]**
The only *pure states* to possess a non-negative Wigner functions are *Gaussian states*.

\[ \psi(x) \propto e^{i(x \theta x + v x)}. \]
How negative is that?
Common exchange at quantum optics conference

- 0.8
Common exchange at quantum optics conference

Wow! That's so non-classical!
Common exchange at quantum optics conference
The quantum information lense

Goals of this program:

▶ “De-mystify” negativity,
▶ build a proper q’info resource theory of negativity,
▶ and pass to discrete systems along the way.

(Bonus: Connections to learnability of low-rank operators)
The math of quantum phase spaces.

(Bear with me).
Canonical position / momentum operators:

\[ [\hat{Q}, \hat{P}] = i\hbar \mathbb{1}. \]

That's a Lie algebra. Exponentiate...
CCR – Weyl – Heisenberg – characteristic function

- Canonical position / momentum operators:

\[ [\hat{Q}, \hat{P}] = i\hbar \mathbb{1}. \]

That's a Lie algebra. Exponentiate...

- ...to get the Weyl operators:

\[ w(p, q) \propto e^{ip\hat{Q}} e^{iq\hat{P}} \]

for \((p, q) \in \mathbb{R}^2\).

\[
\begin{align*}
w(p, 0) & \approx \begin{bmatrix} e^{ipx} \end{bmatrix} \\
w(0, q) & \approx \begin{bmatrix} e^{iqy} \end{bmatrix}
\end{align*}
\]
Weyl operators form a group (up to phases)

\[ w(p_1, q_1) w(p_2, q_2) = w(p_1 + p_2, q_1 + q_2) \exp\{\pi i(p_1 q_2 - q_1 p_2)\\} \]
CCR – Weyl – Heisenberg – characteristic function

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Fun facts:

- The phase factor is symplectic inner product of parameters.
- The group is the Heisenberg group over \( \mathbb{R} \).
- It acts irreducibly on \( \mathcal{H} = L^2(\mathbb{R}) \).
Fix a density operator $\rho$.

**Def.** The *characteristic function* of $\rho$

$$\chi_{\rho}(p, q) = \text{tr} \, \rho w(p, q)$$

maps phase-space points $(p, q)$ to the expectation value of associated Weyl operator.
Fix a density operator \( \rho \).

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Philosophical point:

- Classically, the char. function is the Fourier transform of the probability density.
- So name makes sense *if* “expanding in Weyl terms of Weyl ops” is some kind of FT...
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**Philosophical point:**

- Classically, the char. function is the Fourier transform of the probability density.
- So name makes sense *if* “expanding in Weyl terms of Weyl ops” is some kind of FT . . .
- . . . but it *is*. E.g. it’s the non-commutative FT over the Heisenberg group.
Fix a density operator \( \rho \).

**Def.** The **characteristic function** of \( \rho \):

\[
\chi_{\rho}(p, q) = \text{tr} \rho w(p, q).
\]

**Def.** The **Wigner function** of \( \rho \)

\[
W_{\rho}(p, q) = \mathcal{F}_{(p', q') \rightarrow (p, q)} \chi_{\rho}(p', q')
\]

is the (usual 2D) FT of the characteristic function.
Fix a density operator $\rho$.

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$$W_\rho(p, q) = \mathcal{F}(p', q') \to (p, q) \chi_\rho(p', q')$$

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Philosophical point:
Let’s go discrete.
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<th>Continuous</th>
<th>Discrete – $d$-dimensional</th>
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Weyl operators

Continuous:

\[ w(p, 0) \equiv \begin{bmatrix} e^{ipx} \end{bmatrix} \quad w(0, q) \equiv \begin{bmatrix} \frac{\omega}{2} \end{bmatrix} \]

\[ w(p_1, q_1) w(p_2, q_2) = w(p_1 + p_2, q_1 + q_2) e^{\pi i (p_1 q_2 - q_1 p_2)} \]
Weyl operators

Continuous:

\[ \omega(p, 0) = \begin{bmatrix} e^{ipx} \\ \vdots \\ \vdots \end{bmatrix} \]

\[ \omega(0, q) = \begin{bmatrix} e^{iqy} \\ \vdots \\ \vdots \end{bmatrix} \]

\[ \omega(p_1, q_1) \omega(p_2, q_2) = \omega(p_1 + p_2, q_1 + q_2) e^{\pi i (p_1 q_2 - q_1 p_2)} \]

Discrete – for odd \( d \) – and with \( \omega = e^{2\pi i / d} \):

\[ \omega(p, q) \triangleq \begin{bmatrix} \omega^p \\ \omega^{2p} \\ \vdots \\ \omega^{(d-1)p} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \]

\[ \Rightarrow \omega(p_1, q_1) \omega(p_2, q_2) = \omega(p_1 + p_2, q_1 + q_2) \omega^{p_1 q_2 - q_1 p_2}. \]
Weyl operators

Continuous:

\[
\omega(p, q) = \begin{bmatrix}
\omega^p & 0 \\
0 & \omega^{-p}
\end{bmatrix}
\]

\[
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\omega(p, q) \propto \begin{bmatrix}
\omega^p & 0 & \cdots & 0 \\
0 & \omega^p & \cdots & 0 \\
\vdots & & \ddots & \vdots \\
0 & \cdots & 0 & \omega^p
\end{bmatrix}
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\[\Rightarrow \omega(p_1, q_1) \omega(p_2, q_2) = \omega(p_1 + p_2, q_1 + q_2)\omega^{p_1 q_2 - q_1 p_2} \]

Discrete Heisenberg group = generalized Paulis.
## Dictionary 2

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<td>real FT of c.f.</td>
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Approach very satisfactory. Some shared properties:

- Normalization
  \[ \sum_{p,q} W_\rho(p, q) = 1, \]

- Inner products
  \[ \text{tr} \rho A = \sum_{p,q} W_\rho(p, q) W_A(p, q) \]
Shared properties

Approach very satisfactory. Some shared properties:

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- Inner products
  \[ \text{tr } \rho A = \sum_{p,q} W_\rho(p, q) W_A(p, q) \]

...and also (next slides)...

- symplectic covariance,
- positivity exactly for “Gaussians”,
- described by “displaced parity operators”.
Recall continuous case:

**Thm.** [Hudson, ’74] If \( \rho = |\psi\rangle\langle \psi| \), then \( W_\rho \) non-negative iff \( \psi \) is a Gaussian state:

\[
\psi(x) \propto e^{i(x\theta + vx)} \quad (x \in \mathbb{R}^n).
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My source of early pride:

**Thm.** ("Discrete Hudson") [DG, ’06] If \( \rho = |\psi\rangle\langle \psi| \), then \( W_\rho \) non-negative iff \( \psi \) is a stabilizer state. What is more, stabilizer states are those of the form

\[
\psi(x) \propto e^{i2\pi/d(x\theta + vx)} \quad (x \in \mathbb{Z}^n_d) \quad \text{(at least when restricted to their support)}.
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Let $S$ be a symplectic phase space transformation. (i.e. det-1 matrix for one system). Then there is a unitary $U$ such that

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Remarks:

- In quantum optics, these are the ops of linear optics
- In math-phys $U$ is the metaplectic representation of $S$
- In q’info, these $Us$ are the Clifford group
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- In quantum optics, these are the ops of *linear optics*
- In math-phys $U$ is the *metaplectic representation* of $S$
- In q’info, these $U$s are the *Clifford group*
- The ops preserve positivity $\Rightarrow$ map Gaussians to Gaussians and stabs to stabs.
Parity operators

For every $p, q$, the map $\rho \mapsto W_\rho(p, q)$ is linear in $\rho$, i.e. there is a phase space point operator $A(p, q)$ such that

$$W_\rho(p, q) = \text{tr} \rho A(p, q).$$

Short calculation:

$$A(p, q) = w(p, q) A(0, 0) w(p, q)^\dagger,$$

with $(A(0, 0) \psi)(x) = \psi(-x)$. In particular, the $A(p, q)$'s are unitary (and hermitian).
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In particular, the $A(p, q)$’s are unitary (and hermitian).
## Summary

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Nice’ish. But looks like a kiddo-project ending up undercited in J. Phys. A. Which grown-up problems does it solve?

A few:

- Shows that Spekken’s episodic toy theory is actually stabilizer QM represented as Wigner functions
- Lead to some simulability results for mixed many-body dynamics [U Sydney, ongoing]
- Featured in construction of certain quantum expanders [DG, Eisert ’07]

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▶ Featured in construction of certain *quantum expanders* [DG, Eisert ’07]
▶ But the real deal is...
The Resource Theory of Stabilizer Computation

[Veitch, Housavin, Gottesman, Emerson ’13
Some of the above + Ferrie, DG ’12]
Magic State Model

Recall that Clifford operations on stabilizer states

▶ Are efficiently simulable
▶ Cheap to implement fault-tolerantly.
Magic State Model

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- However, scheme becomes *universal* if augmented by occasional injection of non-stab “magic states”.
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![Diagram of magic state model](image-url)

Practically: Error-correction thresholds


For mixed states?
Magic State Model

Recall that Clifford operations on stabilizer states

- Are efficiently simulable
- Cheap to implement fault-tolerantly.
- However, scheme becomes *universal* if augmented by occasional injection of non-stab “magic states”.

Of interest

- Practically: Error-correction thresholds
- Conceptually: “What drives putative QC speedup?” – in part. for mixed states?
Which states qualify as magic resources?

- Call $\rho$ *muggle* if it is the convex combination of stabs.
- Otherwise, $\rho$ is *magic*.
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<td>PPT</td>
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<td>???</td>
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Re-Visit magic state circuit

Looking at computation in Wigner rep... 

... it’s plain that

▶ Inputs are positively represented,
▶ Cliffords preserve that (symplectic covariance),
▶ Measurements are contractions with positive functions...
Re-Visit magic state circuit

Looking at computation in Wigner rep...

\[ \begin{array}{c}
\text{Inputs are positively represented,} \\
\text{Cliffords preserve that (symplectic covariance),} \\
\text{Measurements are contractions with positive functions...} \\
\end{array} \]

Hence...

\[ \begin{array}{c}
\text{...entire scheme \textit{efficiently simulable} unless resource states introduce negativity!} \\
\end{array} \]
Negativity in mixed states

For mixed states: positive Wigner /muggle

- Continuous case [Brocker, Werner ’95]
- Discrete case [DG ’06]
Negativity in mixed states

For mixed states: positive Wigner / muggle

- Continuous case [Brocker, Werner ’95]
- Discrete case [DG ’06]

But nicest argument by [Waterloo gang]:

\[ \text{muggle} \]
Negativity in mixed states

For mixed states: positive Wigner $\not\equiv$ muggle

- Continuous case [Brocker, Werner ’95]
- Discrete case [DG ’06]

But nicest argument by [Waterloo gang]:

Pos-Wig is simplicial outer approx. of muggle
## Resource Theory 2

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Proof sketch of discrete Hudson

... via phase-space uncertainty relations
Step 1: Parseval

Ingredient 1: Re-scaled $A(p, q)$’s are ONB matrix space:

$$
\text{tr} \left( \frac{1}{\sqrt{d}} A(p, q) \right) \left( \frac{1}{\sqrt{d}} A(p', q') \right) = \delta_{p, p'} \delta_{q, q'}.
$$
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Hence

$$\| \rho \|_2^2 = \sum_{i,j} |\rho_{i,j}|^2 = \frac{1}{d} \sum_{p,q} |W_{\rho}(p, q)|^2 = \left\| \frac{1}{\sqrt{d}} W_{\rho} \right\|_2^2.$$ 

So $\rho$ and $\frac{1}{\sqrt{d}} W_{\rho}$ have “same energy”.
Step 1: Parseval

Ingredient 1: Re-scaled $A(p, q)$'s are ONB matrix space:

$$\text{tr} \left( \frac{1}{\sqrt{d}} A(p, q) \right) \left( \frac{1}{\sqrt{d}} A(p', q') \right) = \delta_{p,p'} \delta_{q,q'}.$$ 

Hence

$$\|\rho\|_2^2 = \sum_{i,j} |\rho_{i,j}|^2 = \frac{1}{d} \sum_{p,q} |W_\rho(p, q)|^2 = \left\| \frac{1}{\sqrt{d}} W_\rho \right\|_2^2.$$ 

So $\rho$ and $\frac{1}{\sqrt{d}} W_\rho =: W'_\rho$ have “same energy”.
Step 2: Uncertainty Relation

Ingredient 2: The energy can’t be highly localized in phase space.
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Ingredient 2: The energy can’t be highly localized in phase space.

- Assume $\rho = |\psi\rangle \langle \psi|$ pure $\iff \|\rho\|_2^2 = 1$,
Step 2: Uncertainty Relation

Ingredient 2: The energy can’t be highly localized in phase space.

- Assume $\rho = |\psi\rangle\langle\psi|$ pure $\Leftrightarrow \|\rho\|^2_2 = 1$,
- and use $\ell_1$-norm as measure of de-localization:

$$\|\chi'_\rho\|_1 = \sum_{p,q} |W'_\rho(p, q)| \in [1, d]$$
Step 2: Uncertainty Relation

Ingredient 2: The energy can’t be highly localized in phase space.

- Assume $\rho = |\psi\rangle \langle \psi|$ pure $\iff \|\rho\|^2_2 = 1$,
- and use $\ell_1$-norm as measure of de-localization:

$$\|\chi'_\rho\|_1 = \sum_{p, q} |W'_\rho(p, q)| \in [1, d]$$

By matrix Hölder inequality,

$$|W'_\rho(p, q)| \leq \frac{1}{\sqrt{d}} \|A(p, q)\|_\infty \|\rho\|_{tr} \leq \frac{1}{\sqrt{d}},$$

which is tight iff $|\psi\rangle$ is an eigenvector of $A(p, q)$. 

Fact: This characterizes stabilizer states.
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- it follows that $\|W'_\rho\|_1 \geq \sqrt{d}$,
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A low-rank matrix cannot have a sparse representation in a matrix basis with small operator norm.
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Final step: Non-negativity implies minimal uncertainty

\[
\sqrt{d} = \sum_{p,q} W'_\rho(p, q) = \sum_{p,q} |W'_\rho(p, q)| = \|W'_\rho\|_1 = \min.
\]

and we are done.
Outlook

Message: *uncertainty relations* more fundamental than positivity.
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with equality if \( \rho \) is a stabilizer *code*.
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Advantages:

- Non-trivial also for mixed states,
- works for qubits, too.

Q: Measures of magic based on char. function uncertainties?
Thanks for your attention.