Compressibility of positive semidefinite factorizations and quantum models

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Fundamental task

Consider: experiment allowing...

- the preparation of states $\rho_1, \ldots, \rho_X$ (unknown),
- the performance of measurements "y" described by POVMs $E_{y1}, \ldots, E_{yZ}$ (unknown; $y \in [Y]$).

Given:

- $\hat{\mathcal{D}}_{xyz} \approx \mathbb{P}[z|xy]$.

Objective: Learning of *effective models*

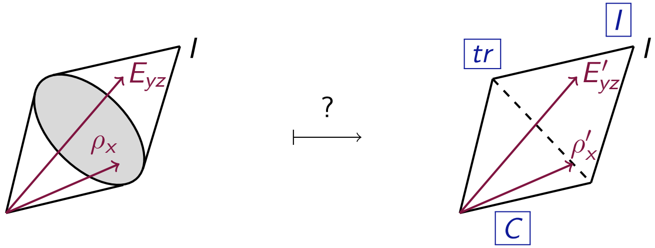
$$\begin{aligned} \min \quad & d \\ \text{s.t.} \quad & \exists\ d\text{-dimensional states } \rho_x \text{ and POVMs } (E_{yz})_z \\ & \text{such that } \hat{\mathcal{D}}_{xyz} \approx \operatorname{tr}(\rho_x E_{yz}). \end{aligned}$$

Low-dimensional descriptions

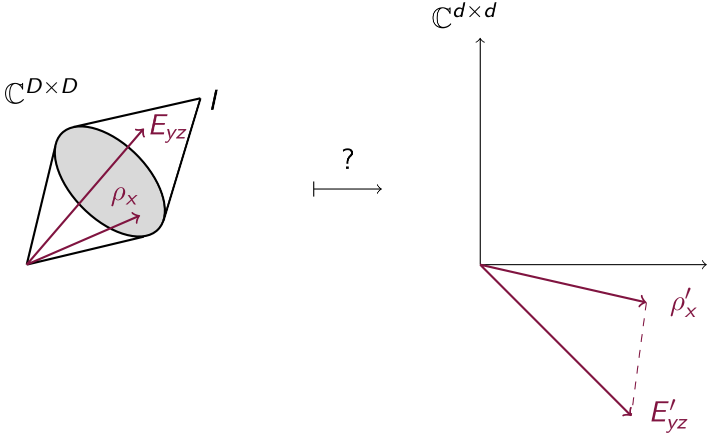
- Often: $\exists$ low dimensional descriptions
- Coincidence?
- Not if

  $\forall\ (\mathbb{C}^D, (\rho_x)_x, (E_{yz})_{yz})$
  $\exists\ (\mathbb{C}^d, (\rho'_x)_x, (E'_{yz})_{yz})$ with $d \ll D$
  s.t. $\mathrm{tr}(\rho'_x E'_{yz}) \approx \mathrm{tr}(\rho_x E_{yz})$.
- True?

Compression of quantum models



Relaxed compression problem



Relaxed compression problem addressed by...

Theorem (Johnson Lindenstrauss). Consider

$$v_1, \dots, v_S \in \mathbb{C}^D,$$

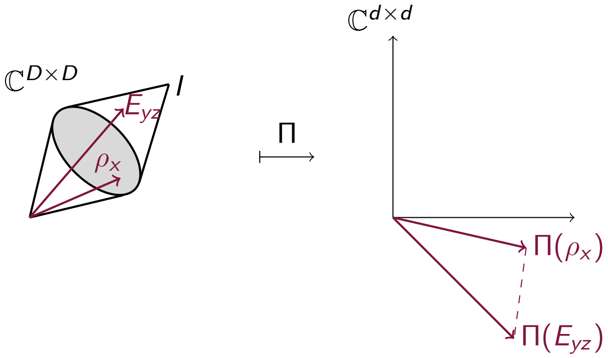
and assume

$$\Pi \in \mathbb{C}^{d \times D} \text{ iid Gaussian.}$$

where $d \ll D$. Then, with high probability

$$(1-\varepsilon)\|v_i - v_j\|_2 \leq \|\Pi v_i - \Pi v_j\|_2 \leq (1+\varepsilon)\|v_i - v_j\|_2$$

Relaxed compression problem



- Q: $\rho'_x = \Pi(\rho_x),\ E'_{yz} = \Pi(E_{yz})$?
- No.
- Q: consequences of boundary conditions $\boxed{C}$, $\boxed{tr}$ and $\boxed{I}$?

Consequences of boundary conditions: $\boxed{I}$, $\boxed{C}$ $\Rightarrow$ LB

- $\mathcal{D}_{xyz} = \mathrm{tr}(\rho_x E_{yz})$
- $\underbrace{\left\| \sum_z E_{yz} \right\|_1}_{\mathcal{D}\text{-based LB} \rightsquigarrow} = \|I\|_1 = \underbrace{d}_{\mathcal{D}\text{-based LB}}$
- $\mathcal{D}_{xyz} = \mathrm{tr}(\rho_x E_{yz}) \leq \underbrace{\|\rho_x\|}_{\leq 1} \|E_{yz}\|_1$

Theorem 1. Let

- $(\mathbb{C}^d, (\rho_x)_x, (E_{yz})_{yz})$ s.t. $\mathrm{tr}(\rho_x E_{yz}) = \mathcal{D}_{xyz}$, and
- $c_{yz} = \max\{\mathcal{D}_{xyz}\}_x$.

Then, for all y,

$$d \geq \sum_{z=1}^{Z} c_{yz}$$

- Found independently in [Lee, Wei, de Wolf, 2014].

Consequences of boundary conditions: $\boxed{I}$, $\boxed{C}$ $\Rightarrow$ LB

- **Example.** Let

  $Y = 1,$
  $X = Z,$
  $\forall j \in [Z]\ \rho_j = E_{1j} = |j\rangle\langle j| \Rightarrow \mathrm{tr}(\rho_x E_{1z}) = \delta_{xz}.$

  Then, $c_{1z} = 1$ and therefore

$$\underbrace{d \geq Z}_{\boxed{\text{no comp. below } Z}}$$

- Q: compression down to Z?

Step 1: compression respecting $\boxed{C}$

- To preserve: $\mathrm{tr}(\rho_x E_{yz}) = \sum_{ij} p_i^x \varepsilon_j^{yz} \left|\langle \psi_i^x | \varepsilon_j^{yz} \rangle\right|^2$
- By the polarization identity,

$$\begin{aligned} & \langle \psi | \varepsilon \rangle \\ = \; & \frac{1}{4}\Big(\|\psi + \varepsilon\|_2^2 - \|\psi - \varepsilon\|_2^2 + i\|\psi + i\varepsilon\|_2^2 - i\|\psi - i\varepsilon\|_2^2 \Big) \\ \approx \; & \frac{1}{4}\Big(\|\Pi\psi + \Pi\varepsilon\|_2^2 - \|\Pi\psi - \Pi\varepsilon\|_2^2 \pm \dots \Big), \quad \text{if } \Pi \text{ Gaussian} \\ = \; & \langle \Pi\psi | \Pi\varepsilon \rangle \end{aligned}$$

- Therefore, we expect

$$\mathrm{tr}(\rho_x E_{yz}) \approx \mathrm{tr}(\underbrace{\Pi \rho_x \Pi^*}_{=:\rho_x'} \underbrace{\Pi E_{yz} \Pi^*}_{=:E_{yz}'})$$

Careful analysis $\rightsquigarrow$ *compression of psd factorizations*

Theorem 2. Let

$$M_1, \ldots, M_J \in S^+(\mathbb{C}^D),$$
$$\varepsilon \in (0, 1/2],$$
$$M_j' = \Pi M_j \Pi^*.$$

Then, w.p. $\geq 1 - 4J^2D^2e^{-\varepsilon^2 d/8}$

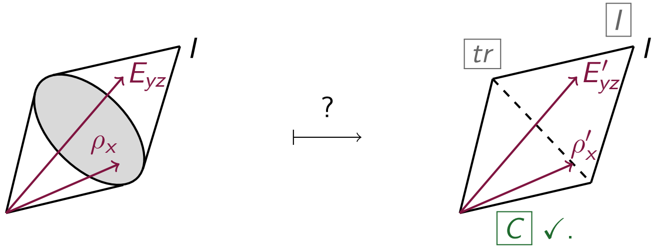
$$\mathrm{tr}(M_i M_j) - 192\varepsilon\, \mathrm{tr}(M_i)\mathrm{tr}(M_j) \leq \mathrm{tr}(M_i' M_j') \leq \mathrm{tr}(M_i M_j) + 192\varepsilon\, \mathrm{tr}(M_i)\mathrm{tr}(M_j).$$

- By union bound (all pairs (i, j)), for

$$d = \frac{16}{\varepsilon^2} \ln(2JD)$$

there exists $M_1', \ldots, M_J' \in S^+(\mathbb{C}^d)$ s.t. error bound ✓.

Where we are...



Remaining boundary conditions

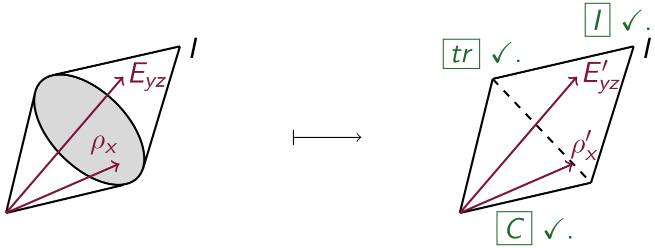
- Q: compression of Q-models?
  - $\boxed{tr}$:

$$\mathrm{tr}(\Pi\rho\Pi^*) = \sum_k p_k \underbrace{\langle\psi_k|\Pi^*\Pi|\psi_k\rangle}_{\|\Pi\psi_k\|_2^2=(1\pm\varepsilon)^2} \in [(1-\varepsilon)^2, (1+\varepsilon)^2]$$

  - $\boxed{I}$: ? $\rightsquigarrow$

- Redefine $E'_{yZ} := I - \underbrace{\sum_{z=1}^{Z-1} E'_{yz}}_{=:E} \rightsquigarrow \boxed{I}$ ✓. $\boxed{C}$?
- Q: $E'_{yZ} \geq 0$?
- By [Haagerup, Thorbjorsen 2003] and Laplace trsf method: true with high probability if $d \geq \frac{32}{\varepsilon^2}\mathrm{rank}(E)$.

Remaining boundary conditions



Quantum compression

Theorem 3. Let

$$J := X + YZ$$

$$d \geq \frac{32}{\varepsilon^2} \ln(4JD) + \frac{32}{\varepsilon^2} \mathrm{rank}\Big(\sum_{z=1}^{Z-1} E_{yz}\Big) \ \forall y.$$

Then, $\exists\ (\rho'_x)_x\ (E'_{yz})_{yz}$ d-dimensional s.t.

- $\left|\mathrm{tr}(\rho_x E_{yz}) - \mathrm{tr}(\rho'_x E'_{yz})\right| \ \leq 200\varepsilon\, \mathrm{tr}(E_{yz})$ if $z \in [Z-1]$
- $\left|\mathrm{tr}(\rho_x E_{yZ}) - \mathrm{tr}(\rho'_x E'_{yZ})\right| \leq 200\varepsilon\, \mathrm{tr}(I - E_{yZ})$.

- Generalization of [Winter, quant-ph/0401060] (specific to rk-1 measurements; quadratic compression)

Implications

- Data analysis
- Dimension witnessing
- 1-way quantum CC

Implications: Data analysis

- Assume $(\mathbb{C}^D, (\rho_x)_x, (E_{yz})_{yz})$ is "*pseudo-low-rank*", i.e.,
  - $\underbrace{E_{y,1}, ..., E_{y,Z-1}}_{\text{rank}=\mathcal{O}(1) \text{ in } D}, E_{yZ}$

  Then, $d = \mathcal{O}(\frac{1}{\varepsilon^2} \ln(D))$.
- For example, in the previous example, if now
  - $Z = 2$, $X = Y = D$, with
  - $\forall j \in [X]\ \rho_j = E_{j1} = |j\rangle\langle j|$, and
  - $E_{j2} = I - |j\rangle\langle j|$,

  Then, $d = \mathcal{O}(\frac{1}{\varepsilon^2} \ln(D)) \ll D \rightsquigarrow$ "Z matters".

Implications: Dimension witnessing

- Let $f^*\big((\mathrm{tr}(\rho_x E_{yz})_{xyz}\big)$ be s.t.
$$f^*\big((\mathrm{tr}(\rho_x E_{yz})_{xyz}\big) \leq D$$

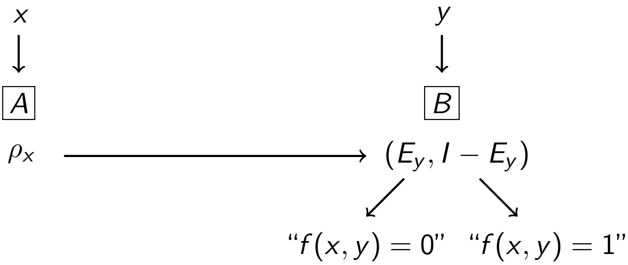
- Robust w.r.t. l_∞ noise if
$$\big|f^*\big((\hat{\mathcal{D}})_{xyz}\big) - f^*\big(\mathcal{M}\big)\big| \leq L\|(\hat{\mathcal{D}})_{xyz} - \mathcal{M}\|_\infty$$

- In particular, if model pseudo-low-rank, we need consider $\mathcal{M} = (\mathrm{tr}(\rho'_x E'_{yz})_{xyz}$. Then,
$$f^*\big(\mathcal{M}\big) \leq d = \mathcal{O}\left(\frac{1}{\varepsilon^2}\ln(D)\right)$$

- Hence, by L-continuity,
$$f^*\big((\mathrm{tr}(\rho_x E_{yz})_{xyz}\big) = \mathcal{O}(\frac{1}{\varepsilon^2}\ln(D) + L\varepsilon) \ll D \Rightarrow \text{gap}.$$

Implications: 1-way quantum CC

- Let $f : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}$, and



- Goal: correct w.p. $\geq 2/3$.
- Set $A \in \mathbb{R}^{2^n \times 2^m}$ s.t. $A_{xy} = f(x,y)$
- $\rightsquigarrow$ Find ρ_x, E_y s.t.
  - $\mathrm{tr}(\rho_x E_{yz}) \geq 2/3$ if $A_{xy} = 1$,
  - $\mathrm{tr}(\rho_x E_{yz}) \leq 1/3$ if $A_{xy} = 0$.

  This is approximate Q-model for A.

Implications: 1-way quantum CC

- Let $(\mathbb{C}^D, (\rho_x)_x, (E_y)_y)$ be valid protocol.
- Set $\mathbf{r} = \max_{y\in\{0,1\}^m} \min_{z\in\{0,1\}} \mathrm{rank}(E_{yz})$.
- By theorem 3, original communication cost $\log_2(D)$ can be compressed to
$$\mathcal{O}\Big(\log\big(nm\mathbf{r}\log(D)\big)\Big).$$

Conclusions

- Psd factorizations can be compressed; error scales with trace.
- Lower bound on compressibility of quantum models.
- Pseudo-low-rank quantum models admit exponential compression.
- Implications in
  - data analysis,
  - robust dimension witnessing,
  - 1-way quantum CC.

Thank you!