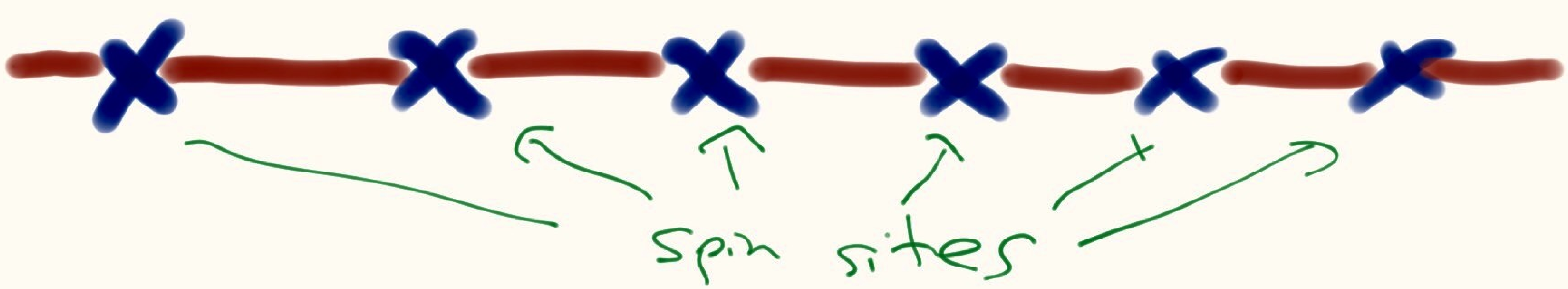
Coogee Lecture.

Motivational idea: subfactors and conformal field theory. New subfactors — Haagerup et al. (Morrison, Peters, Snyder, Bigelow, Grossman, Penneys, ...) $\Longrightarrow$ New CFT's ?

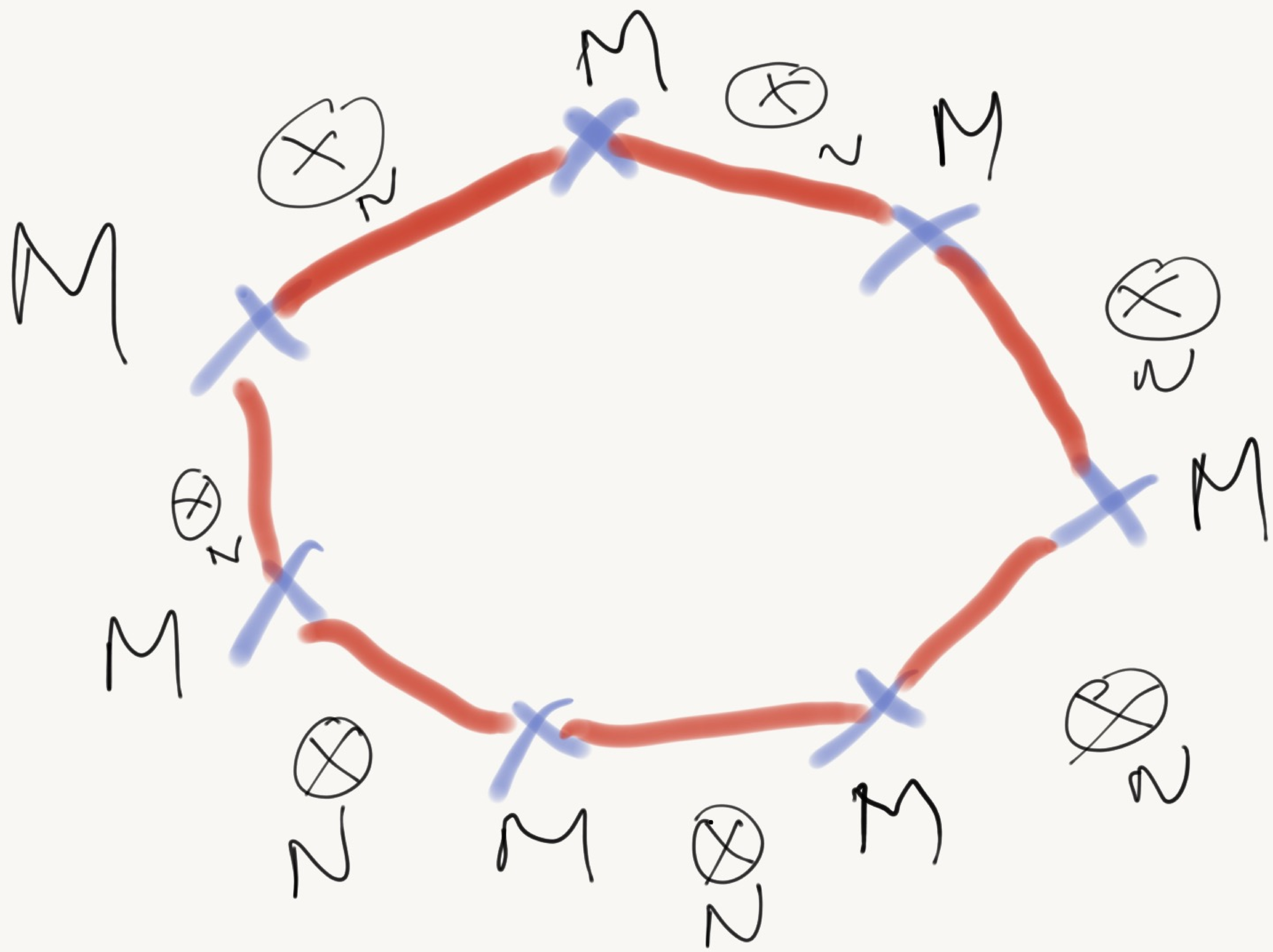
Subfactors "are" quantum spin chains

observables.

$$M \underset{N}{\otimes} M \underset{N}{\otimes} M \underset{N}{\otimes} M \underset{N}{\otimes} M \underset{N}{\otimes} M$$



Periodic boundary conditions:



Recall $M \otimes_N M = \dfrac{M \otimes M}{[m_1 n \otimes m_2 - m_1 \otimes n m_2]}$

A finite dimensional vectorspace.

Example $\quad M = M_2(\mathbb{C})$

$N = \mathbb{C}\,\mathrm{id}$

Quantum spin chain.

Other subfactors give, in simplest case,

Andrews Baxter Forester.

And in general a Rational CFT gives a subfactor from algebras of localised observables.

Conformal invariance comes from local scale invariance of the system (stat mech model at criticality, Lagrangian) and is manifested in a chiral half of the theory by a unitary representation of $\mathrm{Diff}(S^1)$.

For a quantum spin chain we might look for local scale invariance spatially. Obviously $\mathrm{Diff}\, S^1$ does not preserve the lattice so we would have to take continuum limit first. But there is an "intermediate" step – form a lattice with spins on the dyadic rationals. $\left\{ \frac{a}{2^b} , \begin{matrix} a \in \mathbb{Z} \\ b \in \mathbb{Z} \end{matrix} \right\}$

(say in $[0,1]$ or on the circle)

Then the Thompson groups F and T act precisely as local scale transformations.

Definition $F = \{$ homeomorphisms $g : [0,1] \to [0,1]$ that are piecewise linear with finitely many non-smooth points which are dyadic rationals, and such that the slopes $\frac{dg}{dx}$, when they exist, are powers of 2 $\}$

T = same as F except on the circle $\mathbb{R}/\mathbb{Z}$.

F and T obviously preserve the dyadic rationals and act as local scale transformations by definition.

The Hilbert spaces:

Block spin : $R: \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2$ $\left(R(e_i \otimes e_j) = \sum R_{ijk} e_k \right)$

will combine 2 spins into one - tensoring gives a "block spin" map from

$$\bigotimes^{2n} \mathbb{C}^2 \longrightarrow \bigotimes^{n} \mathbb{C}^2 .$$

Take "adjoint" and iterate inclusions

$$\bigotimes^{2^n} \mathbb{C}^2 \hookrightarrow \bigotimes^{2^{n+1}} \mathbb{C}^2 \hookrightarrow \bigotimes^{2^{n+2}} \mathbb{C}^2 \hookrightarrow \cdots$$

Provided these inclusions are Isometries we may put a prehilbert space structure on the direct limit lim $\bigotimes^{2^n} \mathbb{C}^2$ to obtain a Hilbert space $\mathcal{H}$.

Where we have used the notation of spin networks, so for instance the first equation reads

$$\sum_{i,j} R_{ijk} \overline{R_{ijl}} = \delta_{k,l}$$

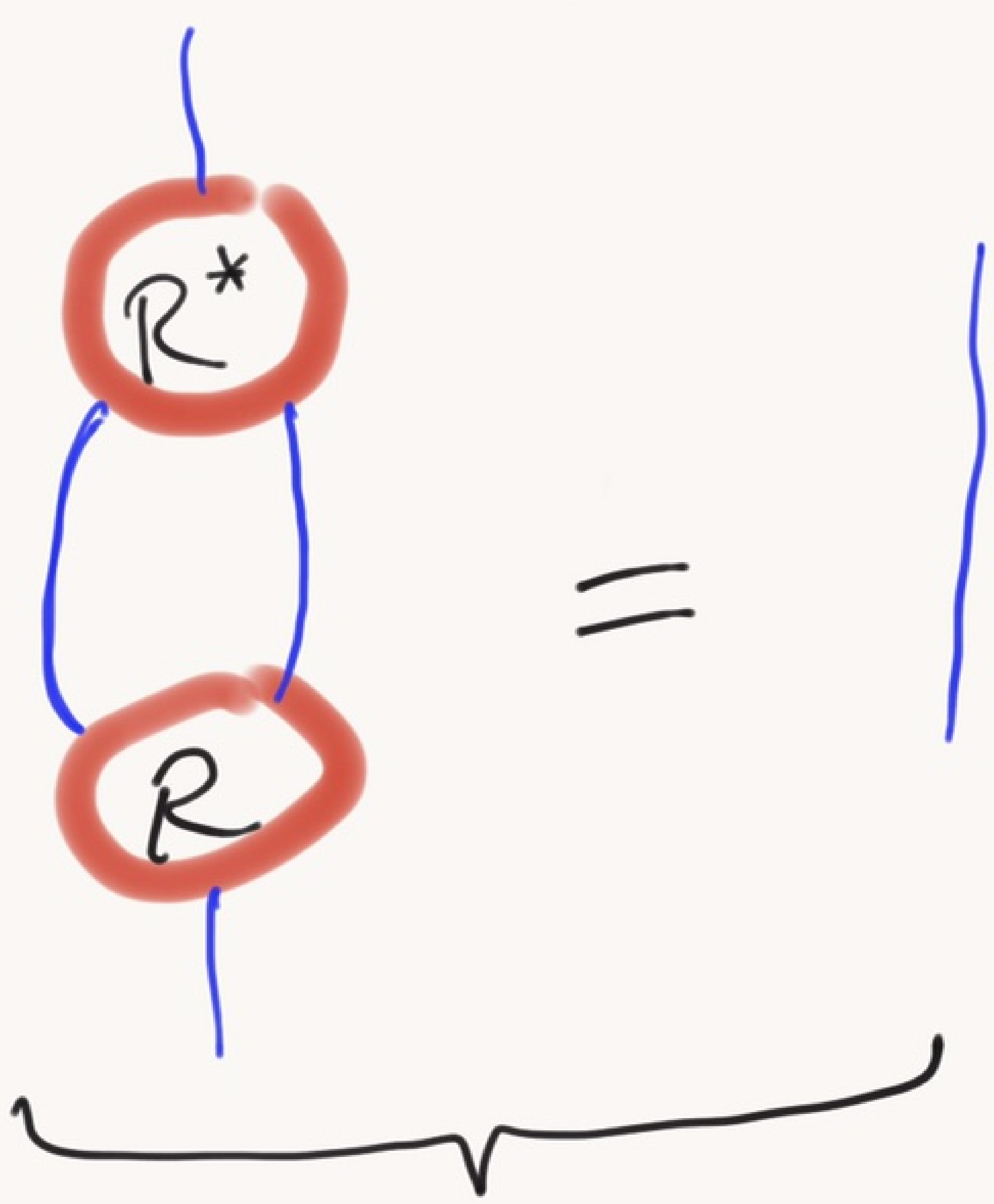
This is the ONLY equation to obtain a Hilbert space.

<u>Theorem</u> The Thompson groups F and T act unitarily in the "obvious" way on the limit hilbert spaces $\mathcal{H}$. "Vacuum" vectors Ω_R may be chosen in $\mathbb{C}^2$ and the coefficients $\langle g\Omega_R, \Omega_R\rangle$ are interesting.

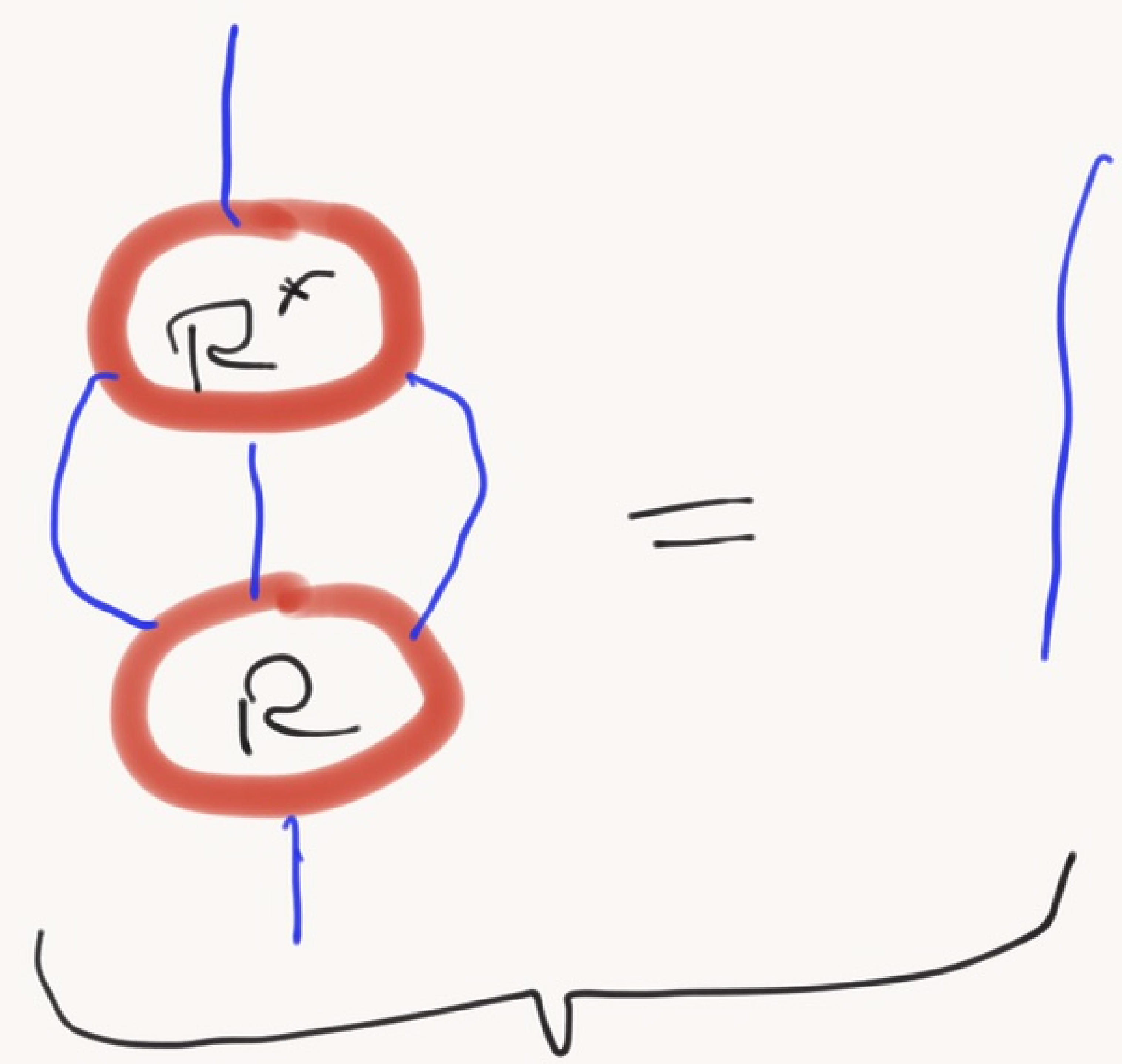
One may also block more spins together by using an R-matrix with more indices.

e.g. given $R: \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2$ one obtains maps $\overset{n}{\otimes} \mathbb{C}^2 \longrightarrow \overset{3n}{\otimes} \mathbb{C}^2 \longrightarrow \overset{3^2 n}{\otimes} \mathbb{C}^2 \dashrightarrow$ as the adjoint of the operation of blocking 3 spins together into one.

The "unitarity" condition is



blocking 2 spins



blocking 3 spins

No-Go theorem:

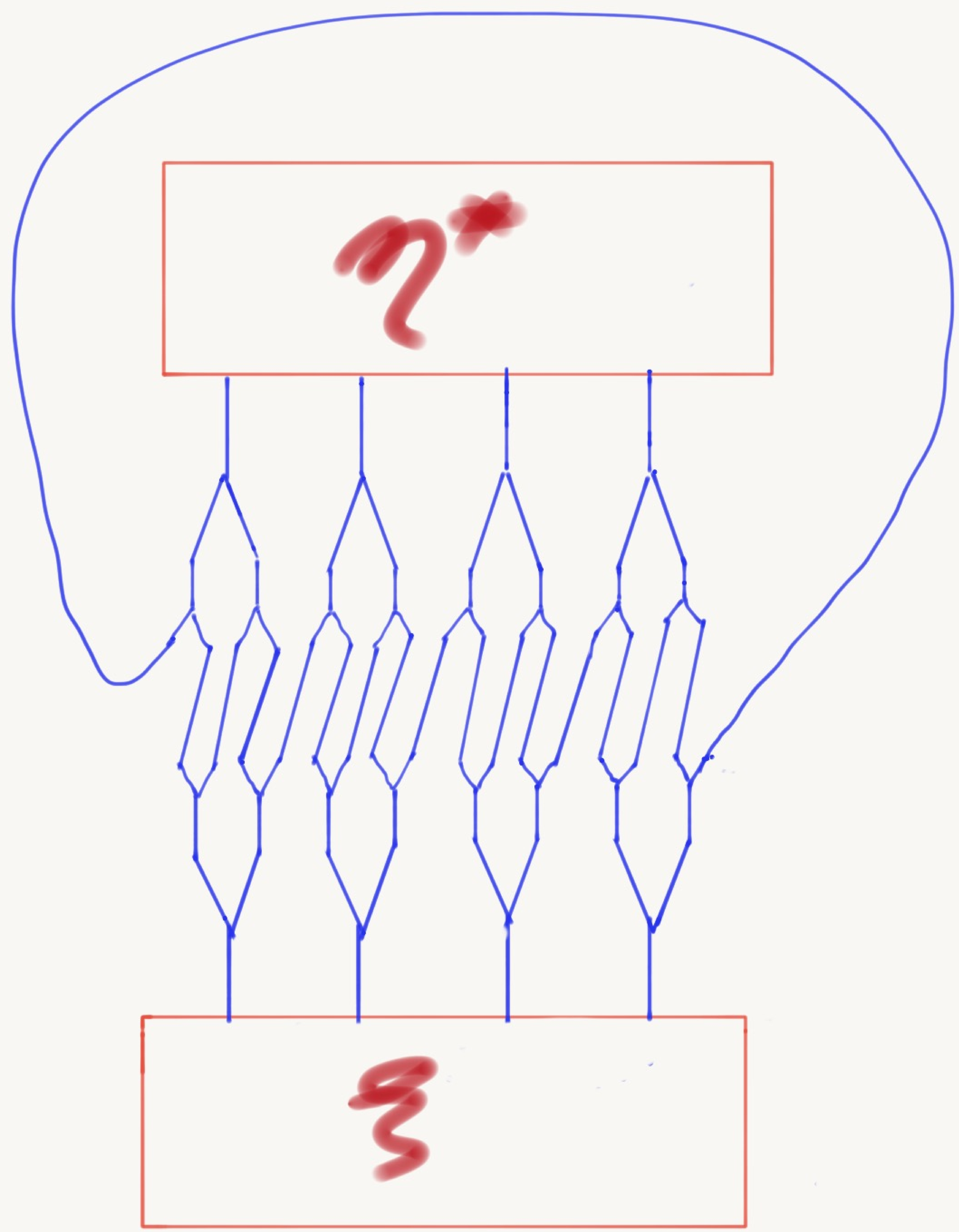
If we want to have time and/or space evolution given by an action of the circle $\mathbb{T}^1$ on the Hilbert space, it cannot be obtained as the limit of the Thompson group representations we have constructed by restricting to the group of rotations by dyadic rationals in T and taking the closure. The theorem says indeed that the image of the dyadic rotations is discrete. More precisely

$$\lim_{n\to\infty} \langle r_{\frac{1}{2^n}} \xi, \eta \rangle = 0$$

for all vectors ξ, η, and where $r_{\frac{1}{2^n}}$ is the dyadic rotation by $1/2^n$

Idea of proof: (rotation $\leftrightarrow$ transfer matrices)

$$\langle r_{\frac{1}{2}}\xi, \eta\rangle =$$



zoom

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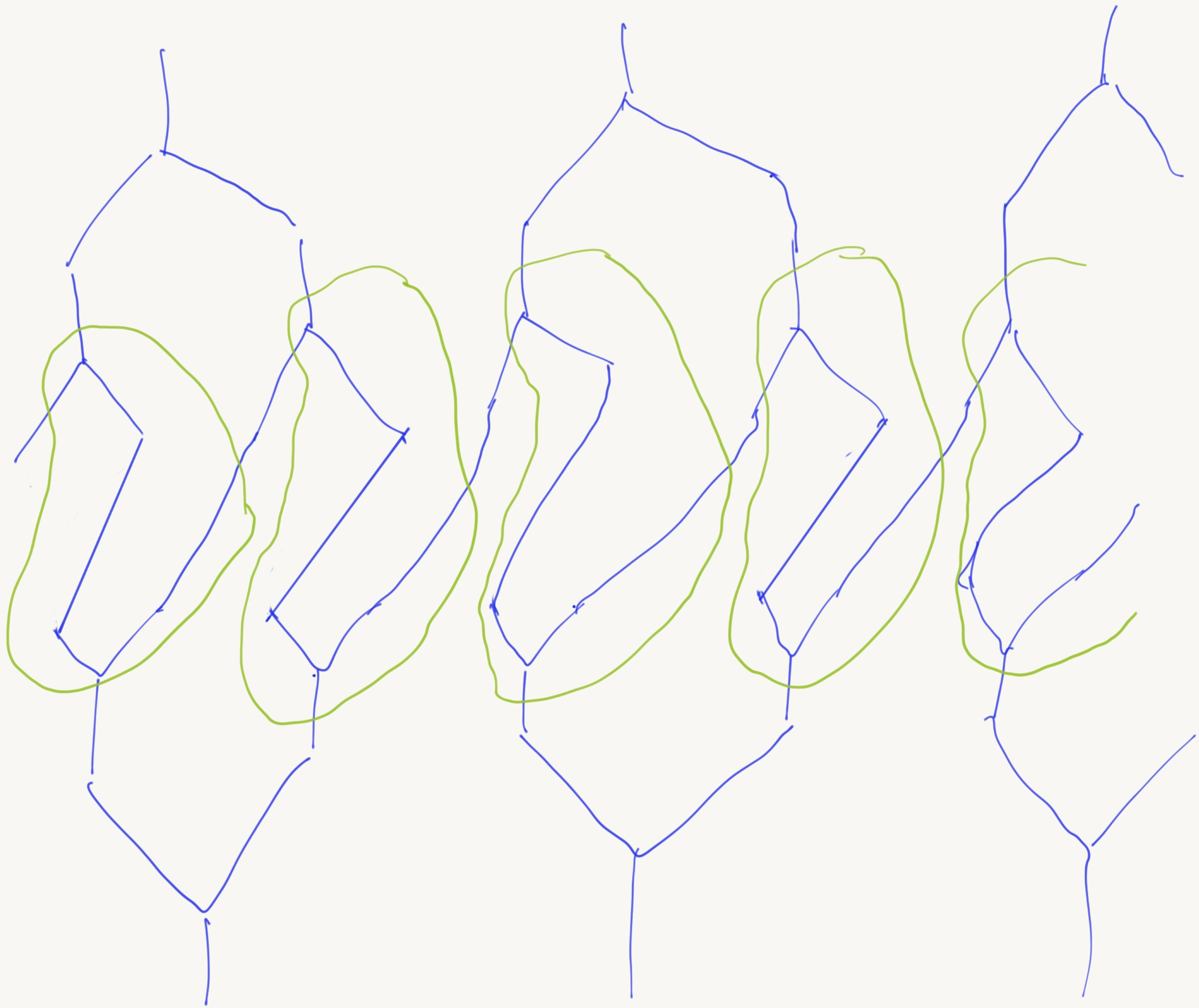

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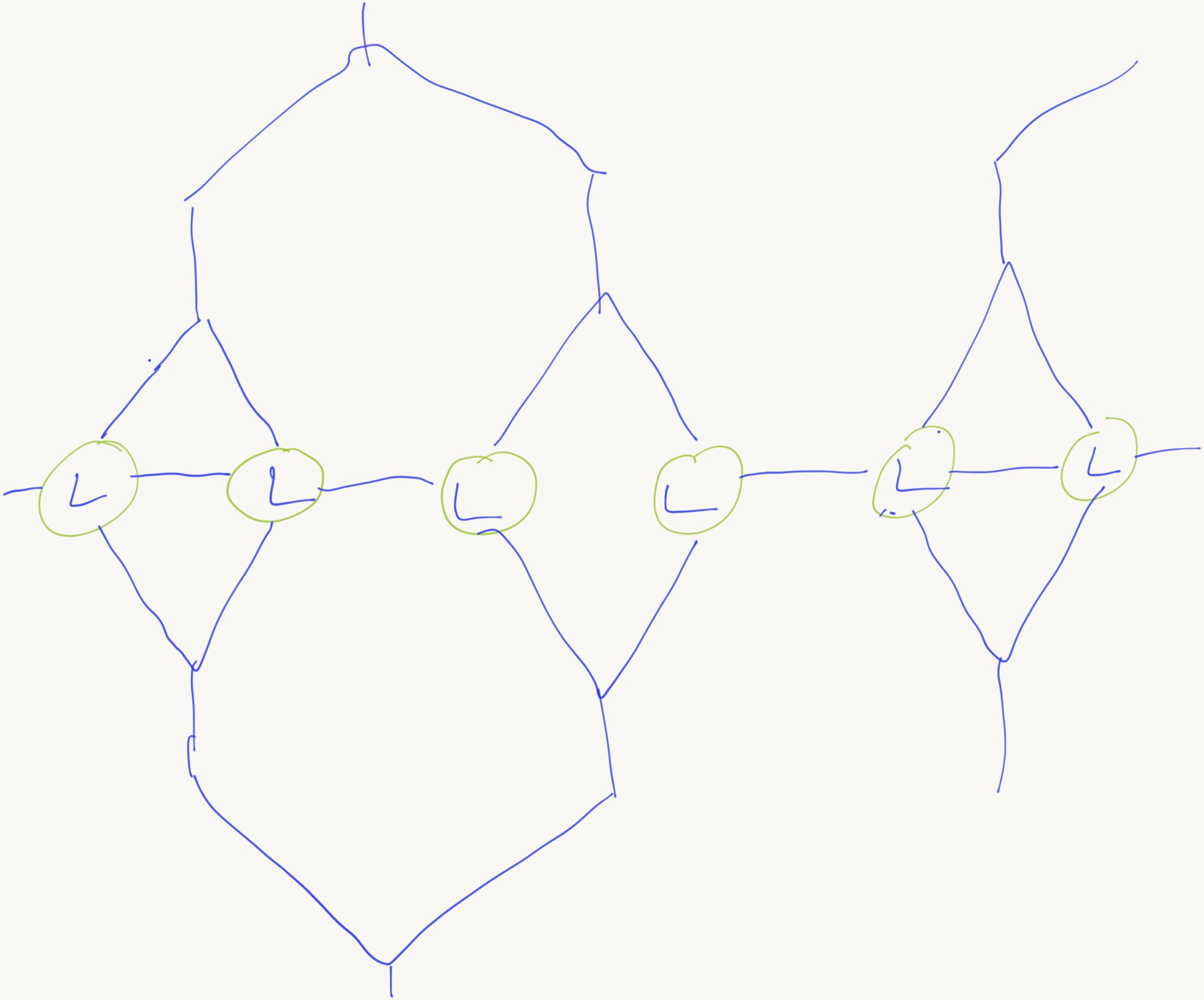
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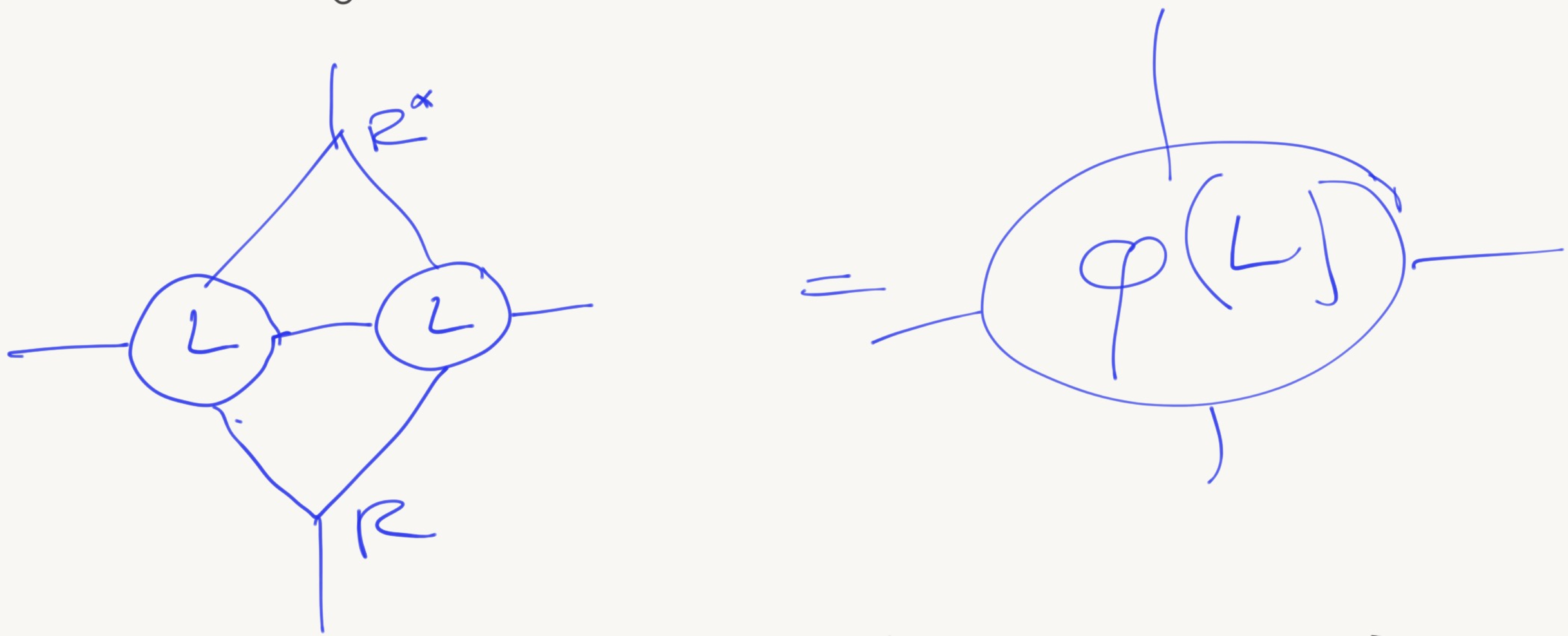
So we focus on the transformation φ (Quadratic) of the "4-box space" given by the "renormalisation"

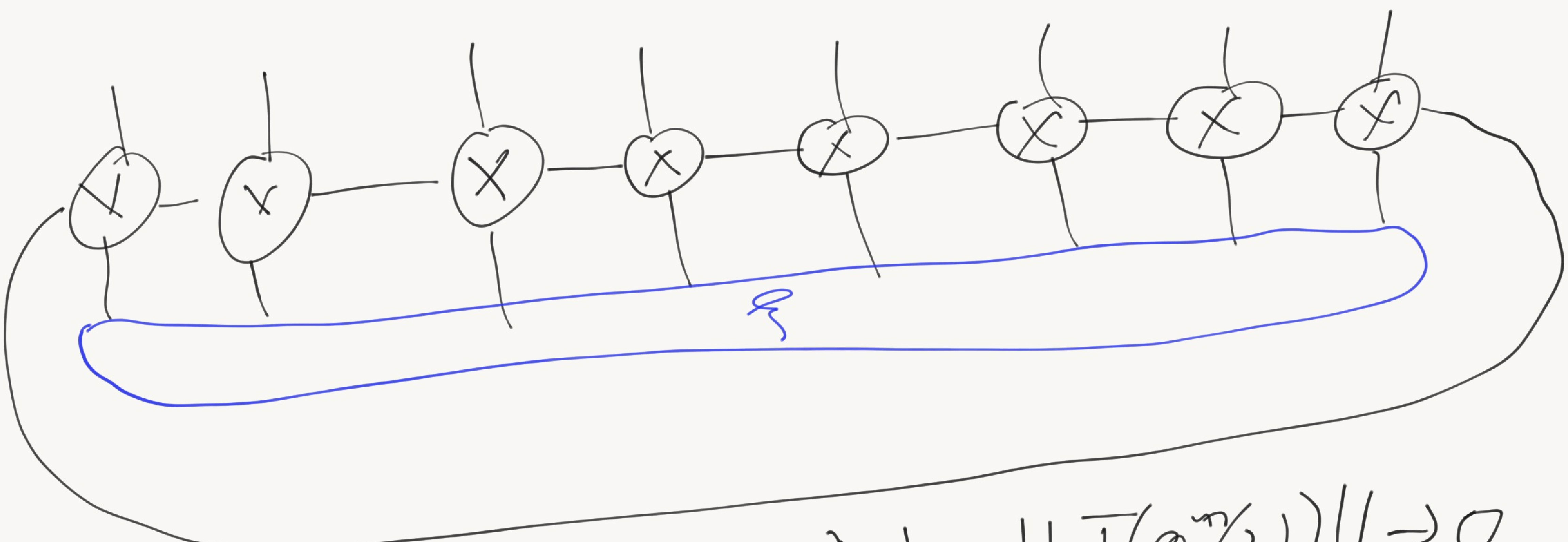


we see that for some large $m(n)$

$$\langle r_{\frac{1}{2^n}}\xi, \eta\rangle = \langle T_{\varphi^{m}(L)}\xi, \eta\rangle$$

where $T(X)$ is the (periodic) transfer matrix

$T(X) =$



Can show (C. Jones, Ghosh) that $\| T(\varphi^n(z)) \| \to 0$ as $n \to \infty$.

What to do??

Give up?

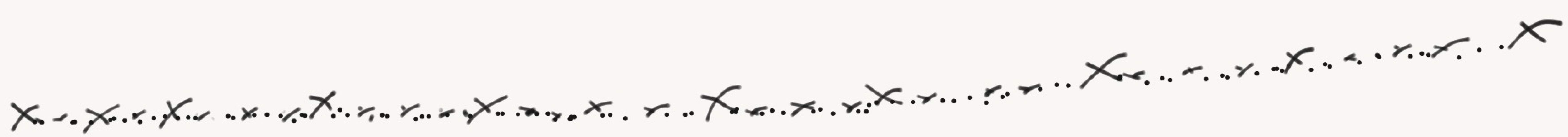
Pasquier/Saleur. Saleur say that the CFT limit should exist. Even conjectured explicit formulae for L_n's of Virasoro, But hard to nail down the Hilbert space. Where is the relevant vacuum vector? Surely not one of the ones we are coming up with.

So.....

At least we should spend some time playing with the structure at hand - it might be very relevant for quantum spin chains.

Which is a better model for a chain of say 10^8 spins:

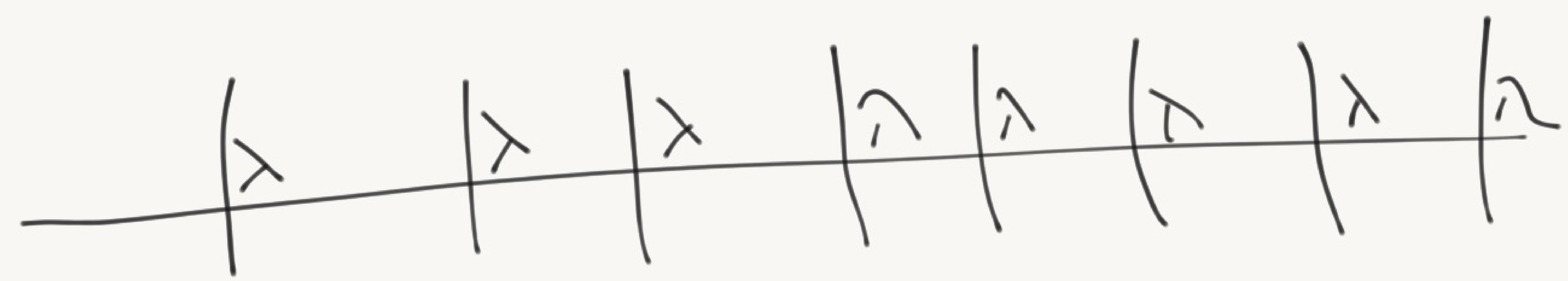
or



?

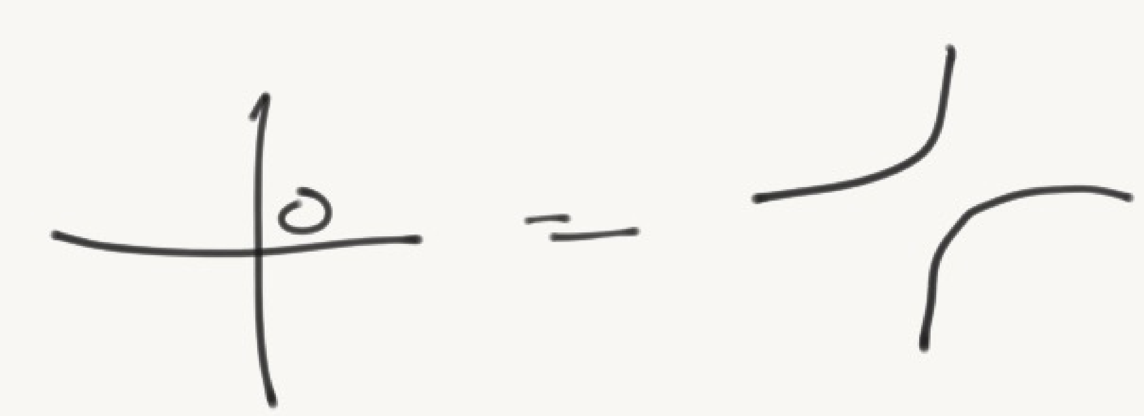
So far only "kinematics". Should be looking for Hamiltonians.

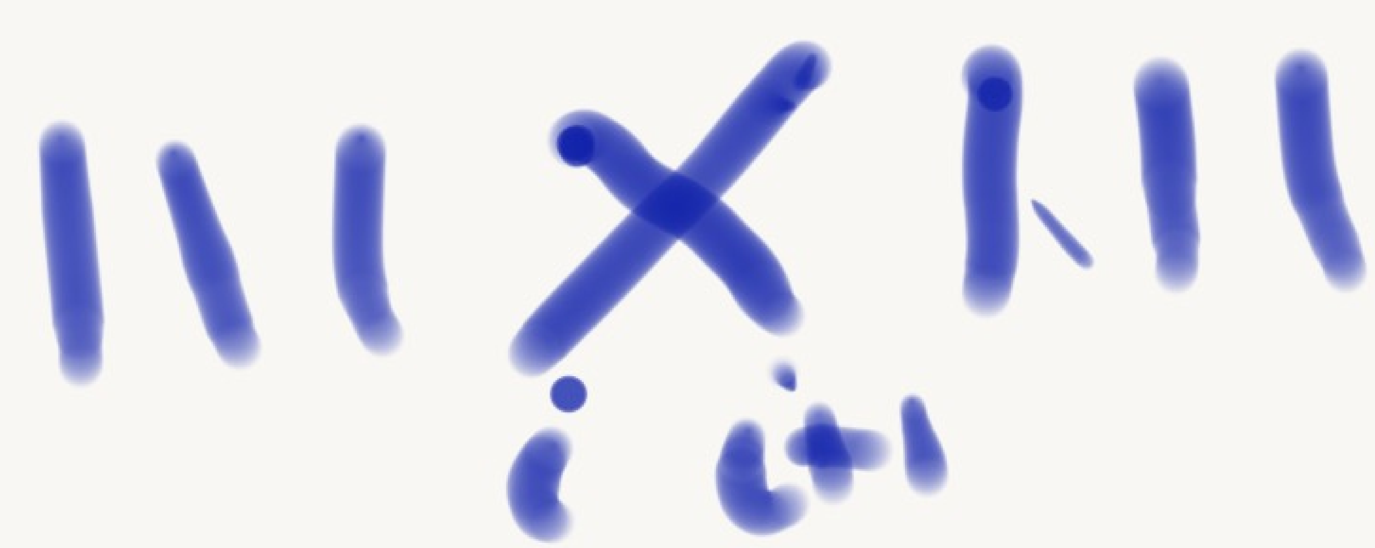
Reminder: the standard procedure:

$T(\lambda)$ 

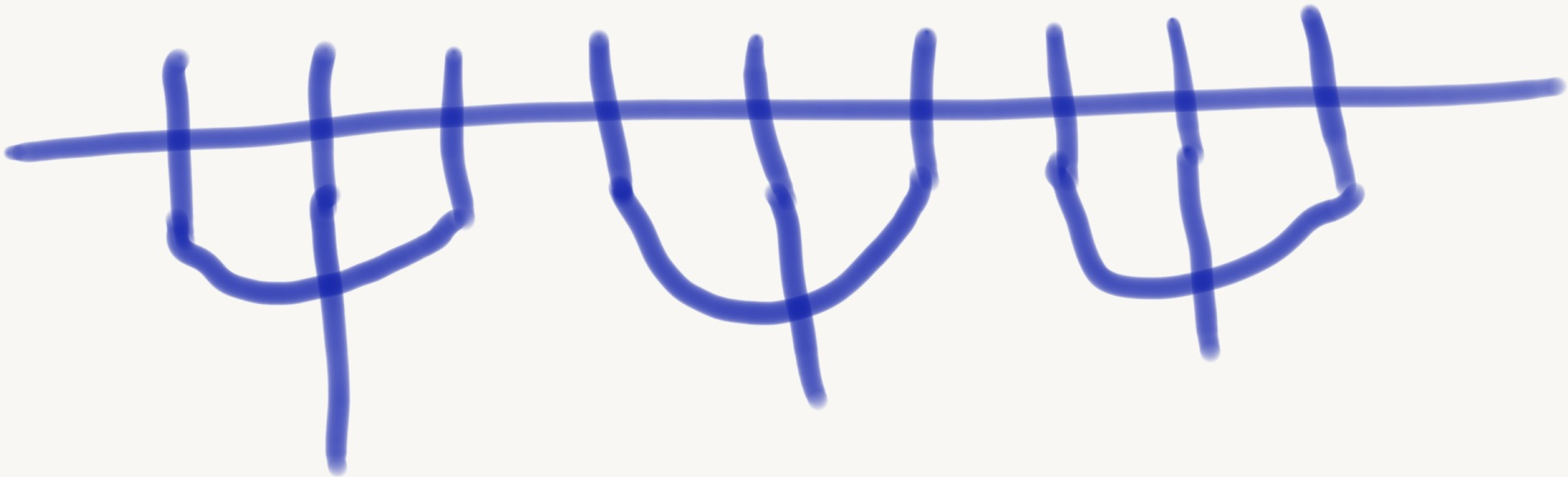
transfer matrix

By product rule for differentiation

$\frac{1}{T(0)} T'(0)$ is a spatially homogeneous local hamiltonian provided 

$\sum_i$ 

Suggests : look for a <u>scale invariant</u> transfer matrix ie.



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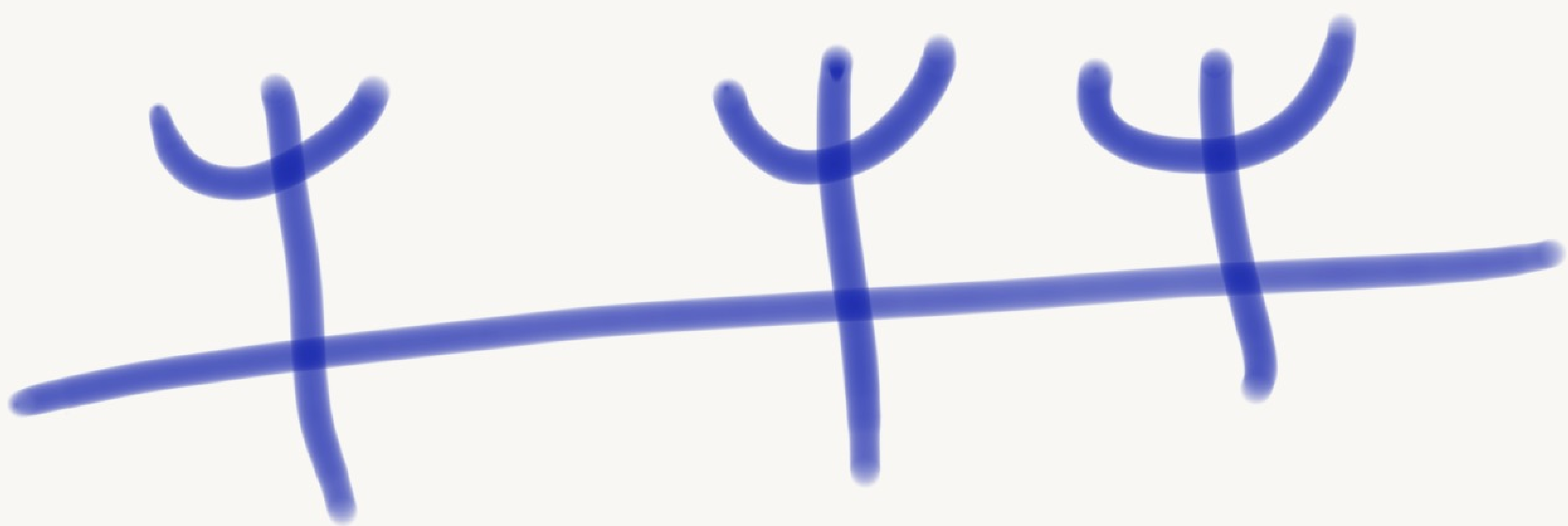
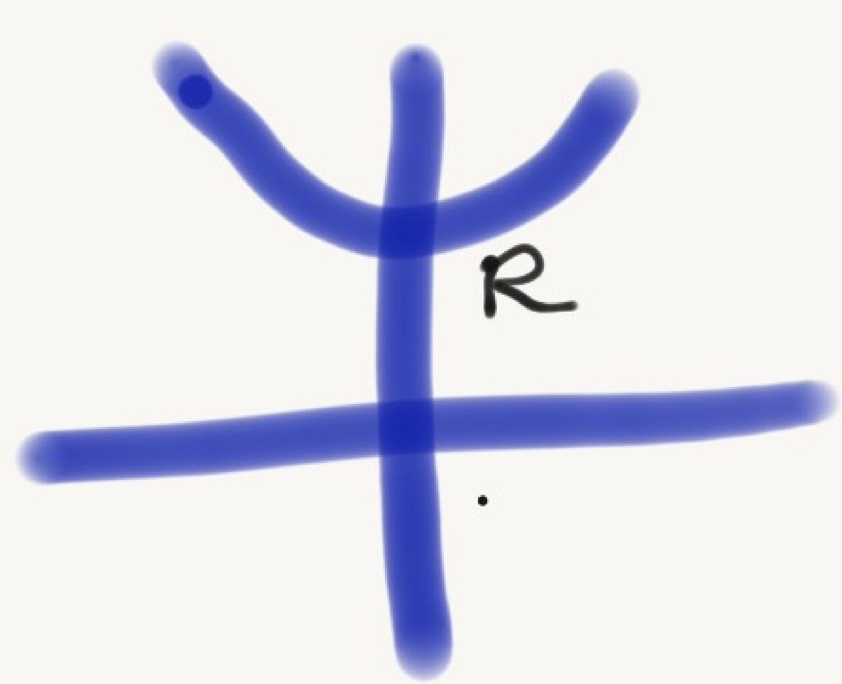


guaranteed by :



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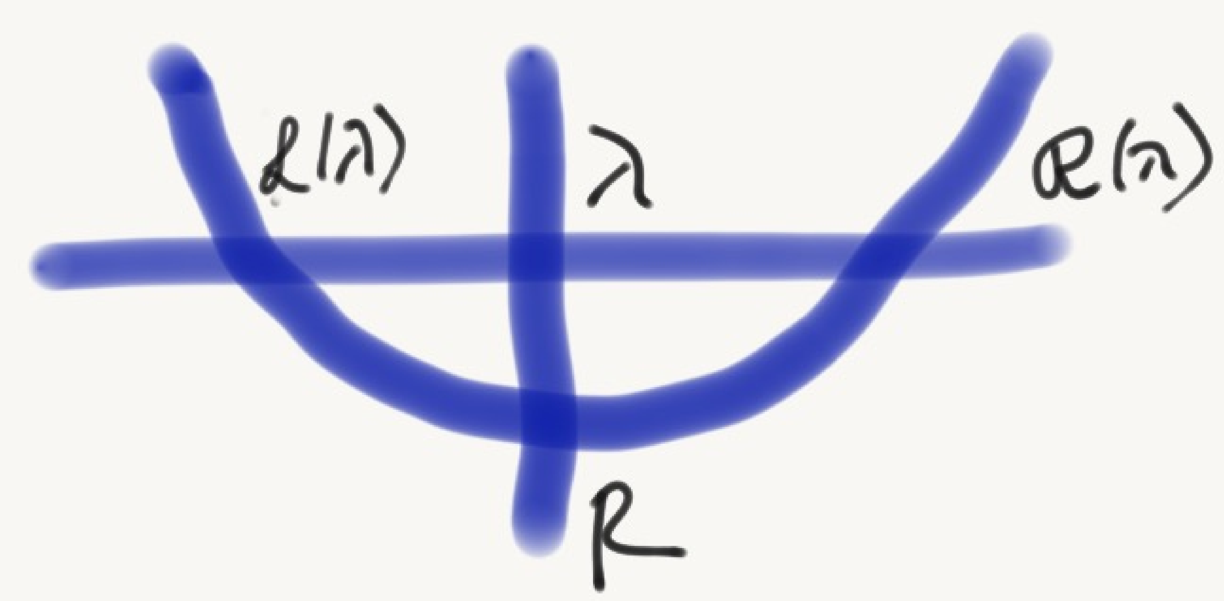


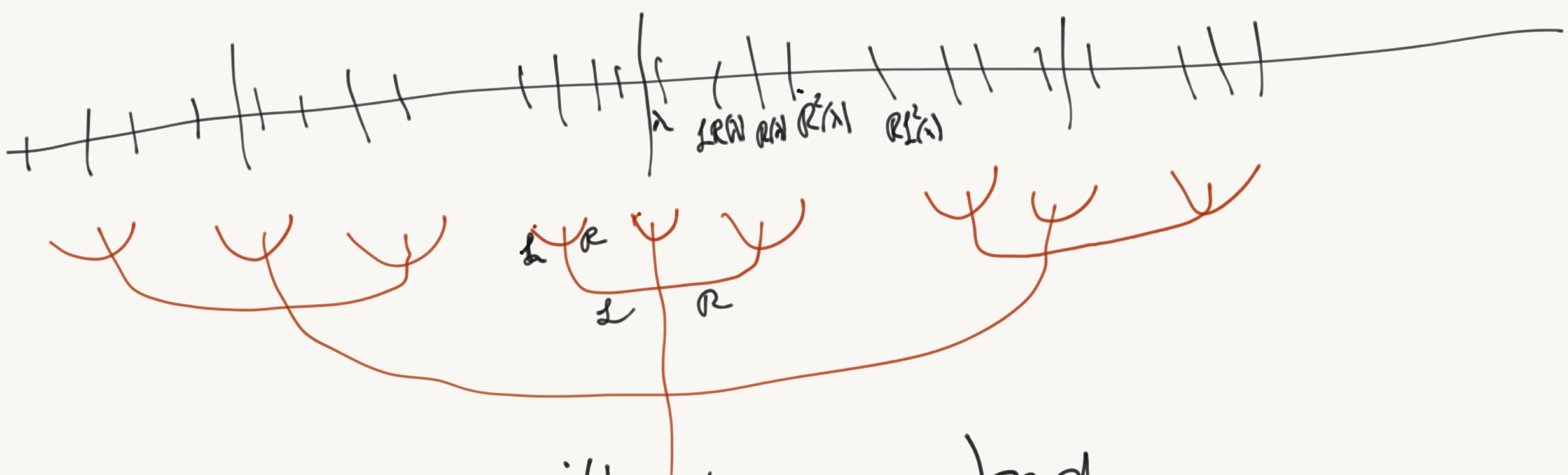
This is the Yang-Baxter Equation.

Solved in the case of "Temperley-Lieb"

- explicit formulae for $\mathcal{L}(\lambda)$ and $\mathcal{R}(\lambda)$.

feature: $\mathcal{L}$ & $\mathcal{R}$ commute

get spatially inhomogeneous T:



- Can't get local hamiltonian - bad
- the $T(\lambda)$'s commute. - good.

Lack of spatial homogeneity means the

$\frac{1}{T(0)} T'(0)$ does not construct a local Hamiltonian.

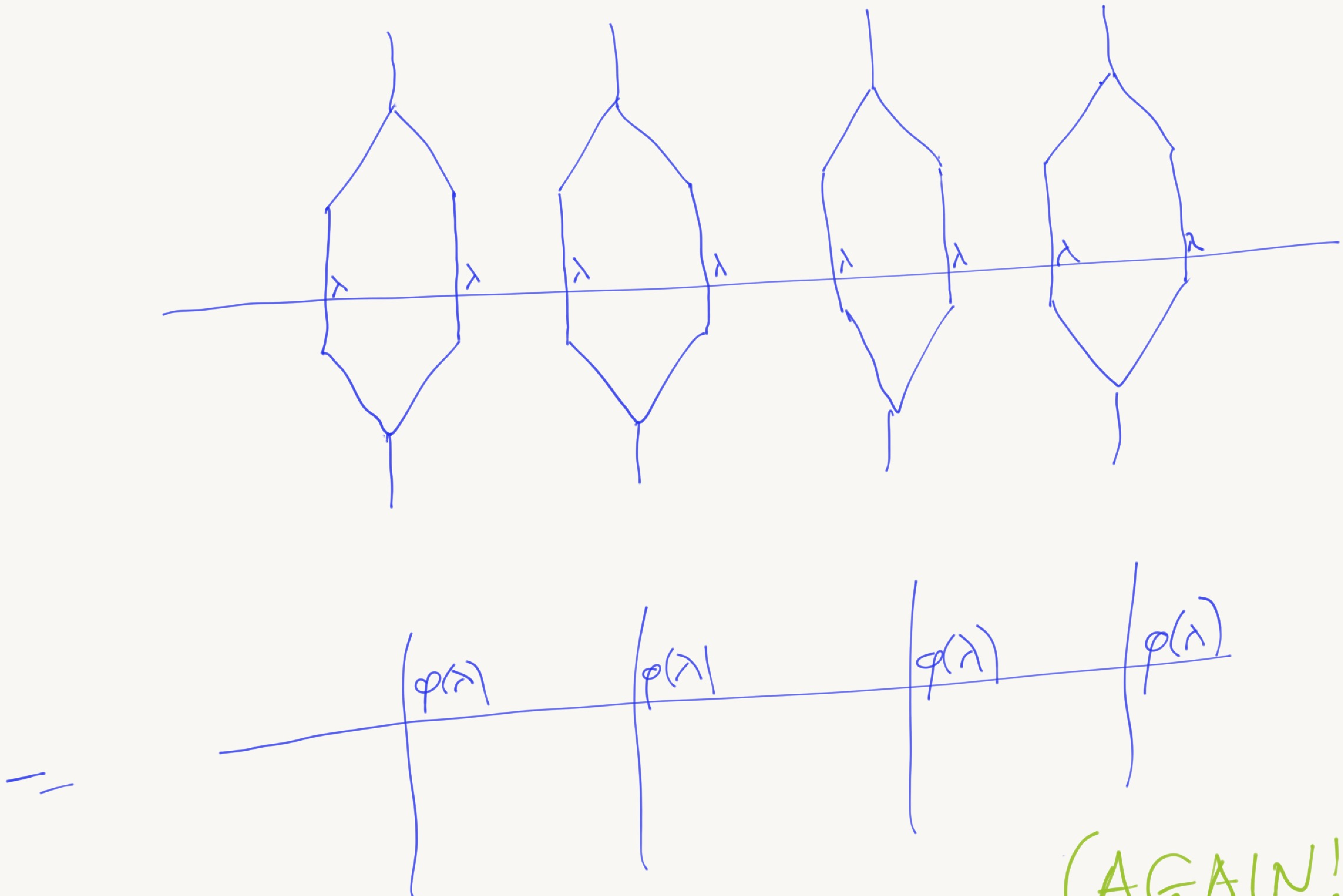
Get a mixture of short and long range interactions.

Another plan. Maybe it's not the Hamiltonian/Transfer matrix that should be scale invariant, but its vacuum expectation value

$$\langle T_n(\lambda)\Omega, \Omega \rangle = \langle T_{n+1}(\lambda)\Omega, \Omega \rangle$$

This will require using local Boltzmann weights for $T_n(\lambda)$ which depend on n:

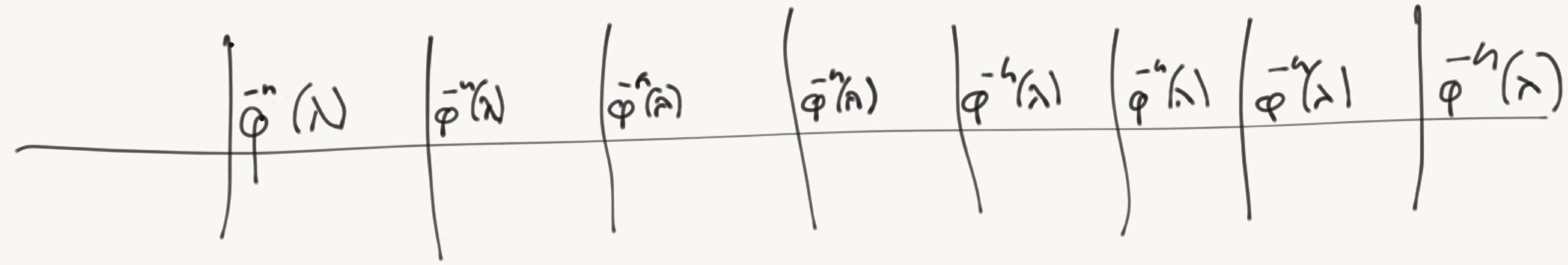
Picture :



(AGAIN!)

Thus if we define

$T_n(\lambda) =$



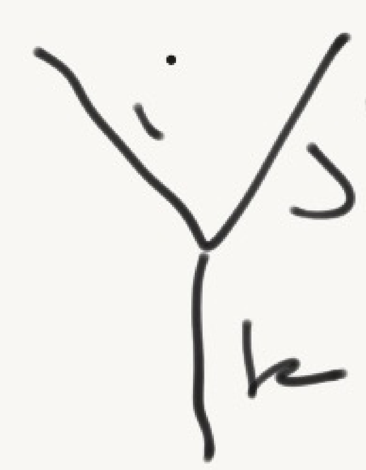
Then the $T_n(\lambda)$ will define an operator on the limit Hilbert space in the sense of quadratic forms : Domain $= \bigcup_{\text{finite dimension}} \mathcal{H}_n$

$$\{\xi, \eta\}_\lambda = \langle T_n(\lambda)(\xi), \eta \rangle \quad \text{for } \xi, \eta \in \mathcal{H}_n$$

where φ^{-n} is any backward iterate of φ.

This is rather speculative so let me end with a couple of the <u>mathematical</u> successes of the Thompson group unitary representations.

1) By considering a very special R matrix for embeddings,

 $= \begin{cases} 0 & i=j,\ j=k \text{ or } i=k \\ 1 & \end{cases}$

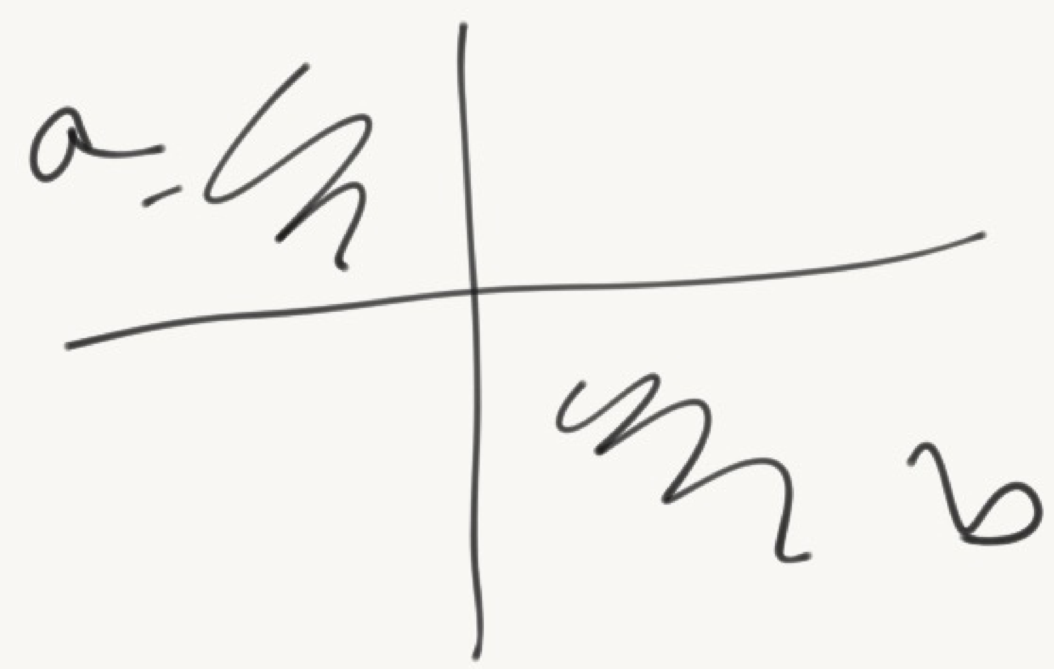
one obtains a unitary rep and $\Omega \in$ Hilbert space such that positivity of <u>all</u> the coefficients $\langle g\Omega, \Omega \rangle$ for $g \in F$ (Thompson) is equivalent to the 4 colour theorem.

(Bacher)

2. By considering other R matrices

$$= \begin{cases} 0 & \text{if } a=b \text{ or } b=c \text{ or } a=b \\ \underline{1} & \text{otherwise} \end{cases}$$

or



$$= \begin{cases} 0 & a=b \\ 1 & \text{otherwise} \end{cases}$$

one obtains reps whose coefficients $\langle g\Omega, \Omega \rangle$ are chromatic polynomials.

In particular for 2 and 3 spin values $\langle g\Omega, \Omega \rangle$ takes only 0 and $\underline{1}$ as values so one obtains subgroups of $\mathbb{F}$ (2 and 3) which are intriguing.

Sapir and Golan have shown that the subgroup for F_2 is actually F_3

and Yunxiang Ren has shown that the subgroup for F_3 is actually F_4. !!!

3) By choosing $R =$  (and setting things up so that scalars are links, one obtains a new way of representing all knots and links as $\langle g\Omega, \Omega \rangle$ for some $g \in F$.

All TQFT invariants in this way and new ones.

F is as good as $\coprod_n B_n$ at producing knots and links.