Many-body entanglement witness

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Coogee, Australia

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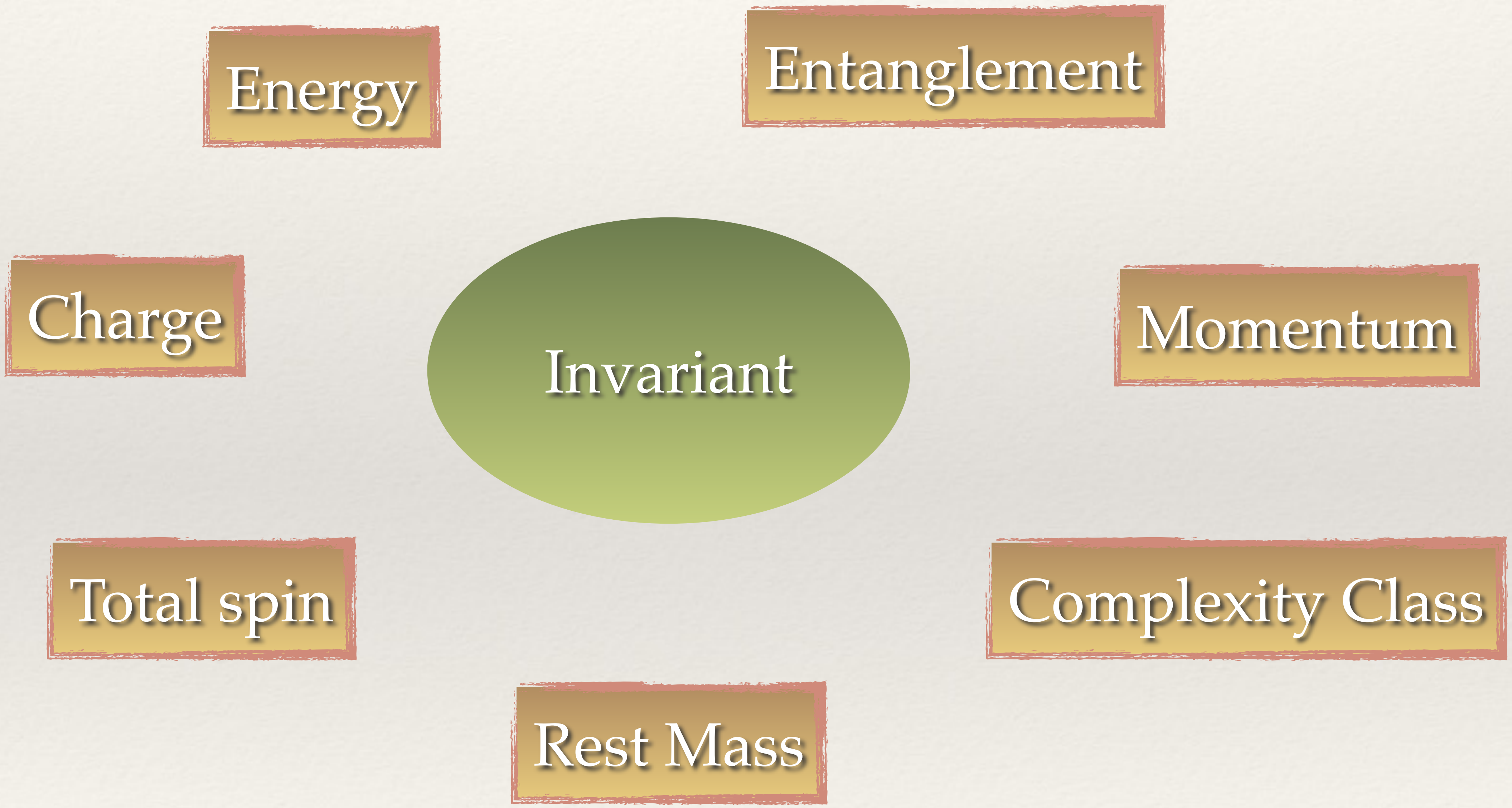
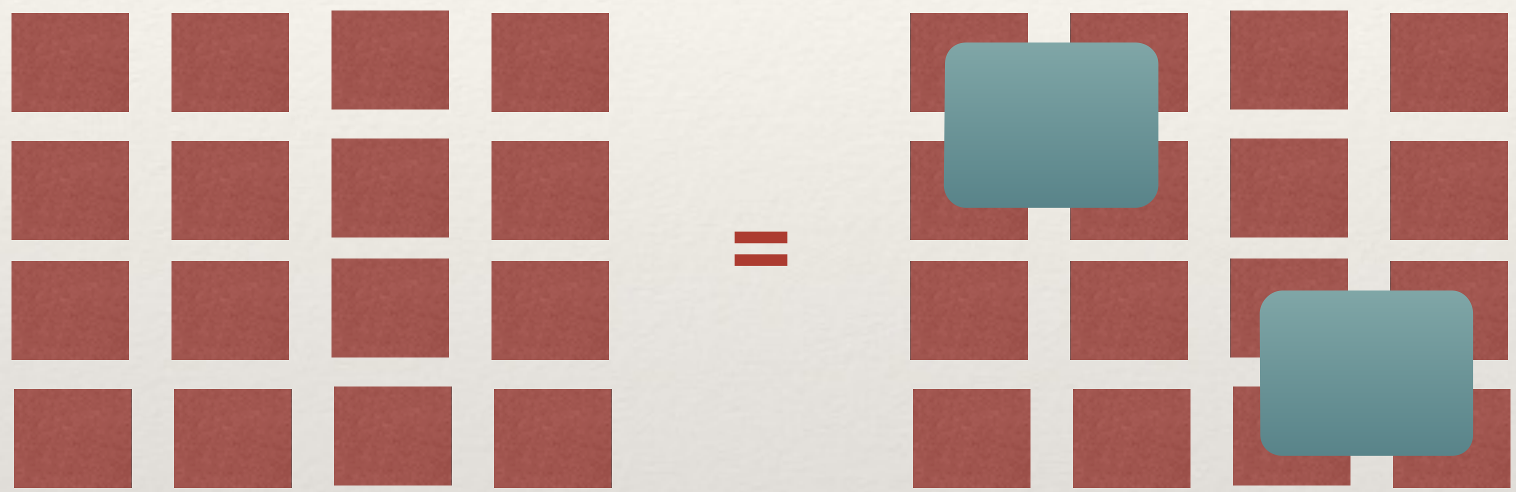
Quiz

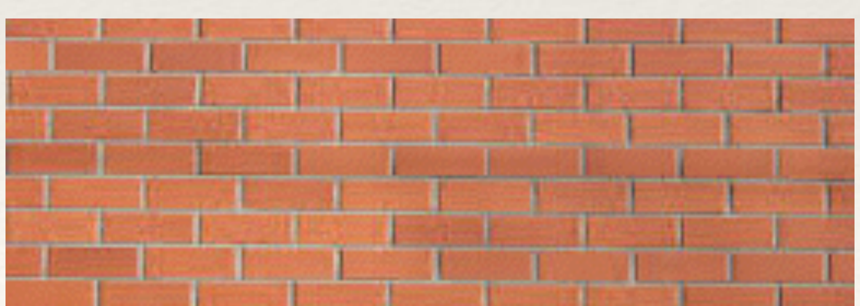

Energy
Entanglement
Charge
Invariant
Momentum
Total spin
Complexity Class
Rest Mass

Many-body Entanglement

- Local entanglement can be washed away by local unitaries.
- Equivalence relation among states:



Transitivity:
If A=B and B=C, then A=C



$$|01001\cdots011\rangle$$

MANY-BODY ENTANGLEMENT
IS AN EQUIVALENCE CLASS
UNDER
SMALL-DEPTH QUANTUM CIRCUITS

Many-body Entanglement

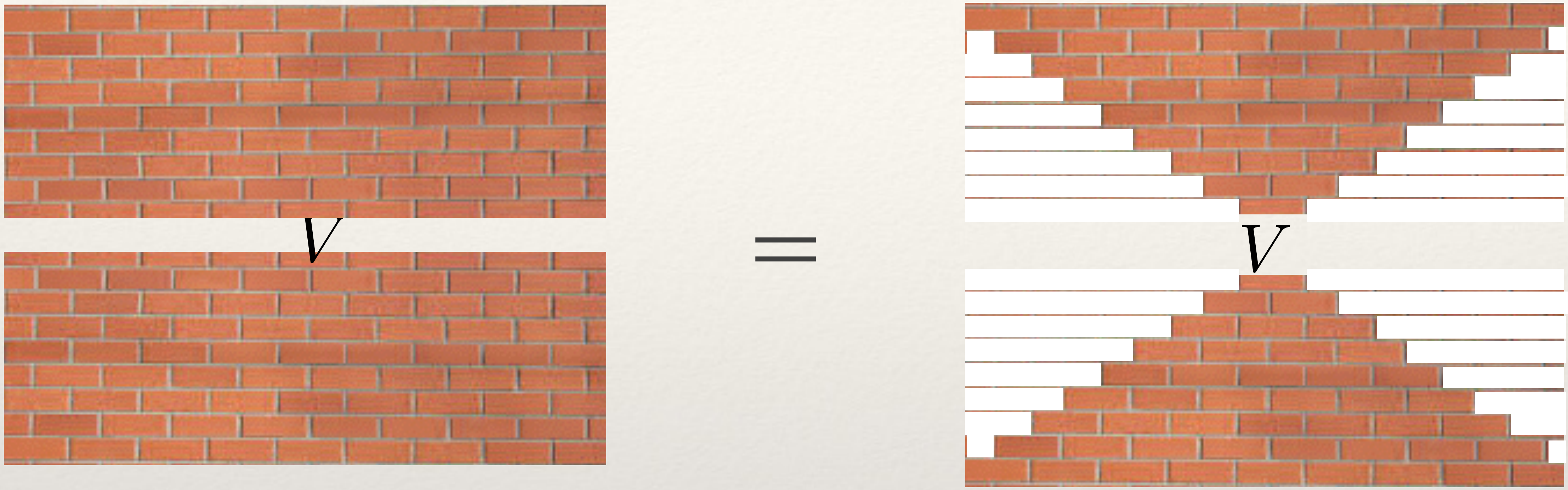
- “Topological order is long-range entanglement pattern.”
- “Topological order is the coarsest structure of the state.”
- Should be easy to detect…
- How would we recognize the pattern?

Guiding Problem

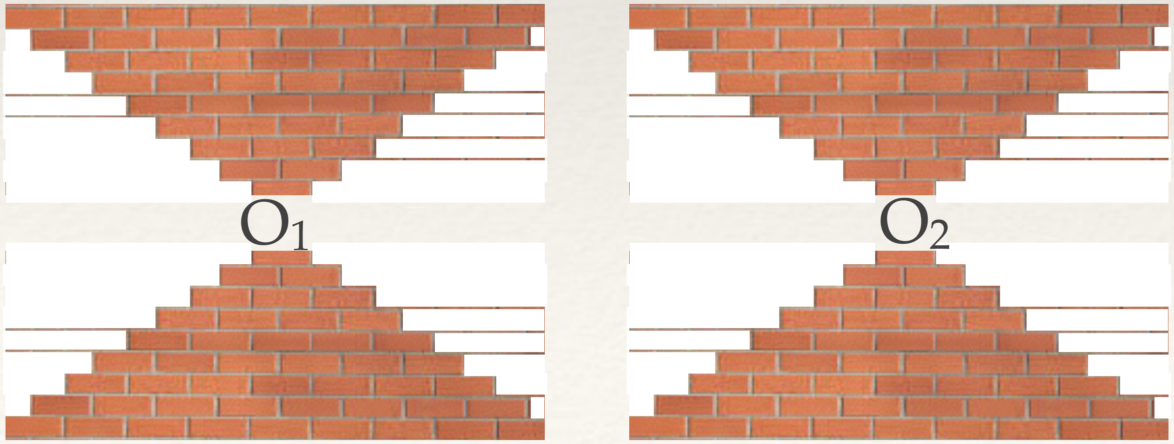
- How deep a quantum circuit must be in order to transform a state to another?
- Can an invariant answer this question by a significant bound?
    - Strength or fineness (opp. coarseness) of the invariant.

0. Long-range order

Quantum circuits



❖ It takes a linear depth-circuit
to build up any long-range correlation.



$$Cor_{|\psi\rangle}(O_1, O_2) \sim 0$$
$$\Leftrightarrow Cor_{W|\psi\rangle}(O_1, O_2) \sim 0$$

Finite correlation length

- Long-range Entanglement ? Long-range correlation
- Many exactly solvable models have commuting Hamiltonian
  - Quantum double models, Levin-Wen model, any Pauli stabilizer code state.

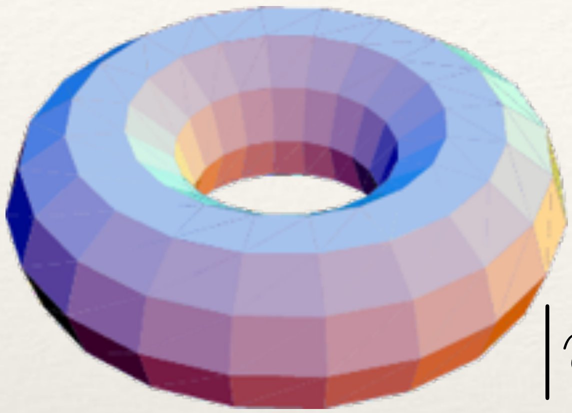
$$\rho_{AB} - \rho_A \otimes \rho_B = 0$$

NOT TOO GOOD

0. Long-range order
1. Local Indistinguishability

Hardness of Generation

[Wolfram MathWorld]



$|\psi_0\rangle$

If =

$|01001\cdots011\rangle$

Bravyi, Hastings, Verstraete (2006)

$|\psi_0\rangle$ $|\psi_1\rangle$

The pair is locally distinguishable.

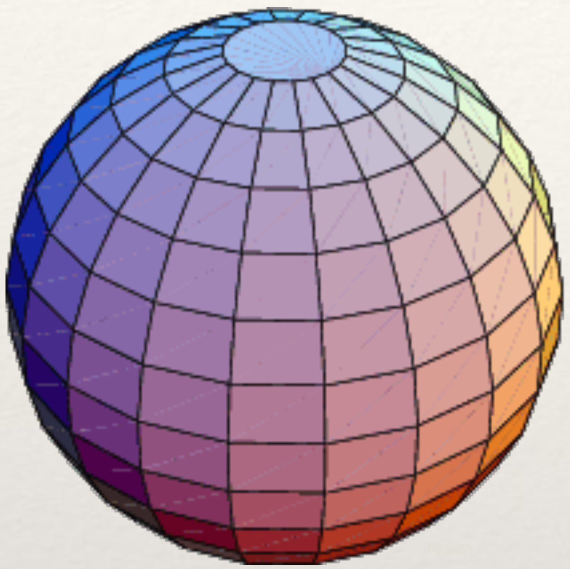
Deep Entanglement

Any orthogonal state is locally distinguishable.

The local indistinguishability is invariant of a pair of states.

A locally indistinguishable partner is an entanglement witness.

Toric code on a sphere



No correlation of local observables.
No pair of locally indistinguishable states.

$$H = -\sum_{e \in \square} \sigma_e^z - \sum_{e \ni v} \sigma_e^x$$

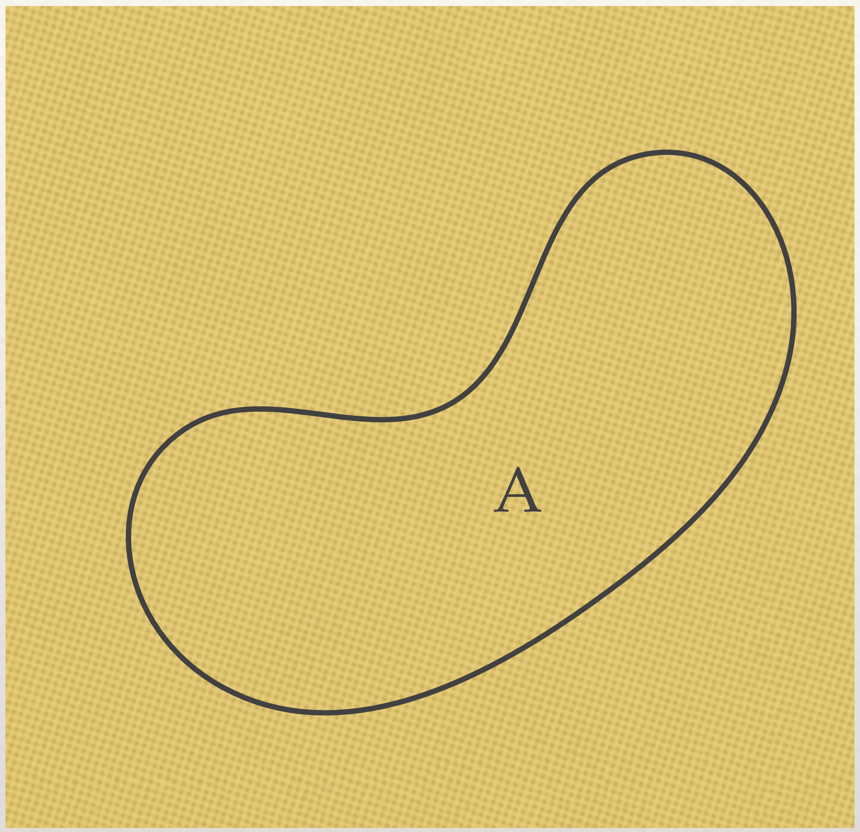
NOT TOO GOOD

What is the complexity of generation?
Is there "deep entanglement"?

0. Long-range order
1. Local Indistinguishability
2. Topological Entanglement Entropy

Topological Entanglement Entropy

Kitaev, Preskill; Levin, Wen (2006)

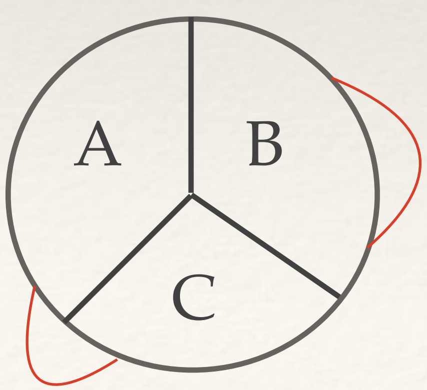


$$S_A = \alpha L - \gamma$$

$$\gamma = \log \sqrt{\sum_a d_a^2}$$

total quantum dimension

Kitaev-Preskill Argument



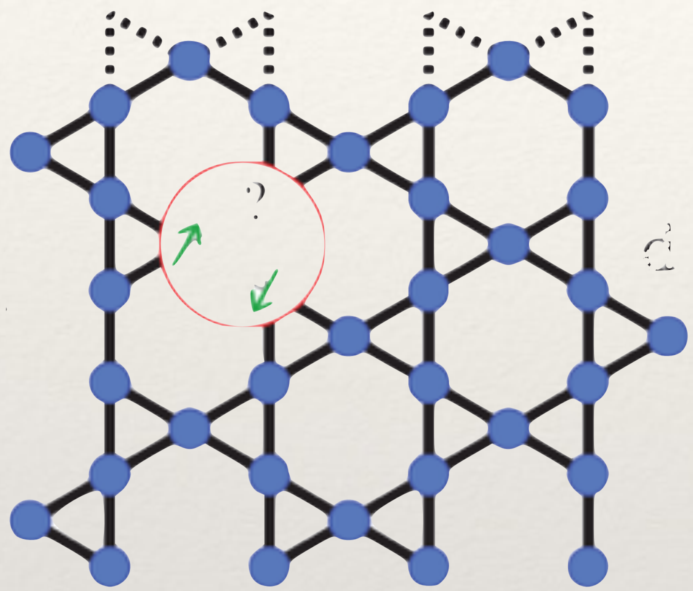
$$S_A + S_B + S_C - S_{AB} - S_{BC} - S_{CA} + S_{ABC}$$

Topological Entanglement Entropy

$$S_A = \alpha L - \gamma$$

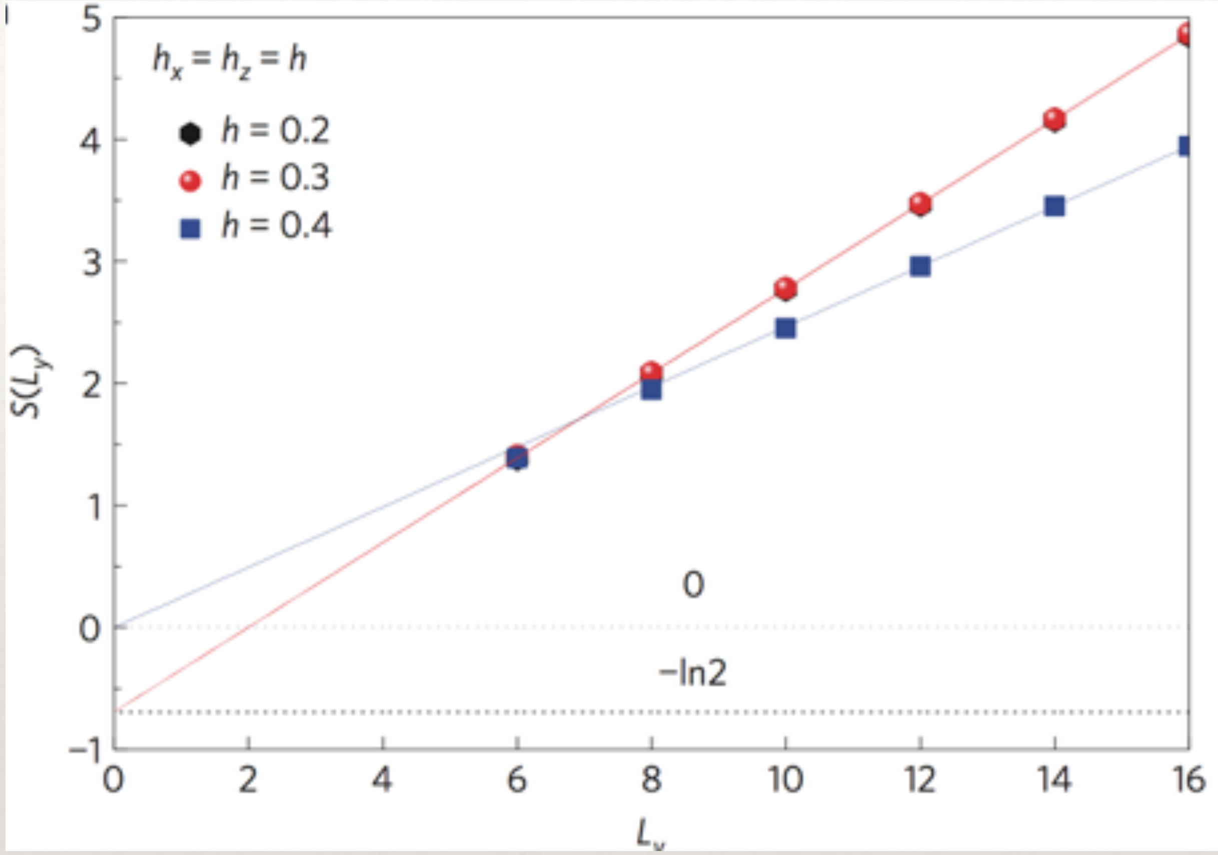
- (Simply) Computable in the bulk
- Quantitative Many-body entanglement witness
- Connected to abstract anyon theory
- Specific to 2D

AntiFerroHeisenberg on Kagome



Yan, Huse, White (2011)

Found no ordering under perturbations



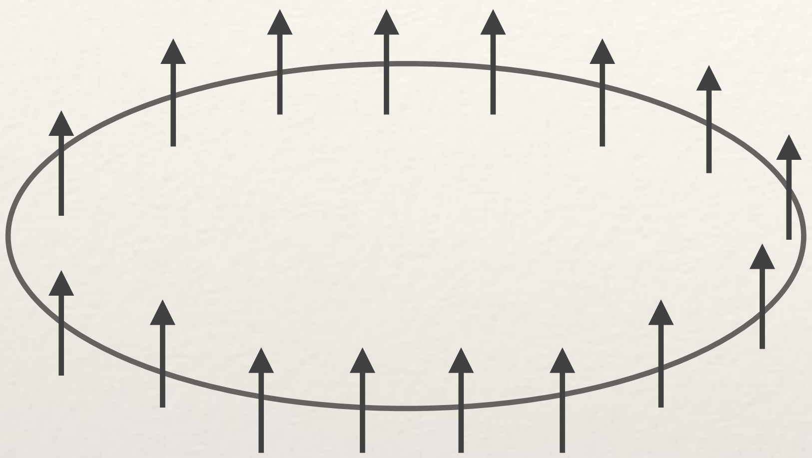
Jiang, Wang, Balents (2012)

Computed topological entanglement entropy

Strong evidence of topological order.

Bravyi's Counterexample

From his talk in 2008

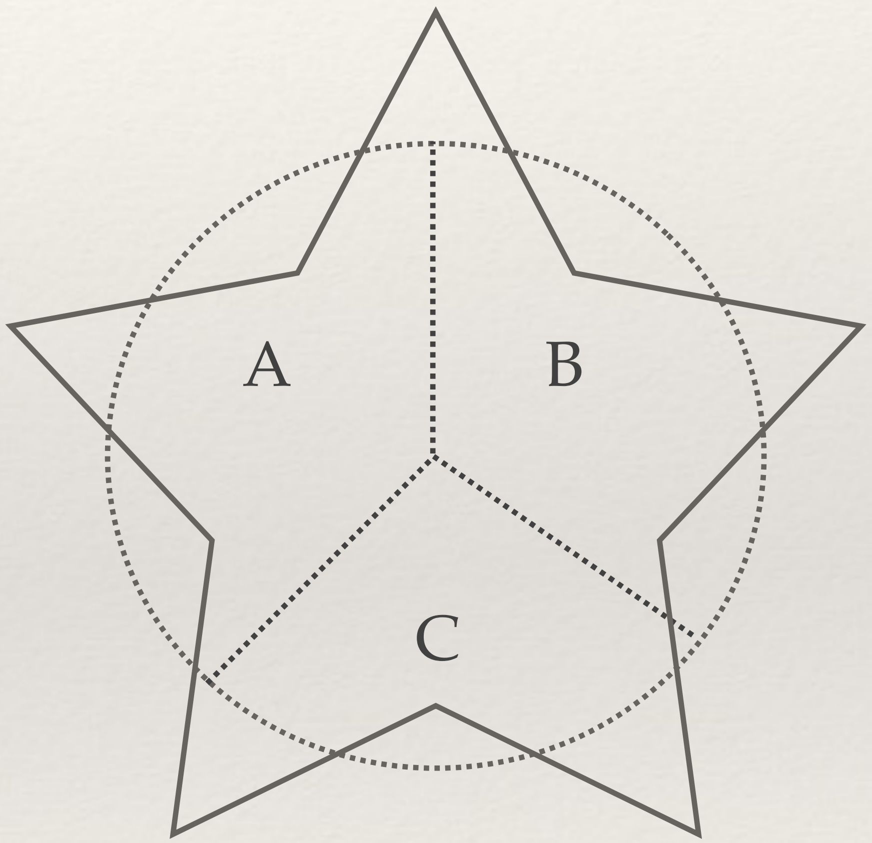


$$H = -\sum_i X_i$$

QC of depth 2

$$H = -\sum_i Z_{i-1} X_i Z_{i+1}$$

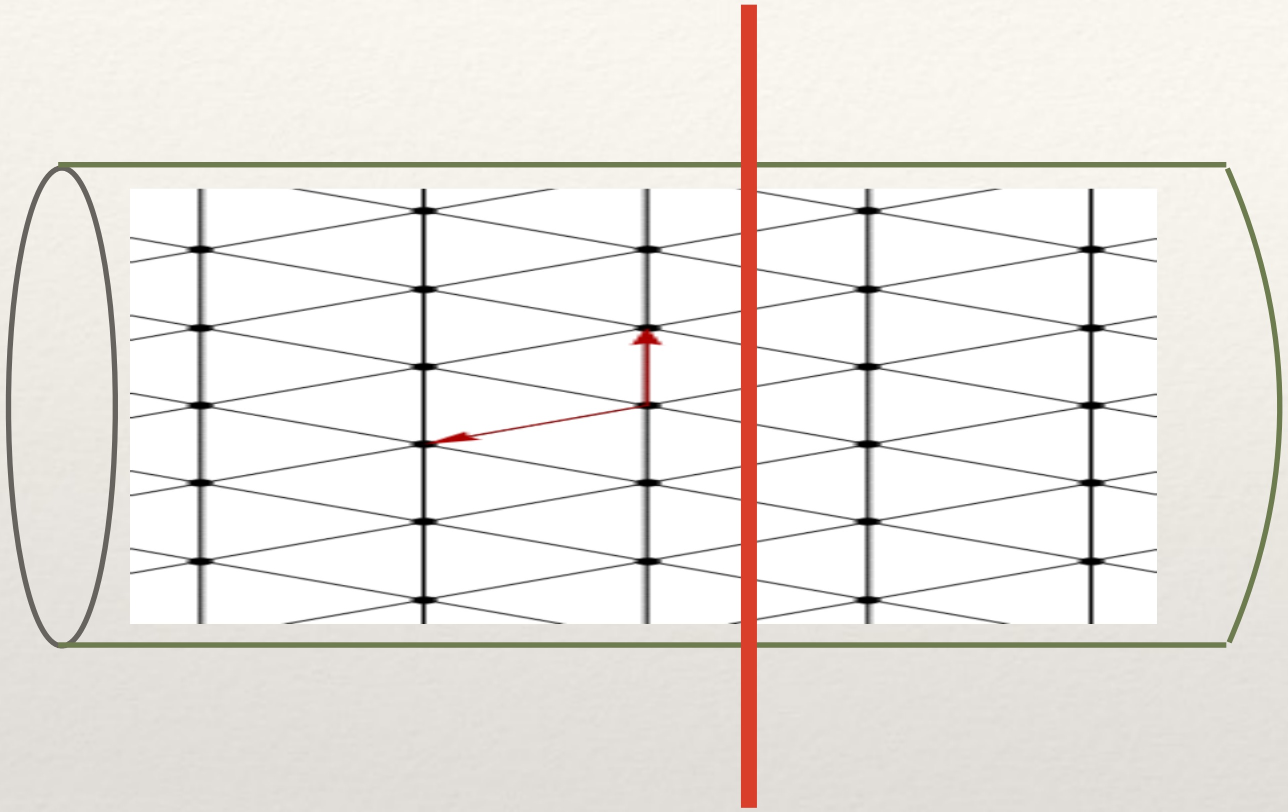
$$S_{\text{even}} = L/2 - 1$$



$$S_{\text{Kitaev-Preskill}} = -\log 2$$

2D cluster state on triangular lattice

[Zou, Haah, Senthil, in preparation]



- S = L - 1
- Sub-leading term of E.Entropy can be contaminated.
- It can even fluctuate.

$$S(L) = L - \gcd(L, n)$$

- Consequence of 1D SPT under a product group

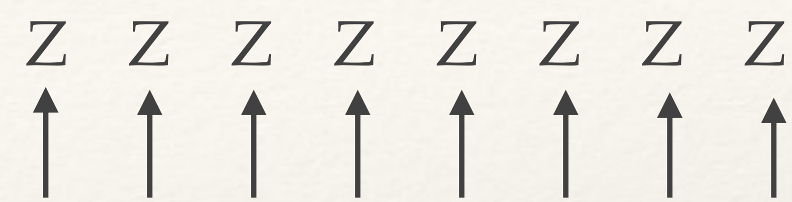
NOT TOO GOOD

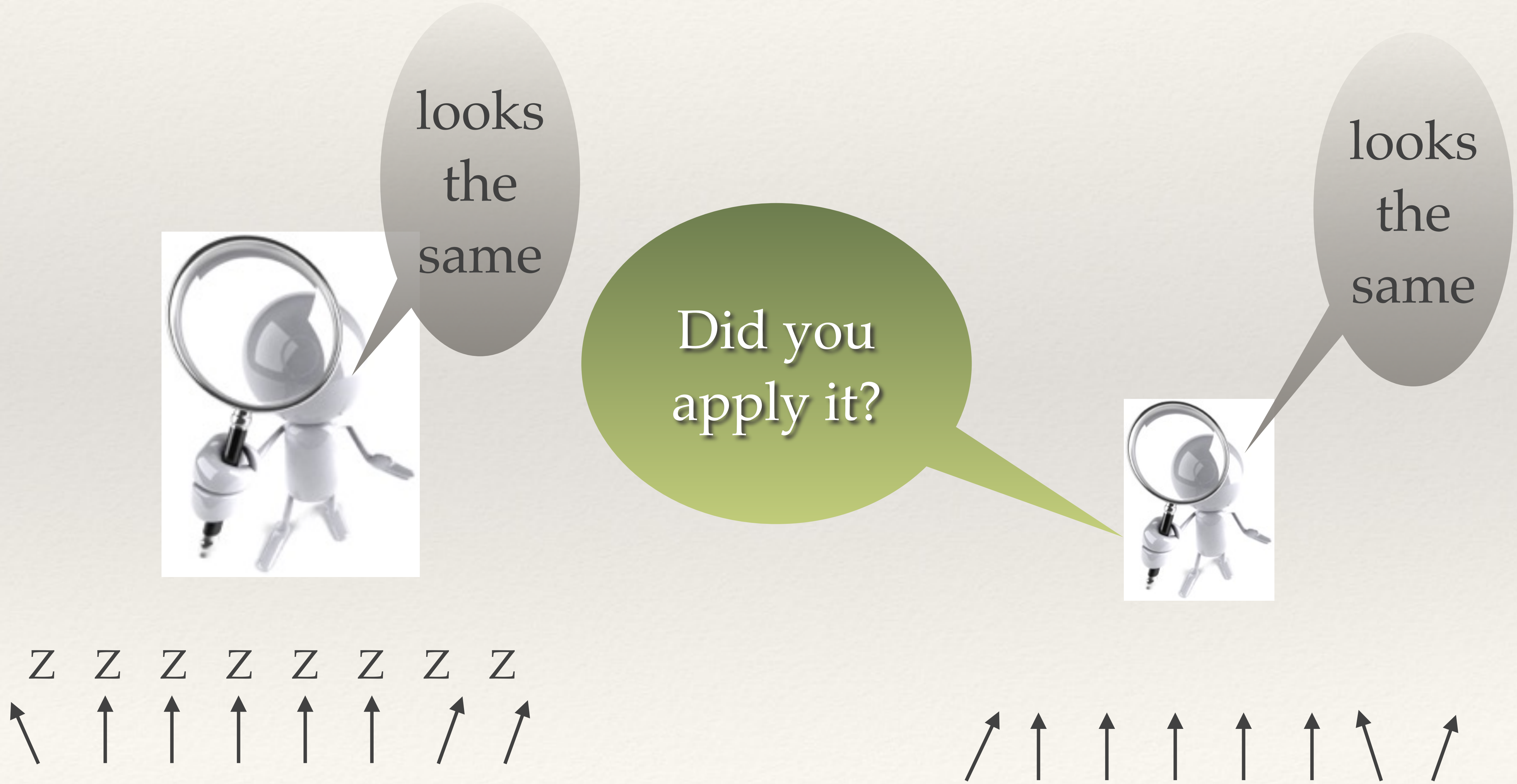
Can we say that TEE is an evidence for topological order?

0. Long-range order
1. Local Indistinguishability
2. Topological Entanglement Entropy
3. Small-depth stabilizers

Small-depth Stabilizers

- They are locally invisible.





Locally invisible operator

$$A \subset B$$

- Def.: O is (A,B)-***locally invisible*** with respect to $|\psi\rangle$

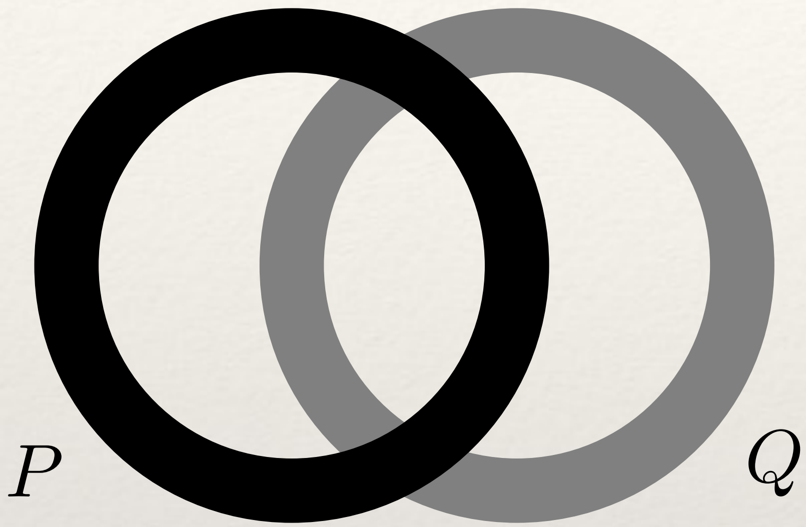
$$\mathrm{Tr}_{B^c}[|\phi\rangle\langle\phi|] = \mathrm{Tr}_{B^c}[|\psi\rangle\langle\psi|$$

$$\Rightarrow \mathrm{Tr}_{A^c}[O|\phi\rangle\langle\phi|O^\dagger] \propto \mathrm{Tr}_{A^c}[|\psi\rangle\langle\psi|]$$

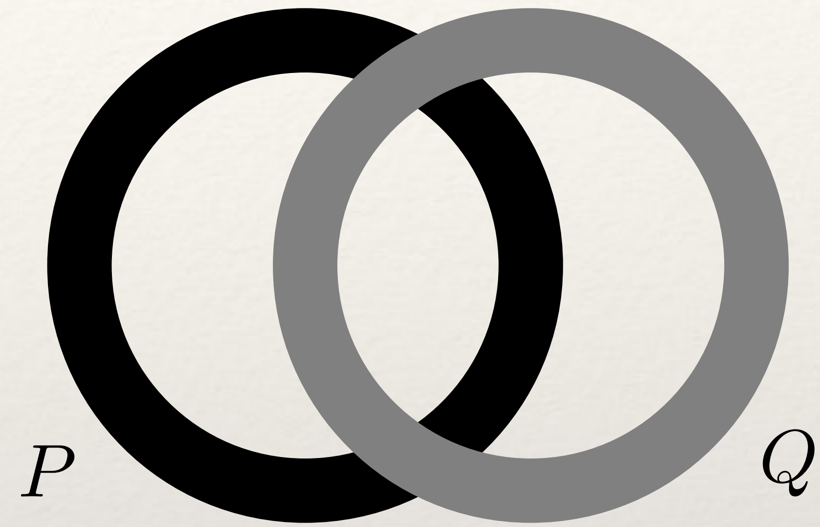
Small-depth stabilizing quantum circuit is (A,A+r)-locally invisible.

Twist product

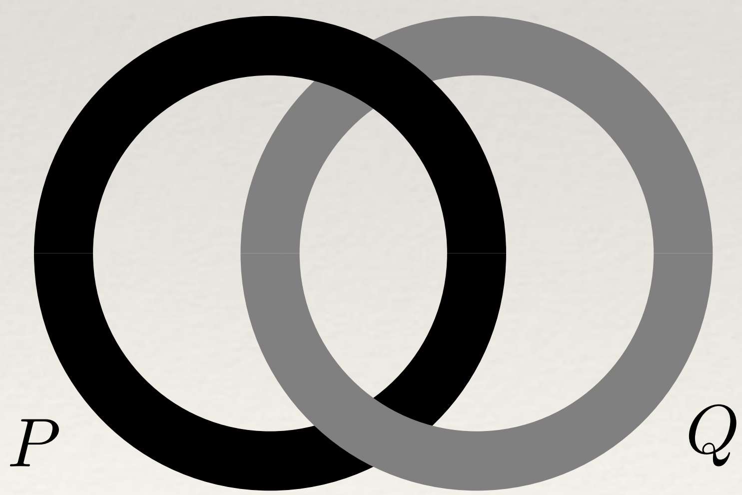
Ordinary product PQ



Ordinary product QP



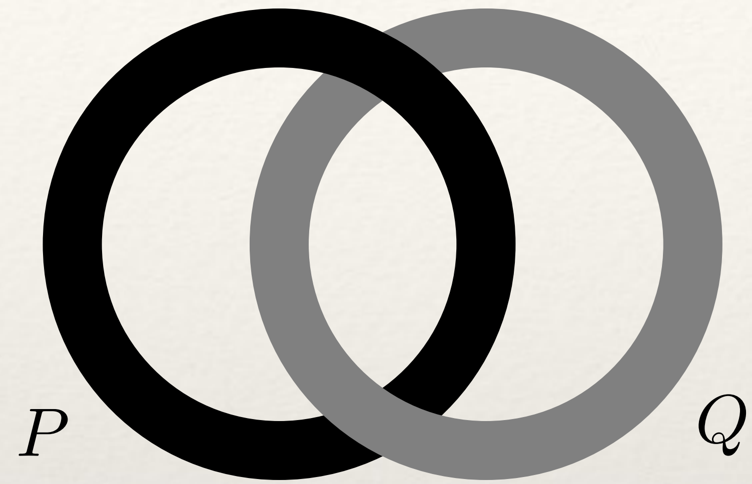
Twist Product



$$\sum_{ij} P_{\mathrm{up}}^{(i)} Q_{\mathrm{up}}^{(j)} \otimes Q_{\mathrm{down}}^{(j)} P_{\mathrm{down}}^{(i)}$$

Well-defined as long as intersection is separated.

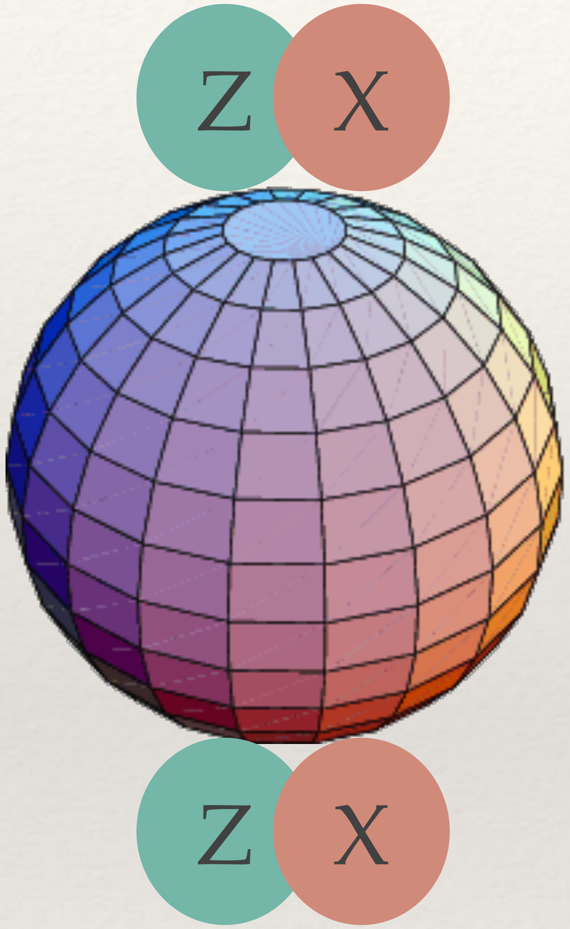
For product states



$$\langle\psi|P\infty Q|\psi\rangle = \langle\psi|P|\psi\rangle\langle\psi|Q|\psi\rangle$$

- Any pair of locally invisible operators whose twist pairing is nontrivial, is a witness of deep entanglement.

Examples



Far-separated Bell pair

Optimal bound
on generating circuits!

$$H = -\sum_{e \in \square} \sigma_e^z - \sum_{e \ni v} \sigma_e^x$$

Toric Code state

Witness, nice!

0. Long-range order
1. Local Indistinguishability
2. Topological Entanglement Entropy
3. Small-depth stabilizers
4. Topological Charges

Topological S-matrix

Quantum amplitude of braiding process

$$\mathcal{D}^2 = \sum_a d_a^2$$

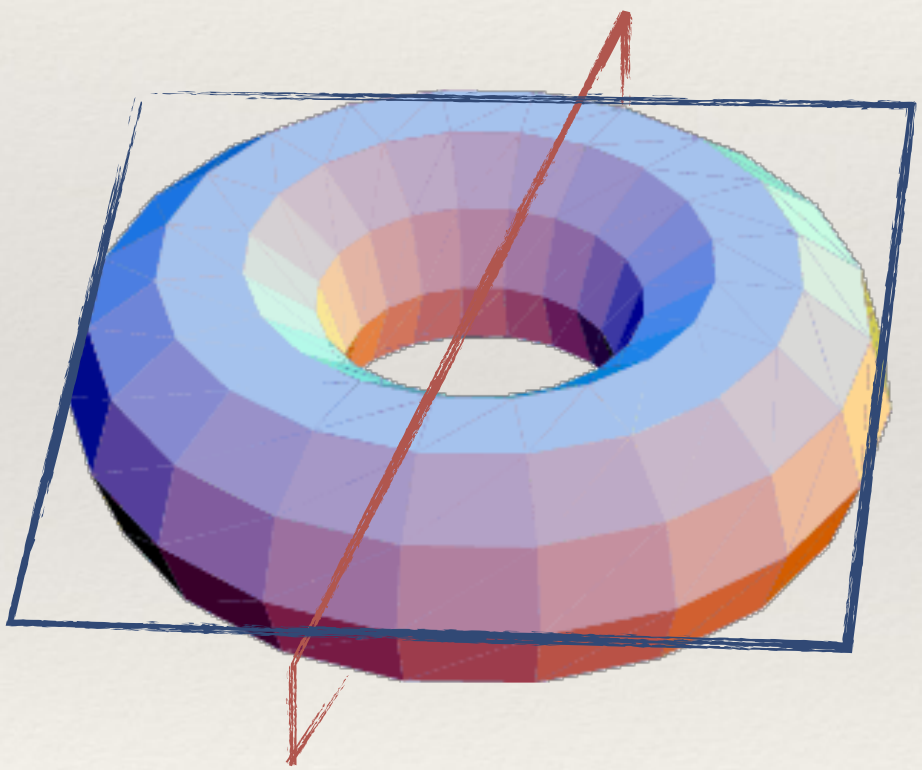
$$S_{ab} = \frac{1}{\mathcal{D}}$$

$$\langle\psi| \bigcirc_a |\psi\rangle = d_a$$

Invariant of Hamiltonian or state?

Minimally Entangled States

Zhang, Grover, Oshikawa, Vishwanath (2012)
Zhang, Grover, Vishwanath (1412.0677)



- Start with full ground space.
- Compute minimal ent. states.
- Compute overlap.

$$S_{ab} = \langle \psi_a^H | \psi_b^V \rangle$$

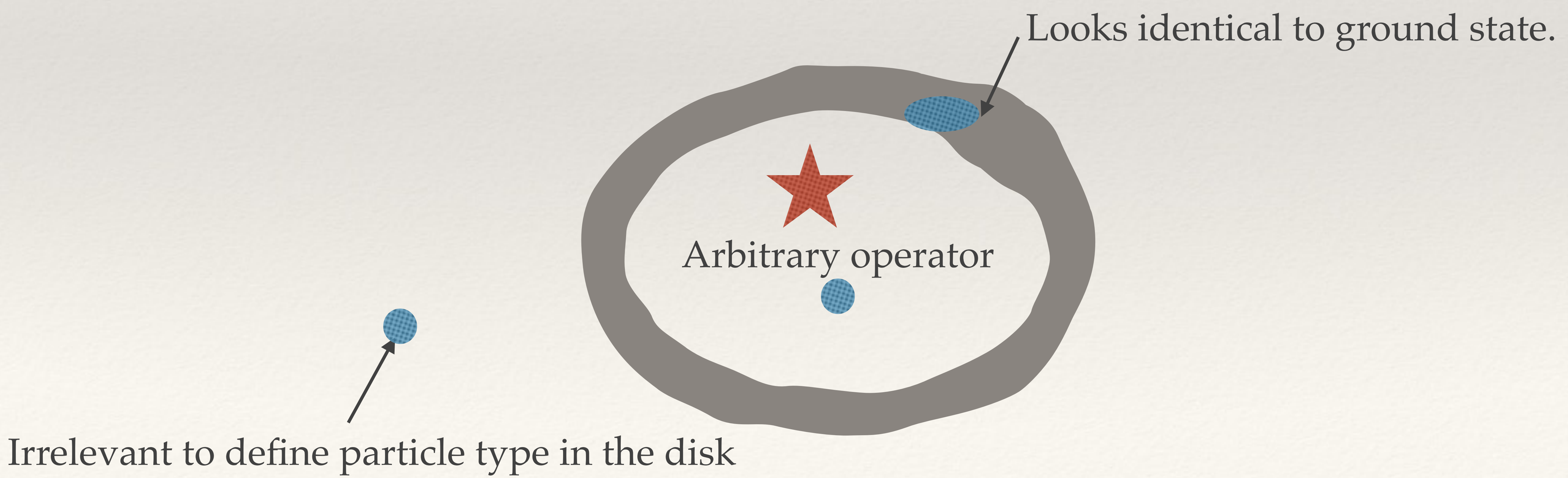
Can we do it in the bulk?

Goal

- Find a quantity such that
  - It is defined by a state.
  - It is independent of boundary conditions.
  - It is invariant under local unitary transformations.
  - (It can be computed given a wave function.)

What is anyon?

- It is a superselection sector.
  - A set of states related by local operators, not necessarily unitary.
  - No symmetry constraint.



Recall: Total spin

$$[J_x, J_y] = iJ_z$$

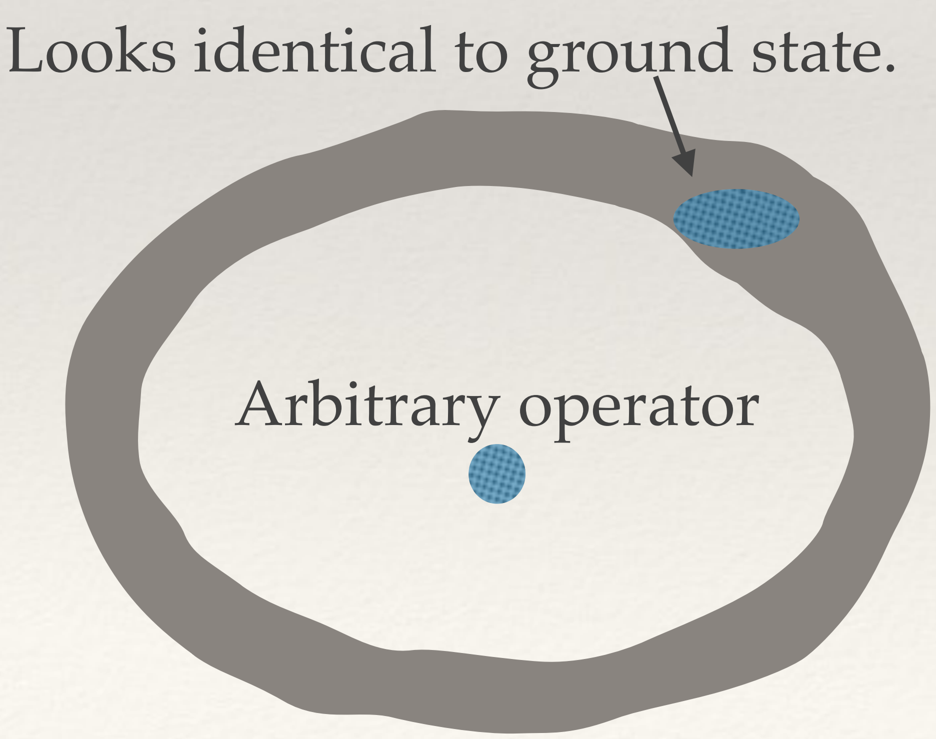
$$J_x^2 + J_y^2 + J_z^2 = j(j+1)$$

- Allowed operators,
- Find an operator in the center of the operator algebra.
- Eigenvalue of the central operator
  = Particle type (spin)
  = Conservation

To define particle types

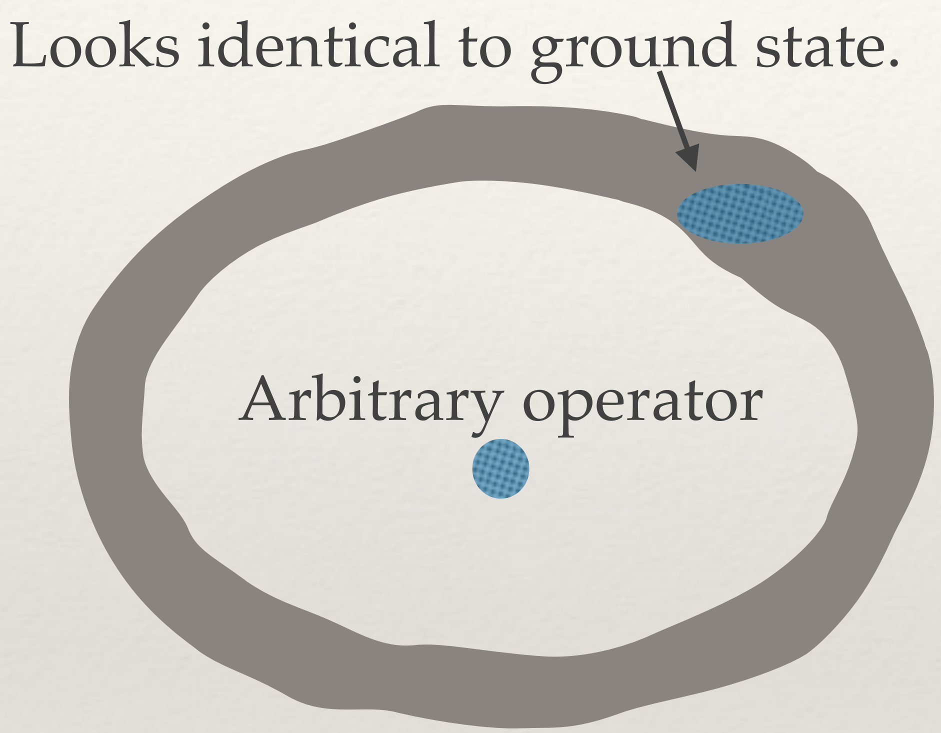
$$Mat(D) \otimes \mathcal{A}$$

Any local term of H should commute



- Allowed operators,
- Find an operator in the center of the operator algebra.
- Eigenvalue of the central operator = particle type (spin)

Null operators



- If any operator on grey annihilates the state, it's like multiplying by 0.
- Factor them out.

$$Mat(D) \otimes \mathcal{A}/\mathcal{N}$$

Operator on grey that annihilates the state

Any local term of H should commute

C^*-algebra

- Algebra over complex numbers (finite dimensional)
- Enough to think of matrix algebra closed under dagger.
- Completely decomposes into (a direct sum of) full matrix algebras
- Projections onto components generate the center.

Structure of C*-algebra

$$UCU^\dagger = \begin{bmatrix} * & * & & & & \\ * & * & & & & \\ & & * & * & * & * \\ & & * & * & * & * \\ & & * & * & * & * \\ & & * & * & * & * \end{bmatrix}$$

$$\pi_1 = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_4 \end{pmatrix}. \qquad \begin{cases} \pi_1 + \pi_2 = I \\ \pi_j^2 = \pi_j = \pi_j^\dagger \\ \pi_1 \pi_2 = 0 \end{cases}$$

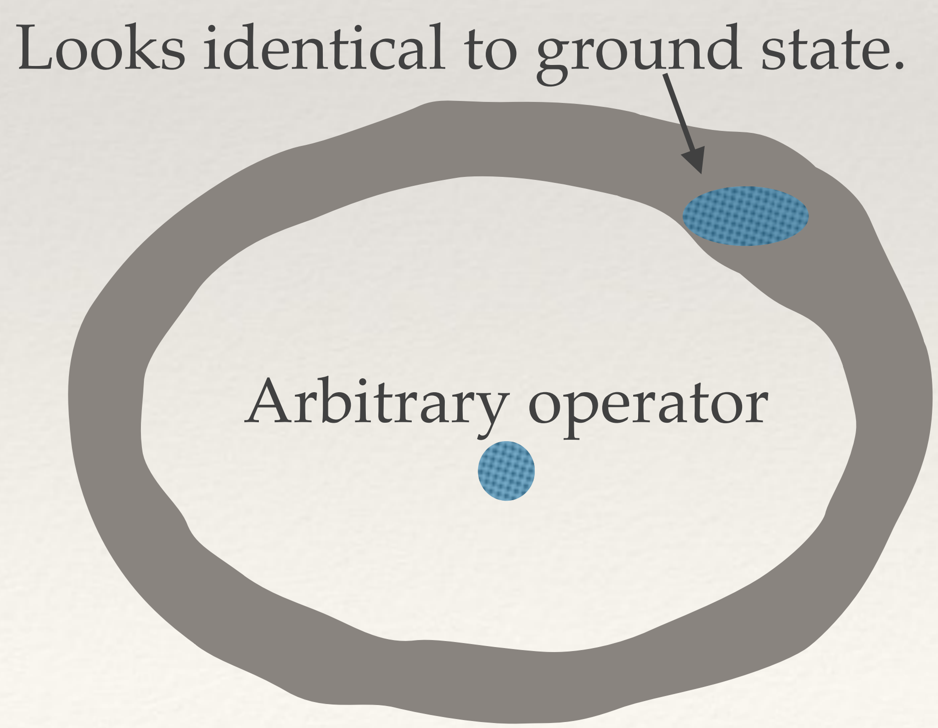
Particle type projectors

- form the canonical basis of the center of

$$Mat(D) \otimes \mathcal{A}/\mathcal{N}$$

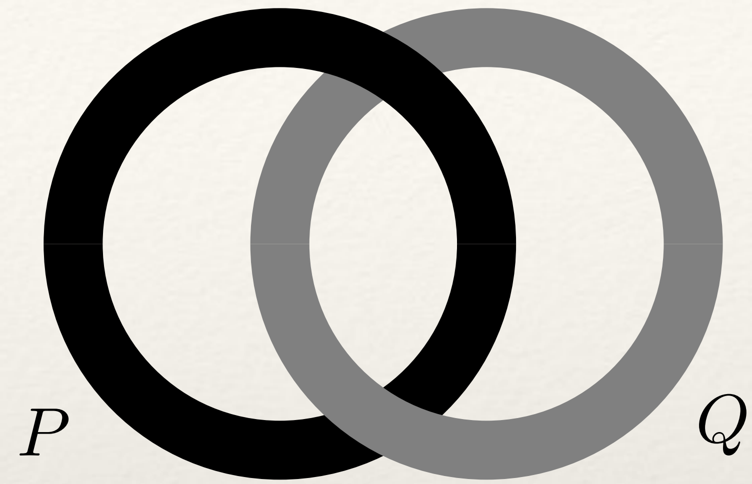
Structure theorem of C*-algebra

- The center lives on the annulus.



My S-matrix

$$\tilde{S}_{PQ} = \langle \psi |$$



$$| \psi \rangle$$

Particle type projectors

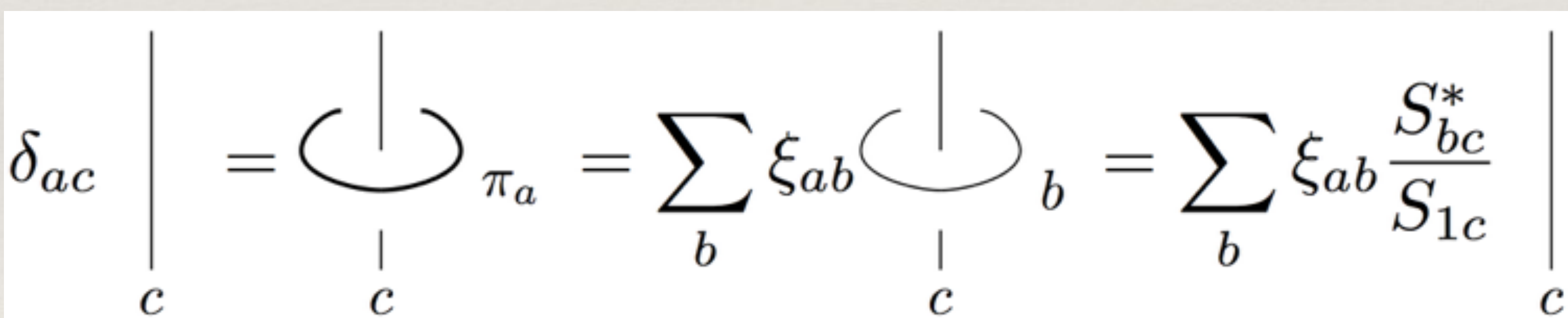
- Input: (commuting) Hamiltonian (ground state)
- No special boundary; just some large disk.
- No phase ambiguity.
- The trivial particle ("1") projector is distinguished.

Relation to ord. S-matrix

$$\tilde{S}_{ab} = \frac{d_a d_b}{D} S_{ab}$$

It contains the same data!

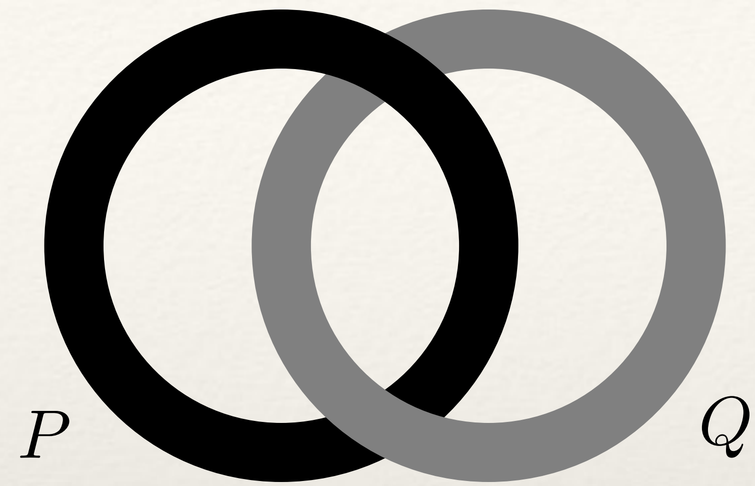
Proof:



$$\pi_a = \frac{d_a}{D} \sum_b S_{ab} \bigcirc_b$$

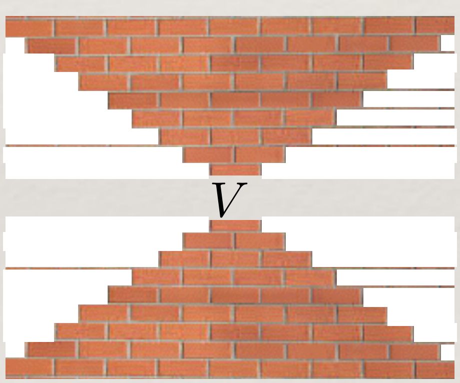
Invariance under local unitaries

$$\langle \psi | \, W^\dagger W$$



$$W^\dagger W \, | \psi \rangle$$

Particle type projectors



$$W(P \infty Q)W^\dagger = (WPW^\dagger) \infty (WQW^\dagger)$$

as long as the depth of W is smaller than the separation of the intersection.

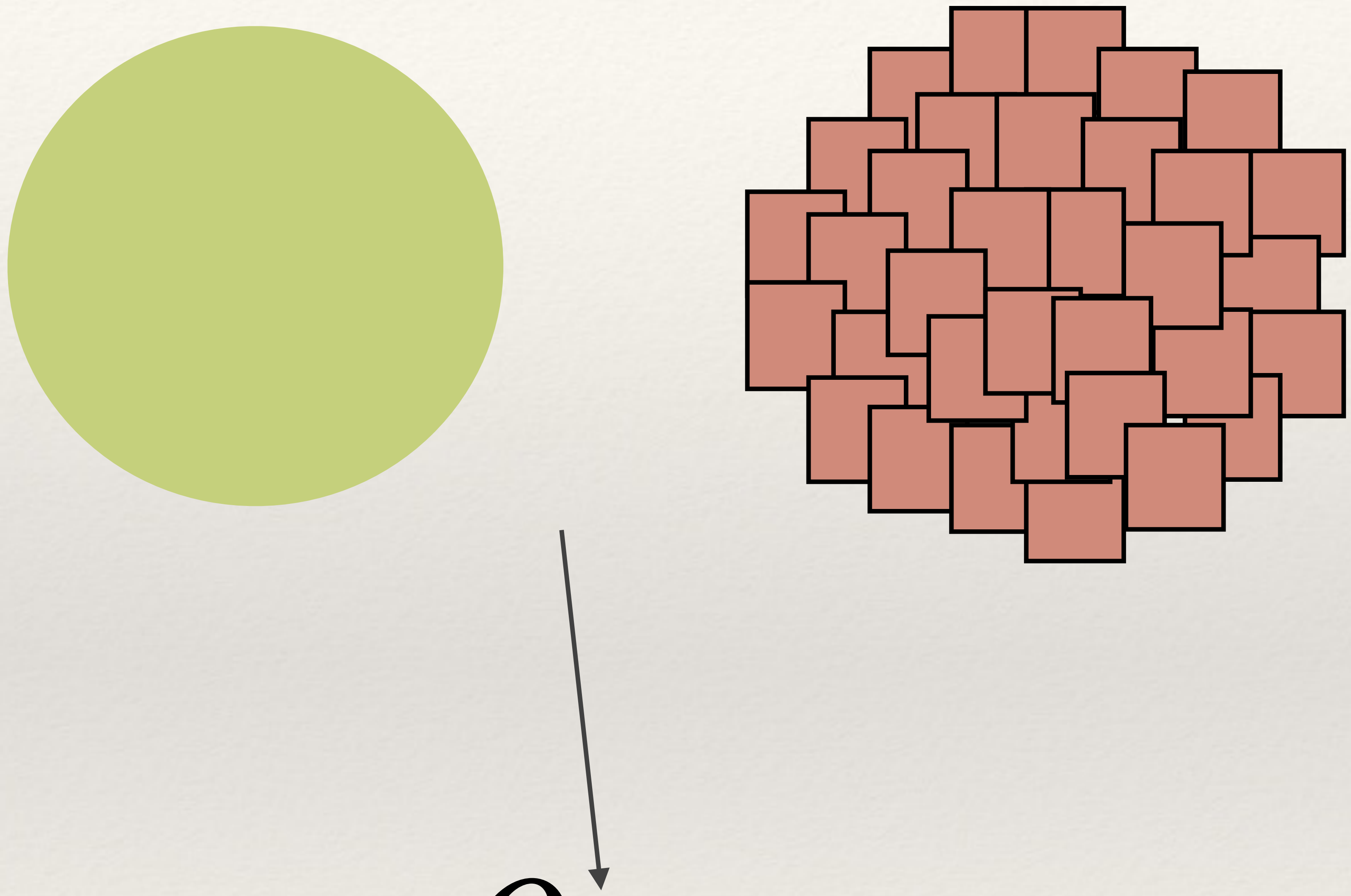
So, invariance is proved if $\mathcal{A}/\mathcal{N}$ is remains isomorphic under W.

This is nontrivial, so I had to assume further.

Assumptions

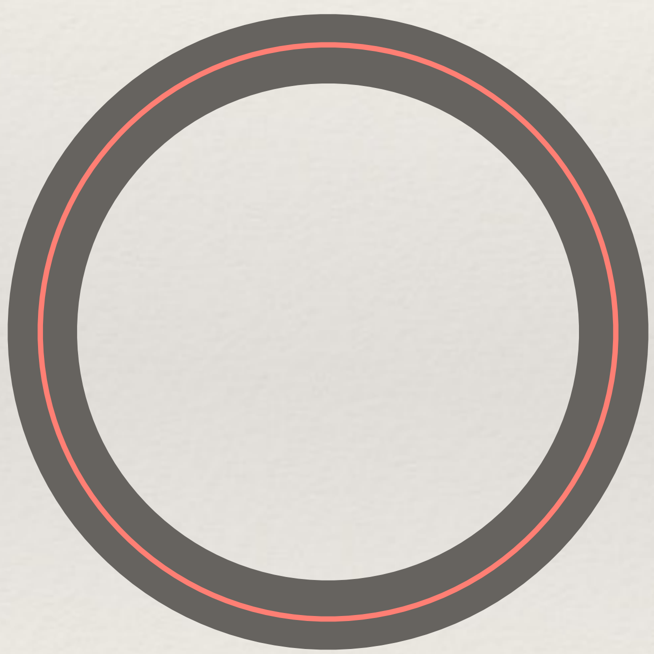
1. Local topological order
   - Local ground state matches the global one
2. Stable logical algebra
   - logical algebra does not depend on the size of the support
   - violated when there are infinitely many particle types.

Local Topological Order



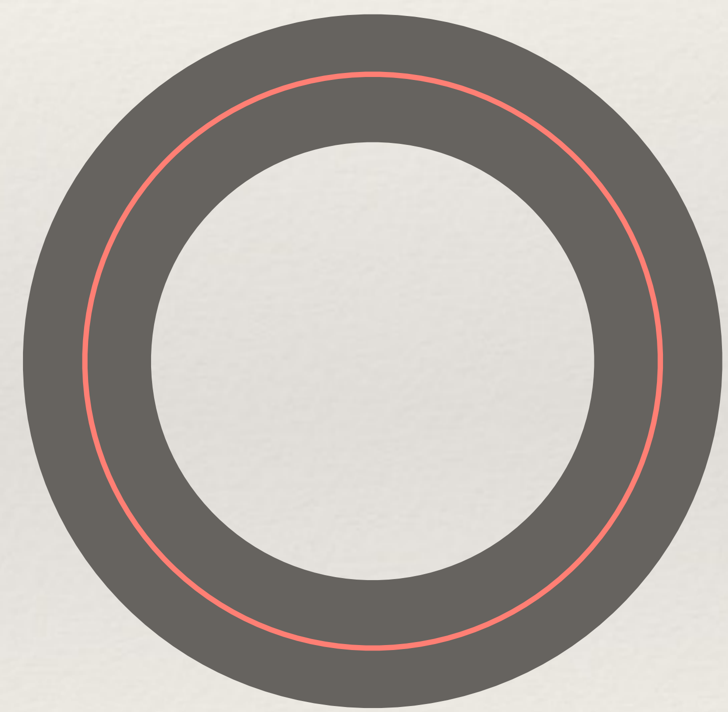
$$\rho_A$$

Stable Logical Algebra



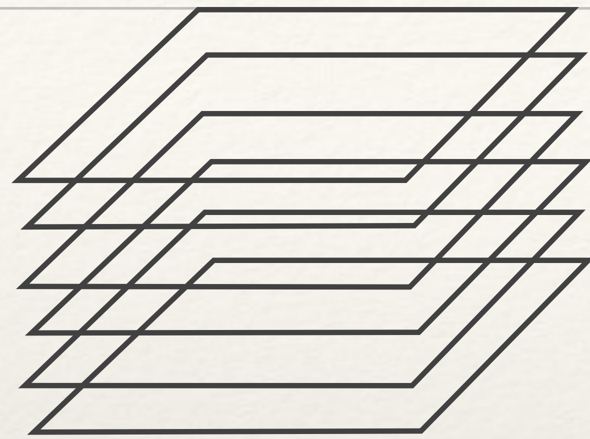
Isomorphic

$$\mathcal{A}/\mathcal{N}$$



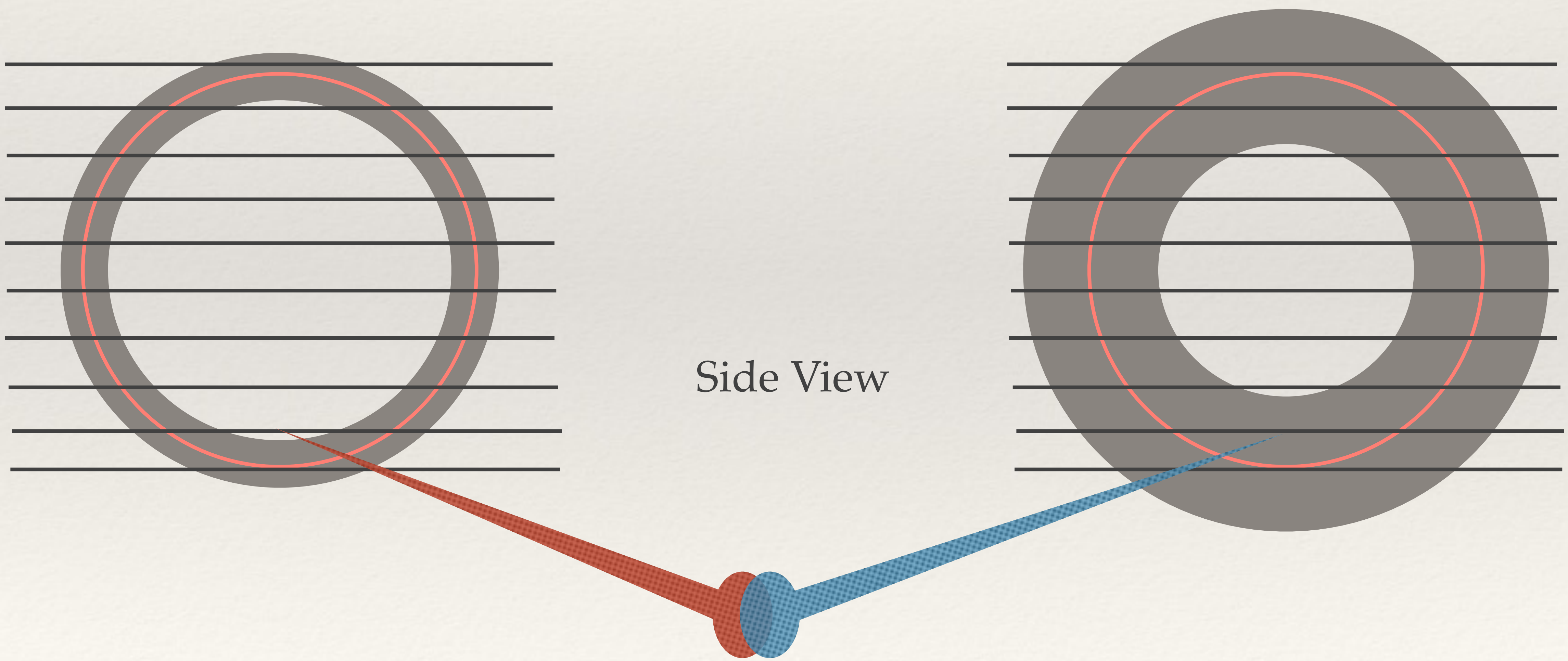
Regardless of the thickness

Finiteness of particle types



Infinite stack of 2D layers

A particle is separated by a sphere with thick wall.



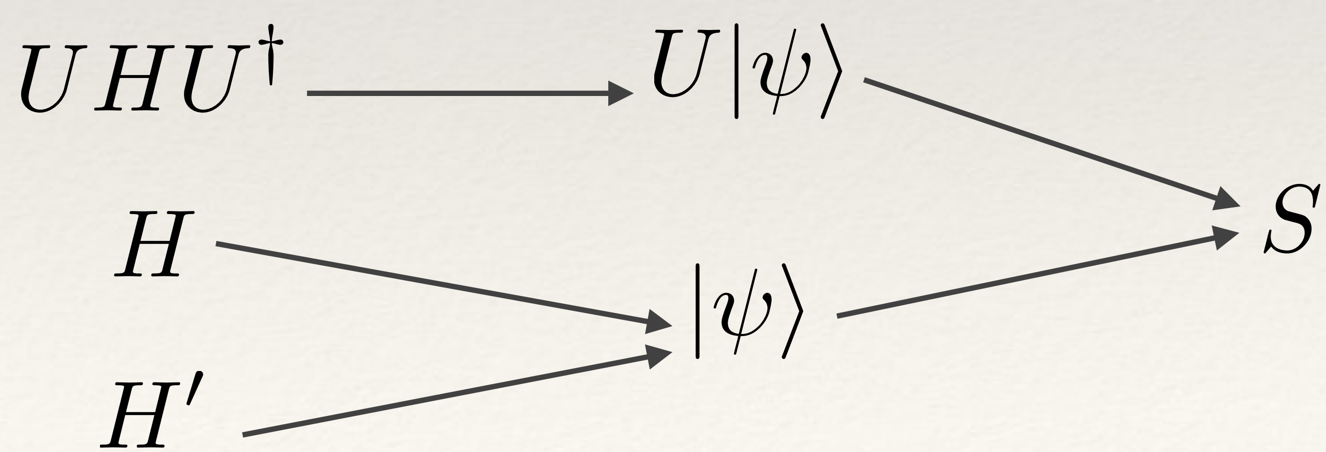
Stable logical algebra is nontrivial assumption in general.

Consequences

$\mathcal{A}/\mathcal{N}$ is in fact independent of Hamiltonian

is invariant under small-depth Q. circuit.

- Therefore, my S-matrix is an invariant of state.



Complexity of transformation

- Any transformation between states with distinct S-matrices requires a deep (linear in diameter) circuit.

$$H_0 \qquad\qquad U H_0 U^\dagger \neq H_1$$

$$|\psi_0\rangle \quad \cdots\cdots\cdots\blacktriangleright \quad U|\psi_0\rangle = |\psi_1\rangle$$

- In view of quasi-adiabatic evolution,
  the energy gap must close at some point in any path between Hamiltonians with distinct S-matrices.

Sketch of independence proof

$$\mathcal{A}_t/\mathcal{N}_t \to \mathcal{I}_t/\mathcal{M}_t \to \mathcal{A}_{t+w}/\mathcal{N}_{t+w}$$

- Logical algebra to locally invisible operators
  - They are naturally invisible thanks to local topological order condition.
- Locally invisible operators to logical algebra
  - "Symmetrize" so locally invisible operators is dressed to commute with the Hamiltonian

$$\mathcal{A}_t^{H_1}/\mathcal{N}_t^{H_1} \to \mathcal{I}_t/\mathcal{M}_t \to \mathcal{A}_{t+w}^{H_2}/\mathcal{N}_{t+w}^{H_2}$$

Toric code state

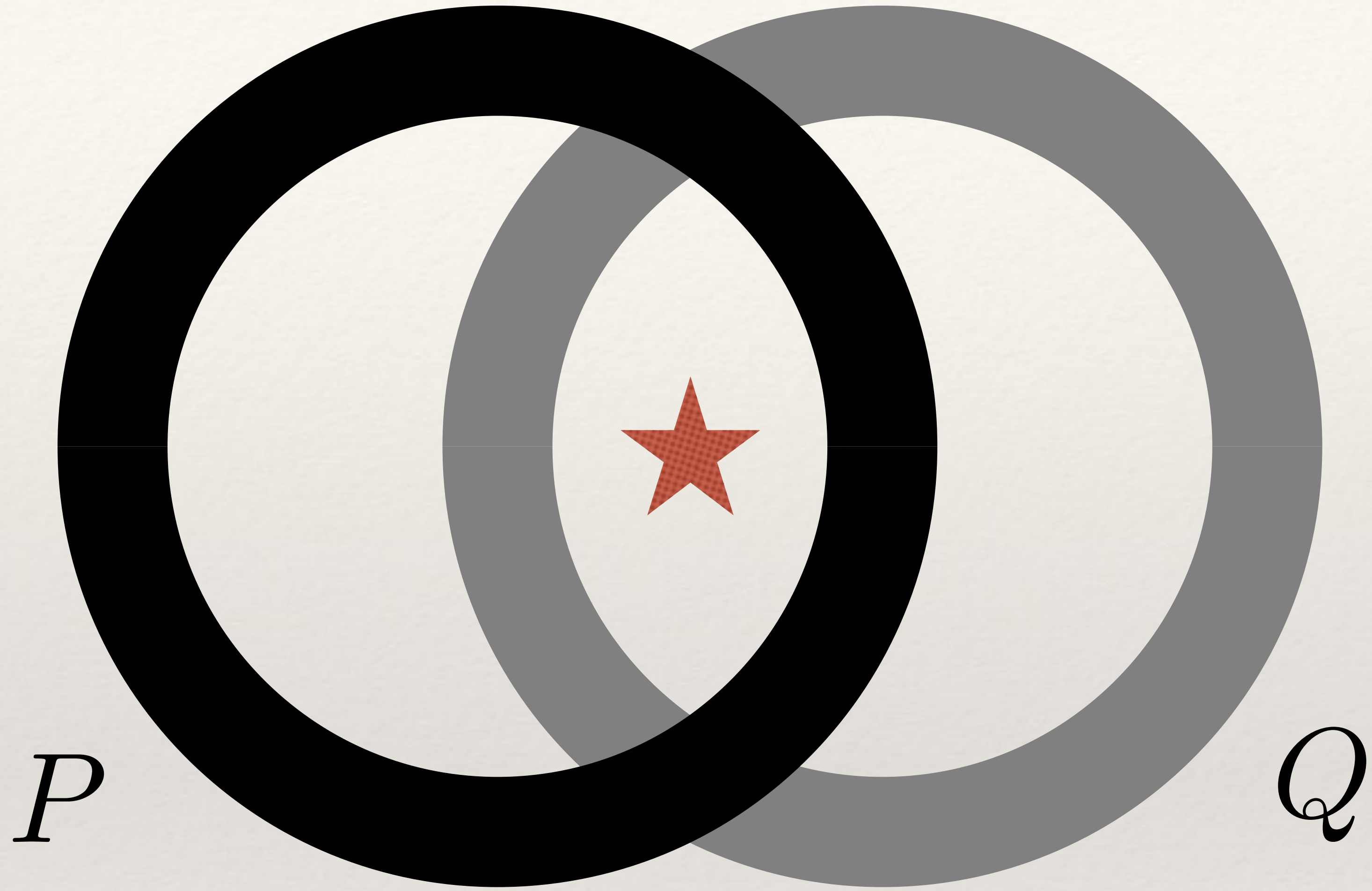
Abelian discrete gauge theory

$\mathcal{A}/\mathcal{N}$ is diagonal matrix algebra of dimension d^2

$$\tilde{S}^{(d)}_{(a_x a_z),(a'_x a'_z)} = \frac{1}{d^2}\omega_d^{a_z a'_x + a_x a'_z}.$$

- Two assumptions are satisfied, as verified by direct computation.
- Rows and columns unsorted except for the distinguished "1".
- Verlinde formula recovers the fusion (group) rules.

Row-column matching

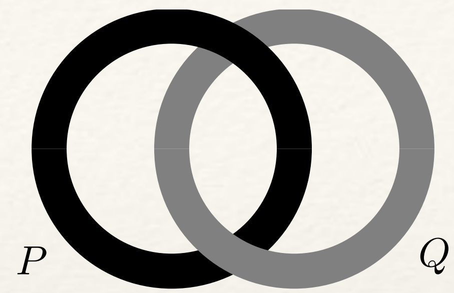


- If projectors jointly stabilize some state, they are matched.

0. Long-range order
1. Local Indistinguishability
2. Topological Entanglement Entropy
3. Small-depth stabilizers
4. Topological Charges

Many-body Entanglement Witness

$$\tilde{S}_{PQ} = \langle \psi |$$



$$| \psi \rangle$$

- We have given a class of ground states, for which S-matrix can be defined.
- Only a patch of a ground state is needed; insensitive to boundary.
- Indeed invariant under perturbations.
- 2D is not particularly used.
- Any heuristic algorithm would be interesting.
- Perhaps, in 2D stable logical algebra assumption is redundant.