Conservation laws from entanglement

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Conservation laws

- Charge conservation
- Energy-momentum conservation

Noether's theorem

Noether's theorem asserts that a continuous symmetry gives rise to conservation laws.

- Conservation laws from symmetry
- It does not matter how complicated the action/Hamiltonian looks like.
- However, the theorem only goes one way; symmetry implies conservation laws, but the converse statement may not be true.

Emergent symmetry

For quantum mechanical many-body systems, physicists have developed a notion of emergent symmetry over the past few decades.

- Topologically ordered system(e.g., toric code) is a canonical example.
- At this point, we do not have an analogue of Noether's theorem for emergent conservation laws.

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- At this point, we do not have an analogue of Noether's theorem for emergent conservation laws.
    - One might argue that all we need to do is to identify the emergent symmetry and apply Noether's theorem, but this is not satisfactory.
        - Any consistent fusion rules for boson/fermion must be equivalent to the multiplication rules of some irrep of some compact group. (Doplicher, Roberts)
        - For some anyon models, e.g., Fibonacci anyon model, their fusion rules are not described by a group.

Narrowing down the problem: algebraic theory of anyons

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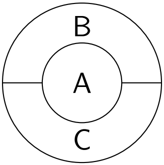
  - No reference to the Hamiltonian either!
  - There exists a body of work in the algebraic quantum field theory literature, but I do not want to assume translational/Lorentz invariance.
  - Kitaev has ironed out such a theory, but he still starts with some axioms.(cond-mat/0506438, Appendix E) Our goal is to derive
    1. the axioms of this theory from **a** property of a single state,
    2. and identify the abstract objects in Kitaev's theory to the physical degrees of freedom.

In relation to Jeongwan's result



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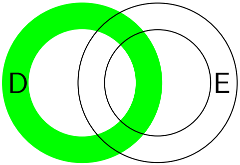
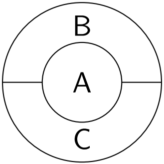


$S(A|B) + S(A|C) \approx 0$ everywhere

$\Pi_D^a \sim \Pi_E^a$

where $\Pi_D^a(\Pi_E^a)$ is a fundamental projector for a topological charge a over an annulus $D(E)$.

In relation to Jeongwan's result



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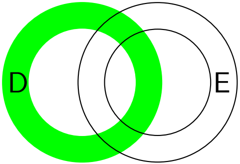

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Physical meaning: Superselection sectors are globally consistent.

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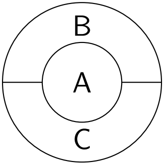

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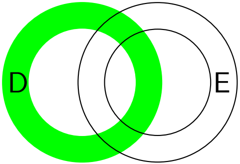

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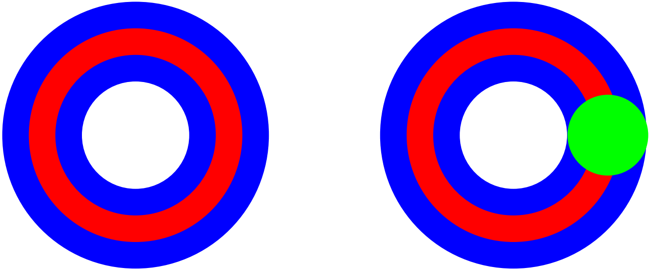
where $\Pi_D^a(\Pi_E^a)$ is a fundamental projector for a topological charge a over an annulus $D(E)$.
Physical meaning: Superselection sectors are globally consistent.
More concretely: There is an isomorphism between the logical algebras over different annuli.

Plan

1. Defining the charge sectors
2. Why charge sectors are globally well-defined
3. Consistency equations

Charge sectors : sets of quantum states

Let's suppose we have a state $P = |\psi\rangle\langle\psi|$. Consider the following sets.



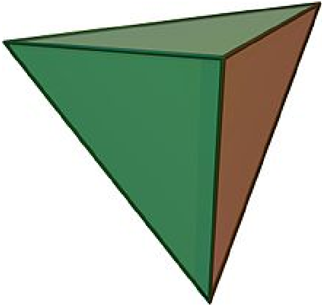
$$S_{\circledcirc} = \{\rho | \mathrm{Supp}(\rho) = \circledcirc, \rho_{\bullet} = \mathrm{Tr}_{\bar{\bullet}} P\}$$

$$S_{\circ} = \{\mathrm{Tr}_{\circledcirc} \rho | \rho \in S_{\circledcirc}\}$$

Charge sectors : convex set

$$S_{\circ} = \{\mathrm{Tr}_{\circledcirc}\rho|\rho \in S_{\odot}\}$$

* $S_{\circ}$ is a convex set, but convex sets come in different sizes and shapes.



Charge sectors: simplex assumption

This is the main weakness of our work at this point. I will need to assume the following statement.

Assumption:

$$S_{\bigcirc} = \{\rho | \rho = \sum_i p_i \rho_i\},$$

where $\sum_i p_i = 1$ and $\rho_i \perp \rho_j \; \forall i \neq j$.
* We call ρ_i as the extreme points of the set.
* This implies that there exists a set of orthogonal projectors $\{\Pi_i\}$ which project onto the support of ρ_i.

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Charge sectors: justifications

Assumption:

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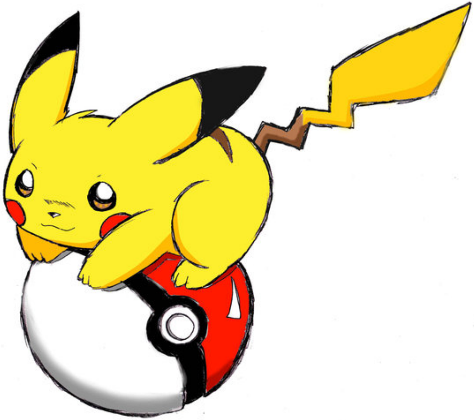
where $\sum_i p_i = 1$ and $\rho_i \perp \rho_j \; \forall i \neq j$.
* This implies that there exists a set of orthogonal projectors $\{\Pi_i\}$ which project onto the support of ρ_i.

- $\{\Pi_i\}$ coincides with the fundamental projectors in Jeongwan's work for exactly solvable models.
- Holevo information of the set is bounded by 2γ ($\gamma < \infty$: topological entanglement entropy).
- Perhaps it is possible to derive this from some condition.

Poké Ball condition



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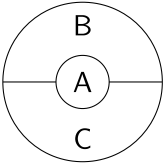
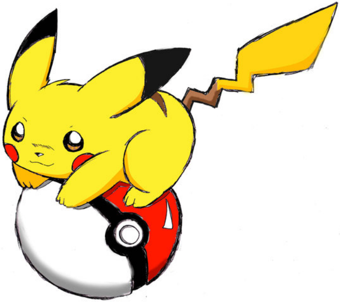
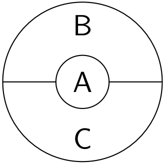


$$S(A|B) + S(A|C) \approx 0$$



$* \; S(A|B) = S(AB) - S(B).$

Why we expect Poké Ball condition to hold



$$S(A) \approx \alpha|\partial A| - \gamma$$

(Kitaev and Preskill 2006, Levin and Wen 2006) gives $S(A|B) + S(A|C) \approx 0$.

A brief recap in the middle

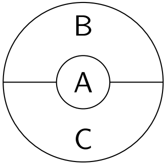
All the things discussed in the remaining slides(starting from the next one) follow from these three assumptions.

1. $\epsilon, \delta \to 0$.
2. Simplex assumption

$$S_{\circ} = \{\rho | \rho = \sum_i p_i \rho_i\},$$

where $\sum_i p_i = 1$ and $\rho_i \perp \rho_j$ $\forall i \neq j$.

3. Poké Ball condition



$$S(A|B) + S(A|C) \approx 0$$

*I actually suspect that 3 implies 2, but I do not know how to show that.

Poké Ball condition $\rightarrow$ Global consistency of charge sectors

First consider a state, $|\psi\rangle$, which satisfies the Poké Ball condition everywhere:

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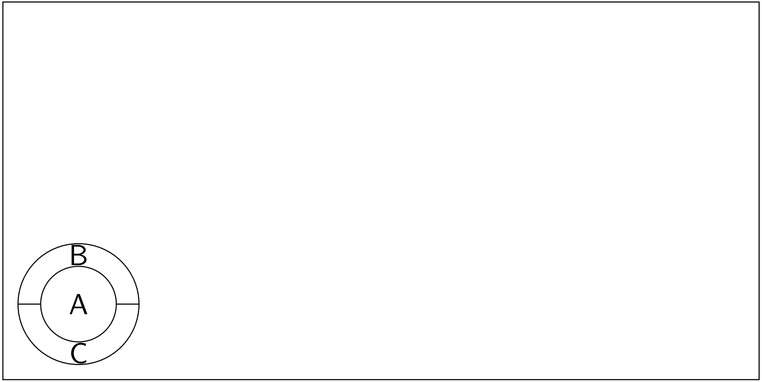

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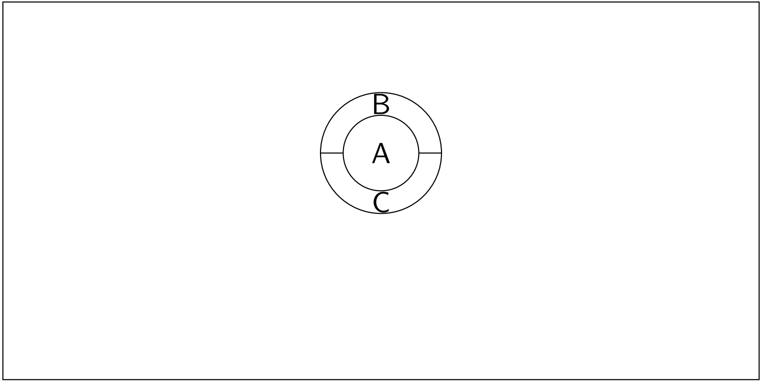

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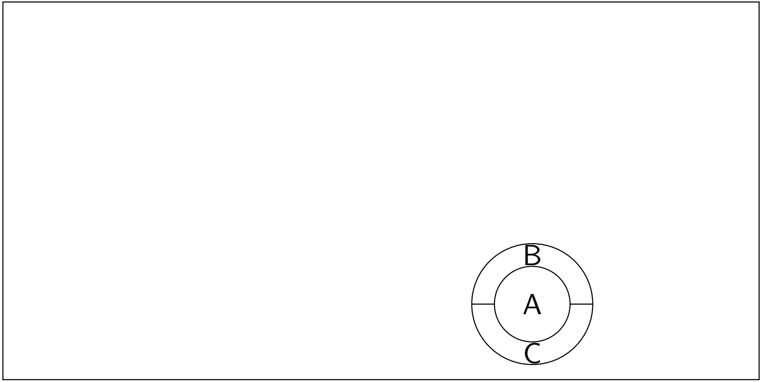

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$$S_{\circ} = \{\mathrm{Tr}_{\circledcirc} \rho | \rho \in S_{\circledcirc}\}$$

Poké Ball condition implies that $S_{\circ}$ for all $\circ$ are isomorphic to each other.

Consistency of the charge sectors: contraction lemma

Lemma

(Contraction lemma): If an annulus A is contained in A',
$\forall \rho, \sigma \in S_{A'}$,

$$\|\rho - \sigma\|_1 \geq \| Tr_{A' \setminus A}(\rho) - Tr_{A' \setminus A}(\sigma)\|_1.$$

Proof.

Trivial $\square$

Consistency of the charge sectors: expansion lemma

Lemma

(Expansion lemma): If an annulus A is contained in A', and the Poké Ball condition is satisfied near A', $\forall \rho, \sigma \in S_{A'}$,

$$\| Tr_{A' \setminus A}(\rho) - Tr_{A' \setminus A}(\sigma) \|_1 \geq \|\rho - \sigma\|_1 - o(1).$$

Proof.

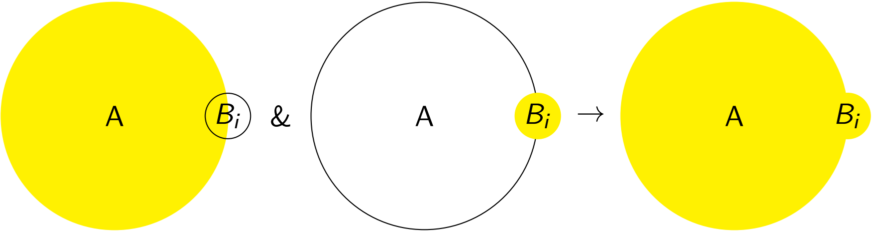
Nontrivial. Uses 1405.0137. $\square$

More on expansion lemma, colloquially

Suppose we have two states, $|\psi_1\rangle$ and $|\psi_2\rangle$, such that

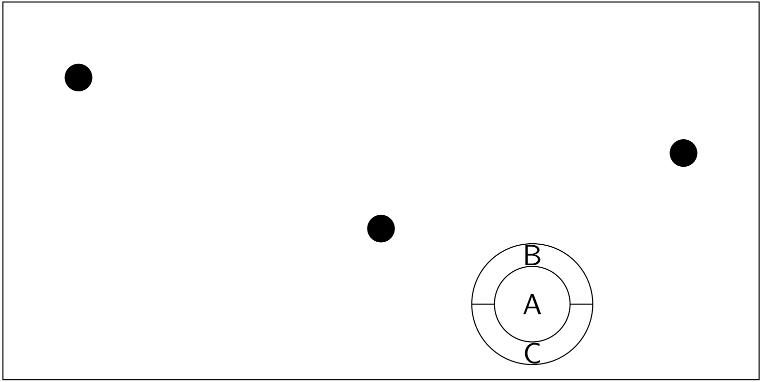
- They are close in trace distance over a subsystem A.
- They are close in trace distance over a set of (bounded) balls $\{B_i\}$ in the neighborhood of A.
- In the balls, the Poké Ball condition is satisfied.

Expansion lemma says that, if $A \cup B_i$ has the same shape as A, then $|\psi_1\rangle$ and $|\psi_2\rangle$ are close in trace distance over $A \cup B_i$.



Consistency of the charge sectors: an intermediate conclusion

$$S(A|B) + S(A|C) \approx 0$$



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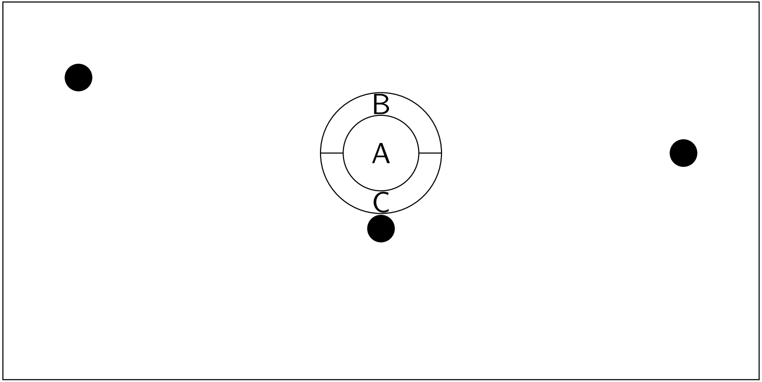

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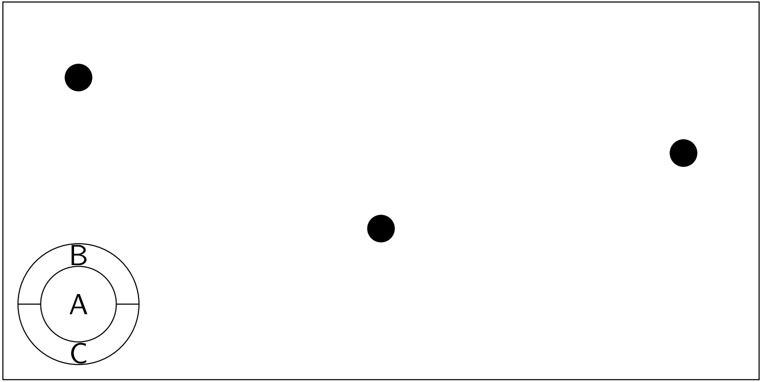

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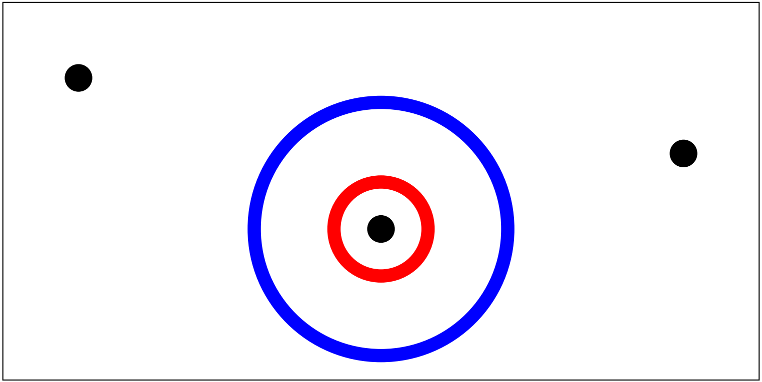
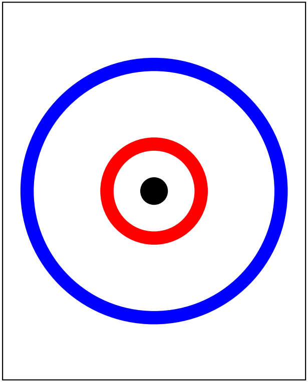
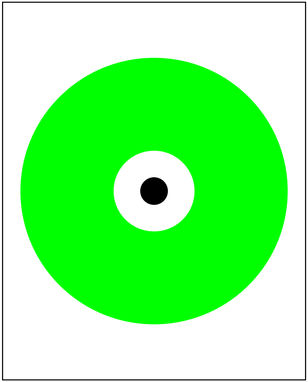


What we want to show: $\forall i$, $\exists j$ such that $\Pi_{\circ,i} \sim \Pi_{\circ,j}$. **Now let's see why.**

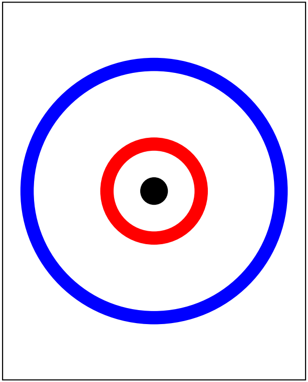
Consistency of the charge sectors: localized excitations



Consistency of the charge sectors: localized excitations



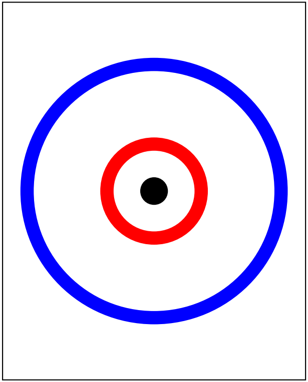
Consistency of the charge sectors: localized excitations



WLOG, choose $\rho_{{\color{red}\circ},i}$ to be one of the extreme points of $S_{\color{red}\circ}$. For $\rho_{{\color{red}\circ},\perp} = \sum_{j\neq i} \frac{1}{N} \rho_{{\color{red}\circ},j}$,

$$\begin{aligned}\|\rho_{{\color{red}\circ},i} - \rho_{{\color{red}\circ},\perp}\|_1 &\leq \|\rho_{{\color{green}\circ},i} - \rho_{{\color{green}\circ},\perp}\|_1 \\ &\leq \|\rho_{{\color{blue}\circ},i} - \rho_{{\color{blue}\circ},\perp}\|_1 + o(1).\end{aligned}$$

Consistency of the charge sectors: localized excitations



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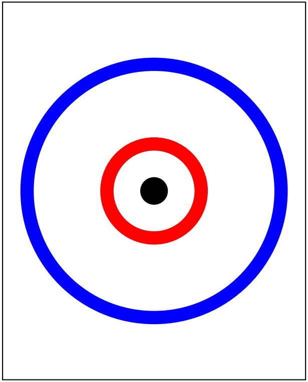
This means that $\|\rho_{{\color{red}\circ},i} - \rho_{{\color{red}\circ},\perp}\|_1 \approx \|\rho_{{\color{blue}\circ},i} - \rho_{{\color{blue}\circ},\perp}\|_1 \approx 2.$

Consistency of the charge sectors: localized excitations



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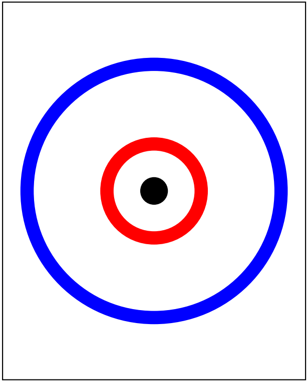
$\|\rho_{{\color{red}\circ},i} - \rho_{{\color{red}\circ},\perp}\|_1 = 2 = \|\rho_{{\color{blue}\circ},i} - \rho_{{\color{blue}\circ},\perp}\|_1 + \epsilon$.

* The projector onto the positive eigenvalue subspace of $\rho_{{\color{red}\circ},i} - \rho_{{\color{red}\circ},\perp}$ picks out $\rho_{{\color{red}\circ},i}$ and annihilates all $\rho_{{\color{red}\circ},j\neq i}$ with error $O(N\epsilon)$. (Same for ${\color{blue}\circ}$.)

Consistency of the charge sectors: localized excitations



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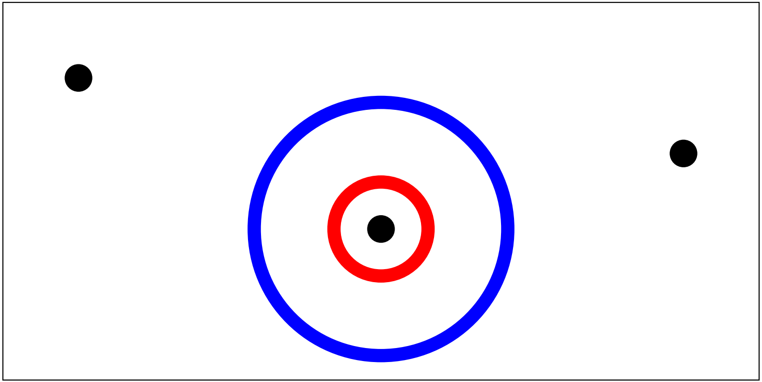
* The projector onto the positive eigenvalue subspace of $\rho_{{\color{red}\circ},i} - \rho_{{\color{red}\circ},\perp}$ picks out $\rho_{{\color{red}\circ},i}$ and annihilates all $\rho_{{\color{red}\circ},j\neq i}$ with error $O(N\epsilon)$. (Same for ${\color{blue}\circ}$.)

* We declare these projectors to be $\Pi_{{\color{red}\circ},i} \sim \Pi_{{\color{blue}\circ},j}$.

Poké Ball condition → Global consistency of charge sectors

* We can now unambiguously label all the charges lying inside an annulus by some *fixed universal set*, as long as the annuli can be deformed into one another without encountering ●.



Existence of the trivial charge

We learned that all the annuli have a canonical set of (almost) orthogonal projectors $\{\Pi_i | i = 1, \cdots, N\}$. The index set, $\{i = 1, \cdots, N\}$, is universal for all annuli. These indices represent the charge sectors.

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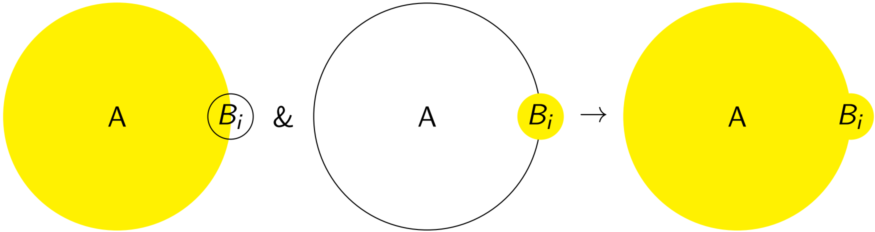
* Let's first recall the expansion lemma.



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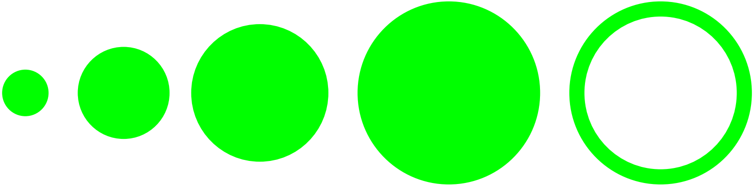
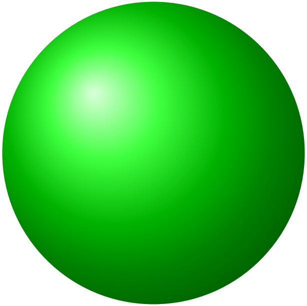


For $\bigcirc$, pick Π_i such that $\Pi_i \rho_{\bigcirc} \Pi_i \approx \rho_{\bigcirc}$.

* You are guaranteed to have only one such Π_i!

* We denote this charge as 1.

Charge of a single localized excitation is trivial (on a sphere)



Consistency equations

Playing the same game, one can show that

- All the charges have their own unique antiparticles.
- Charges can be transported unitarily.

We are almost at the cusp of deriving the anyon theory!

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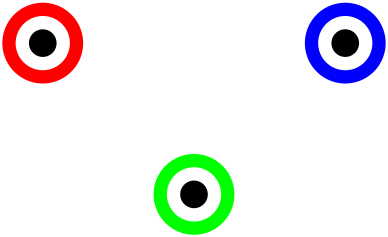
- All the charges have their own unique antiparticles.
- Charges can be transported unitarily.

We are almost at the cusp of deriving the anyon theory! We only need to prove the following list.

- Pentagon equation
- Hexagon equation
- Triangle equation

Consistency equations(flavor of the argument)



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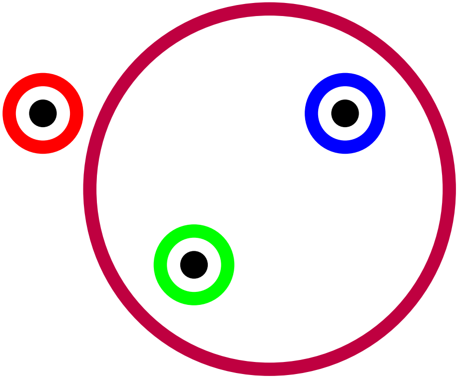

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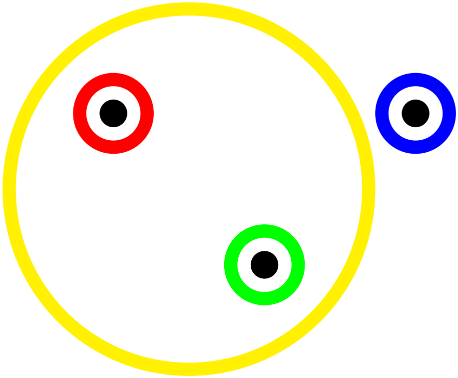

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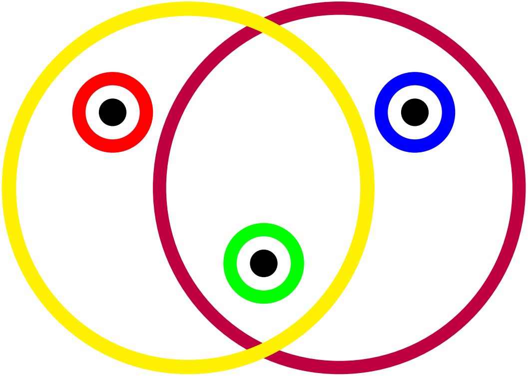


The projectors on and may not commute with each other.
* This is reminiscent to an observation that the eigenvalues of (L^2, L_z) as well as (L^2, L_x) forms a set of good quantum numbers, but the eigenvalues of (L^2, L_z, L_x) do not.

Summary

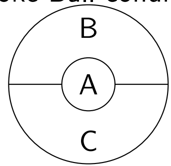
We are at the cusp of deriving all the axioms of cond-mat/0506438 (Appendix E) from the following set of assumptions:

1. $\epsilon, \delta \to 0$.
2. Simplex assumption

$$S_{\circ} = \{\rho | \rho = \sum_i p_i \rho_i\},$$

where $\sum_i p_i = 1$ and $\rho_i \perp \rho_j$ $\forall i \neq j$.

3. Poké Ball condition



$$S(A|B) + S(A|C) \approx 0$$

Future directions

- We are still forcefully injecting the notion of charge from the very beginning.(Simplex assumption) I would like to be able to prove the simplex assumption from the Poké Ball condition.
- Now we can mechanically derive the axioms of the effective theory that describe interesting classes of quantum many-body systems at low energy.
    - I expect this machinery to be applicable in the presence of open boundary condition/higher dimensions.
- Classification of phases?
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Thank you!