THE ABC OF COLOR CODES

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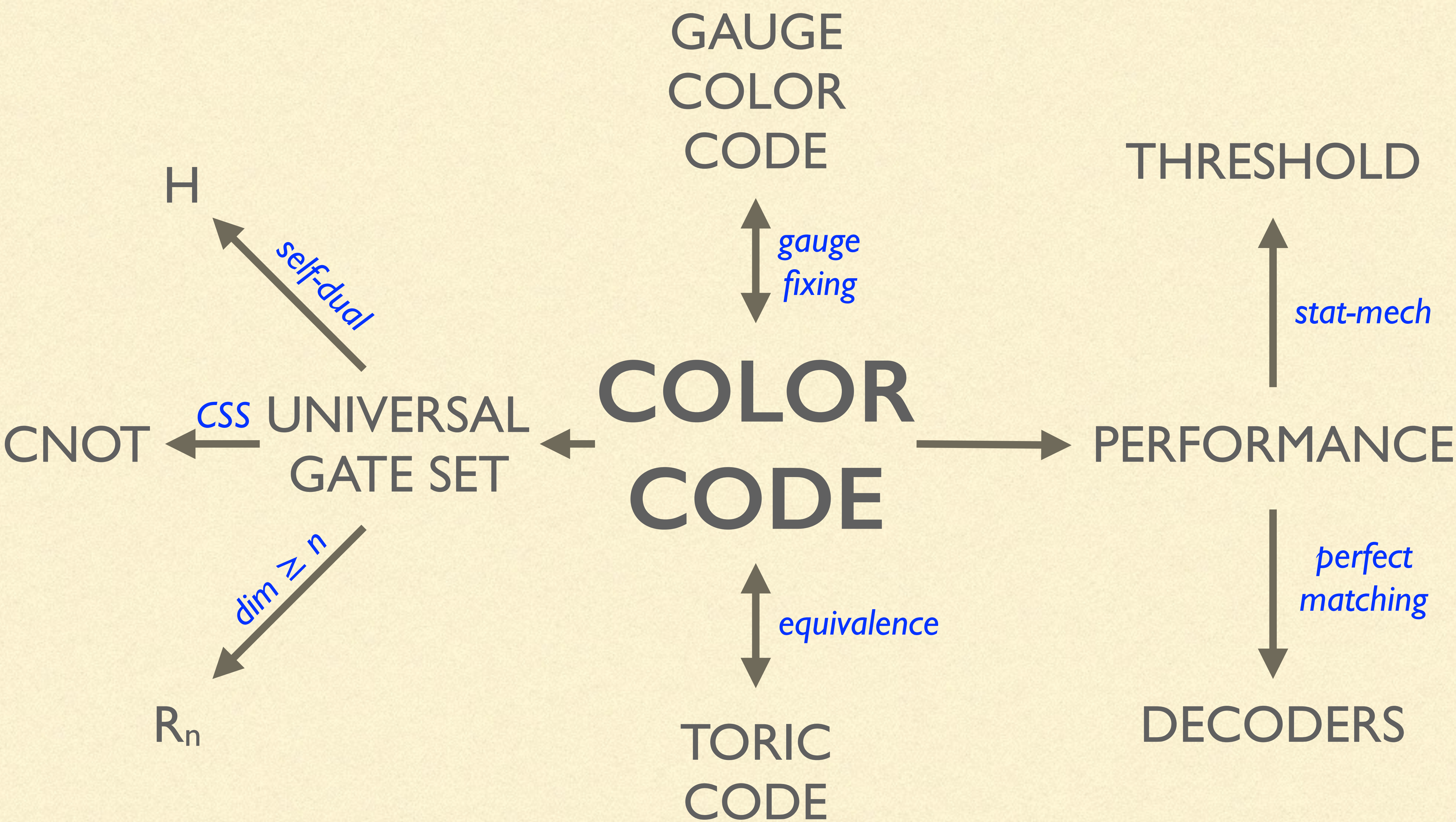
work in progress w/ F. Brandao, K. Svore, N. Delfosse

WHY DO WE CARE ABOUT QUANTUM CODES?

- **Goal**: store information and perform computation.
- Encode to protect from noise.
- Perform computation without corrupting encoded information.
- Color code: many transversal logical gates (Clifford group in 2D).

- Toric code: high threshold, experimentally realizable (2 dim, 4-body terms), scalable.


GAUGE COLOR CODE
gauge fixing
THRESHOLD
stat-mech
H
self-dual
CSS
UNIVERSAL GATE SET
CNOT
COLOR CODE
PERFORMANCE
dim ≥ n
R_n
equivalence
TORIC CODE
perfect matching
DECODERS

STABILIZER CODES

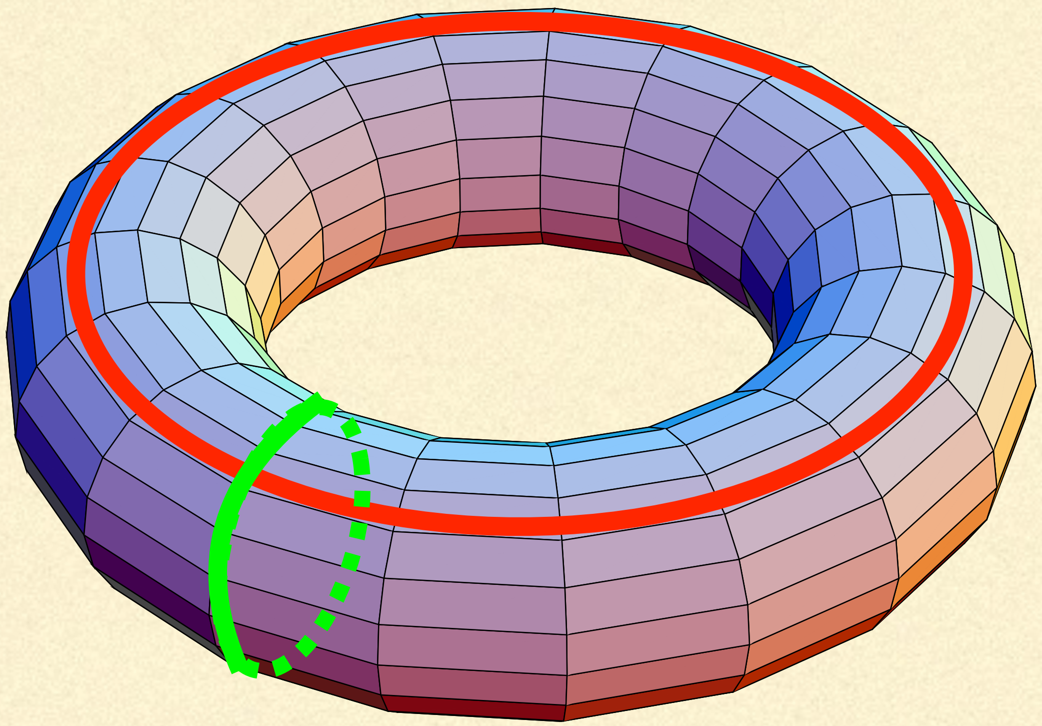
- Stabilizer code - stabilizers are product of Pauli operators and the code space is +1 eigenspace of stabilizers.
- Stabilizer Hamiltonian - commuting terms are products of Pauli operators; ground space corresponds to the code space.

$$H = -\sum_{i=1}^{n} Z_i Z_{i+1}$$

$Z_i Z_{i+1}$

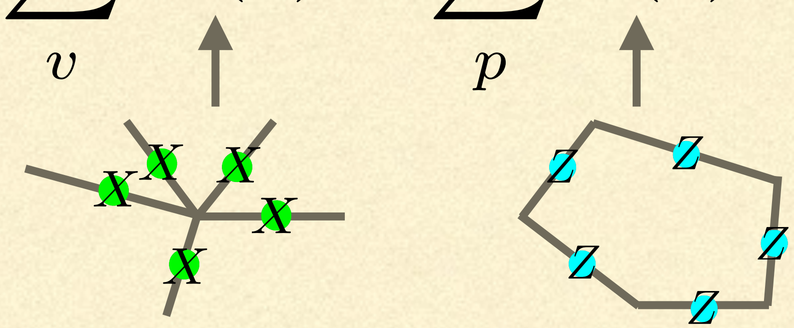
- (Exactly solvable) toy models, e.g. classification of quantum phases.
- Topological quantum codes - encode information in non-local degrees of freedom but have (geometrically) local stabilizer generators.

TORIC CODE IN 2D



- qubits on edges
- X-vertex and Z-plaquette terms

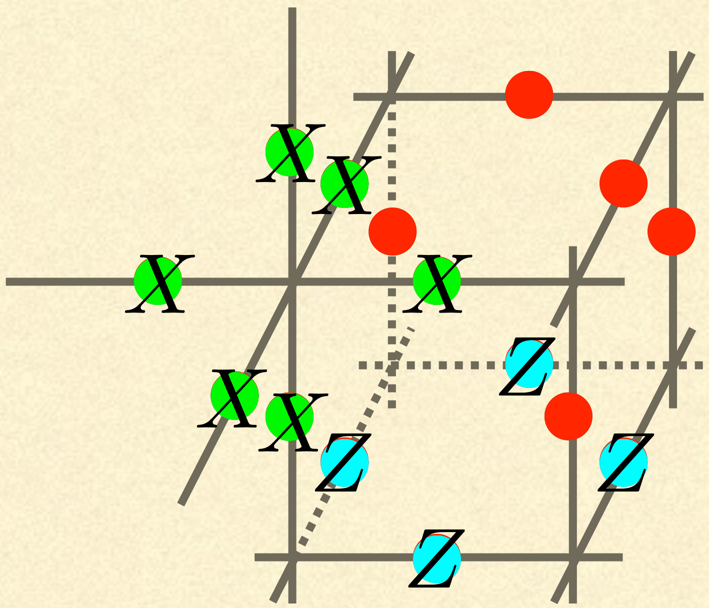
$$H = -\sum_v X(v) - \sum_p Z(p)$$



$$\forall v, p: \ [X(v), Z(p)] = 0$$

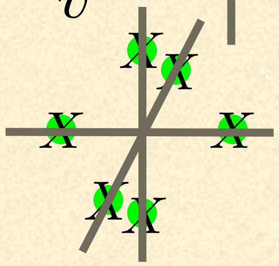
- code space C = ground space of H
- degeneracy(C) = 2^{2g}, where g - genus
- logical operators - non contractible loops of X's or Z's

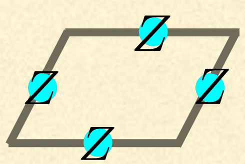
TORIC CODE IN 3D (OR MORE)



- qubits on edges
- X-vertex and Z-plaquette terms

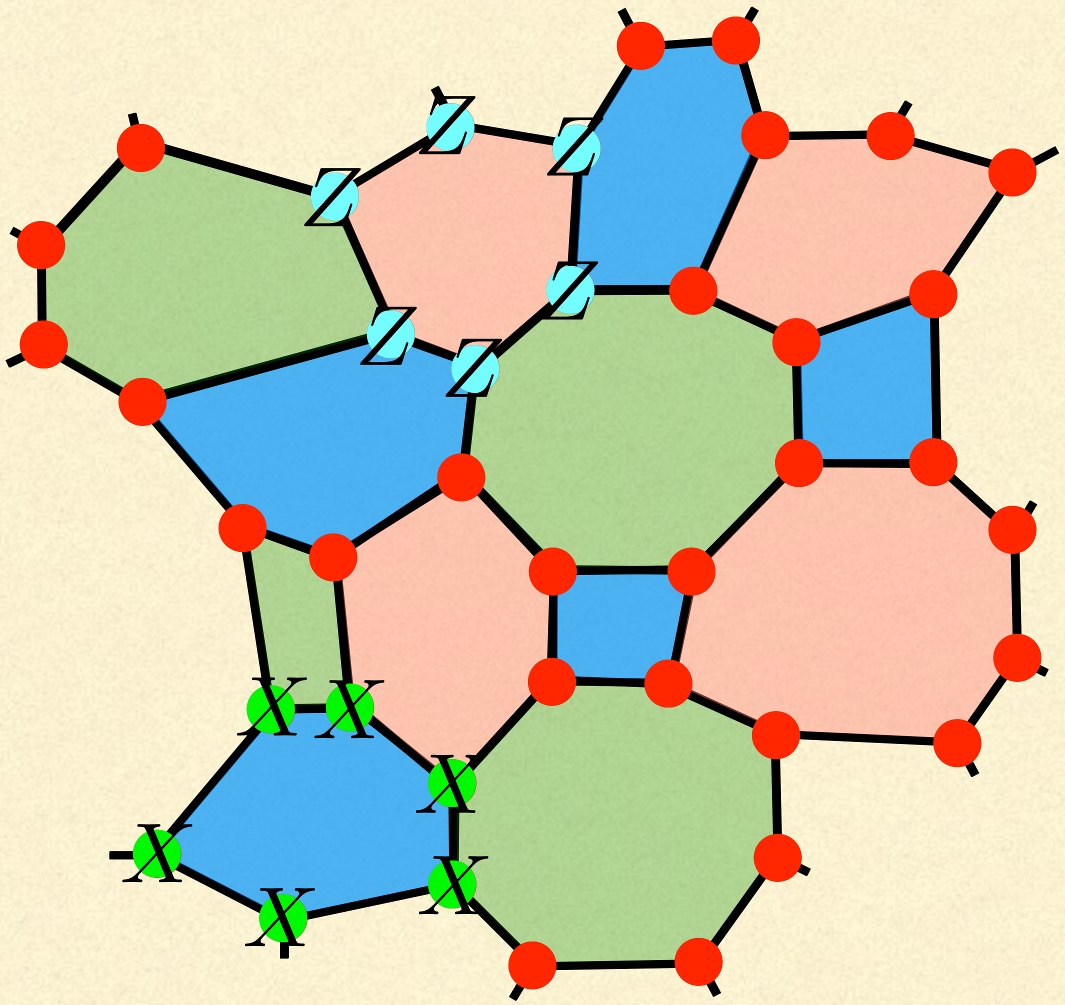
$$H = -\sum_v X(v) - \sum_p Z(p)$$





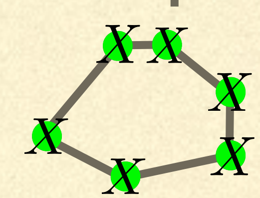
- lattice L in d dim - d-1 ways of defining toric code
- for $1 \leq k < d$, **TC_k(L)**: qubits - k-cells
  X stabilizers - (k-1)-cells
  Z stabilizers - (k+1)-cells

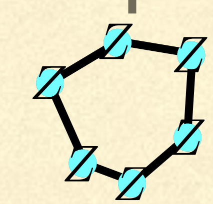
COLOR CODE IN 2D



- 2 dim lattice:
  - 3-valent
  - 3-colorable
- qubits on vertices
- plaquette terms

$$H = -\sum_p X(p) - \sum_p Z(p)$$





$$\forall p, p' : \ [X(p), Z(p')] = 0$$

- code space C = ground space of H
- degeneracy(C) = 2^{4g}, where g - genus

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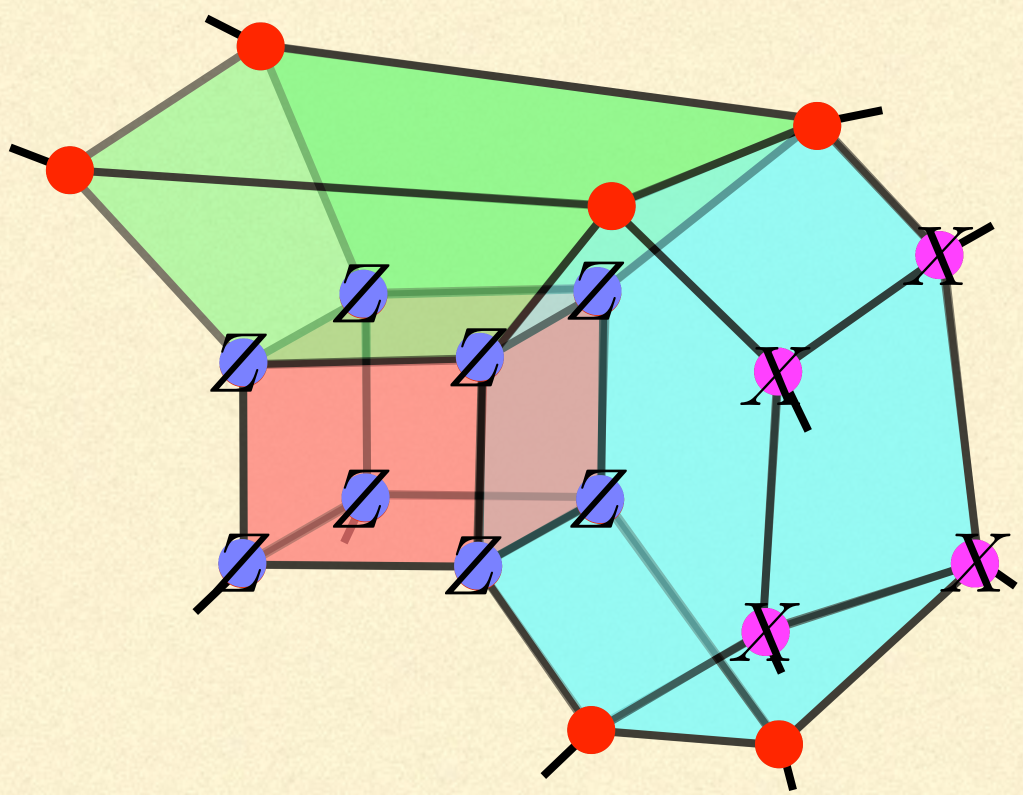
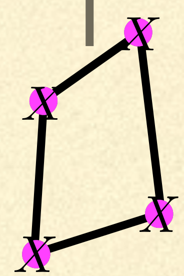
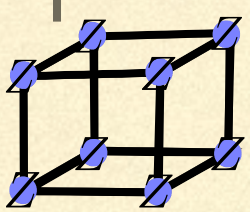
COLOR CODE IN 3D (OR MORE)



- d dim lattice:
  - (d+1)-valent
  - (d+1)-colorable
- qubits on vertices

$$H = -\sum_p X(p) - \sum_c Z(c)$$





- lattice L in d dim - d-1 ways of defining color code
- for $1 < k \leq d$, $\mathbf{CC_k(L)}$: Z stabilizers - k-cells
  X stabilizers - (d+2-k)-cells
  qubits - 0-cells

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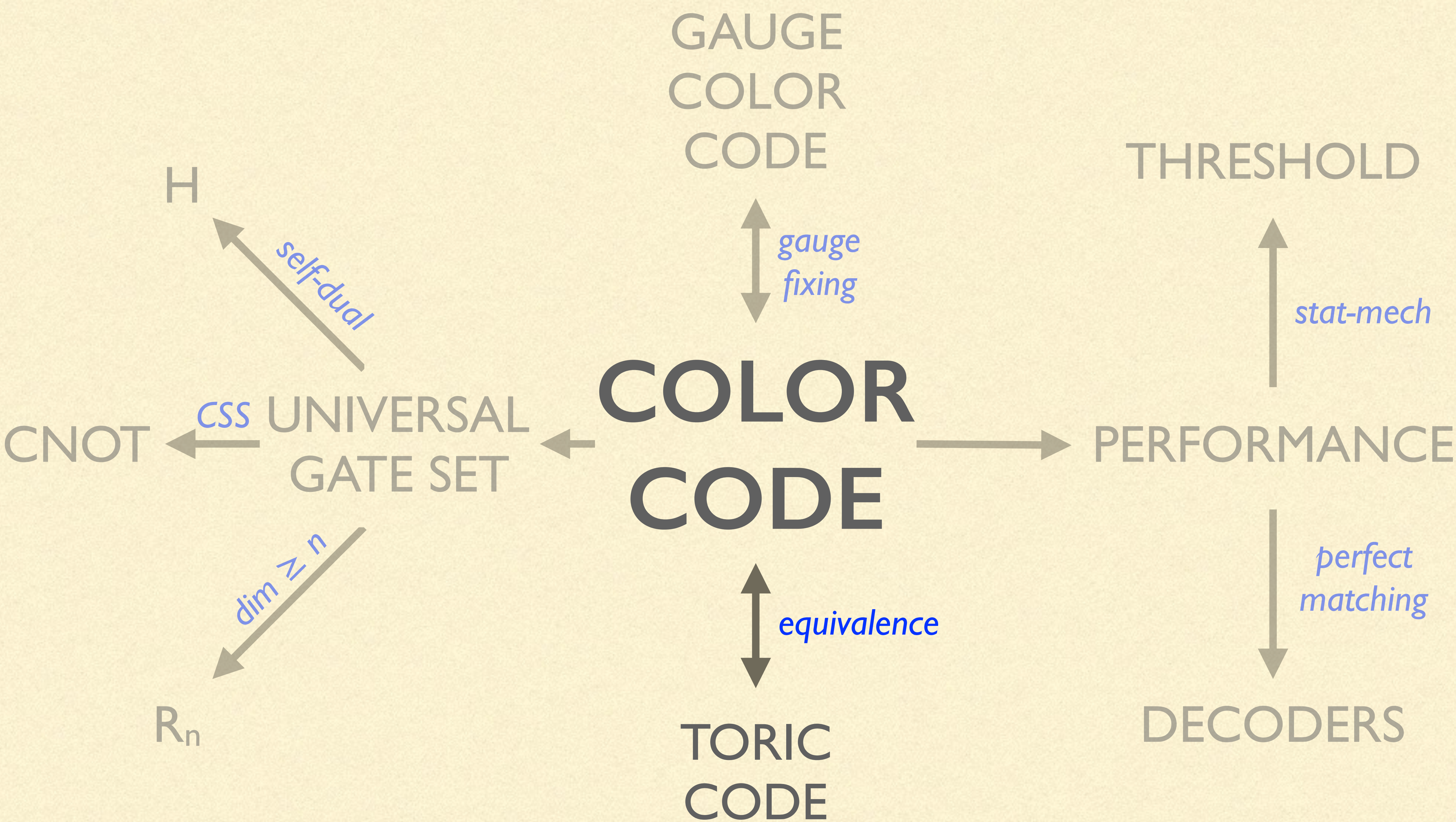
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TORIC CODE

TORIC AND COLOR CODES - HOW ARE THEY RELATED?

- **Yoshida'10, Bombin'11**: 2D stabilizer Hamiltonians w/ local interactions, translation and scale symmetries are equivalent to toric code.

- Equivalence of codes = ground states of stabilizer Hamiltonians in the same quantum phase.

- Two gapped ground states belong to the same phase if and only if they are related by a **local unitary evolution** (**Hastings&Wen'05, Chen et al.'10**).

- **EQUIVALENCE** = local unitaries and adding/removing decoupled ancillas.

- Classification for: **2D**, **stabilizers, translation symmetry**, **no boundaries**.

arXiv:**1503.02065**

EQUIVALENCE IN D DIMENSIONS

Theorem: there exists a unitary $U = \bigotimes_\delta U_\delta$, which is a tensor product of local terms with disjoint support, such that U transforms the color code into $n = \binom{d}{k-1}$ decoupled copies of the toric code.

$$U\,[CC_k(\mathcal{L})]\,U^\dagger = \bigotimes_{i=1}^{n} TC_{k-1}(\mathcal{L}_i)$$

obtained from $\mathcal{L}$ by local deformations

CC$_k$(L): qubits - 0-cells
X stabilizers - (d+2-k)-cells
Z stabilizers - k-cells

TC$_k$(L): qubits - k-cells
X stabilizers - (k-1)-cells
Z stabilizers - (k+1)-cells

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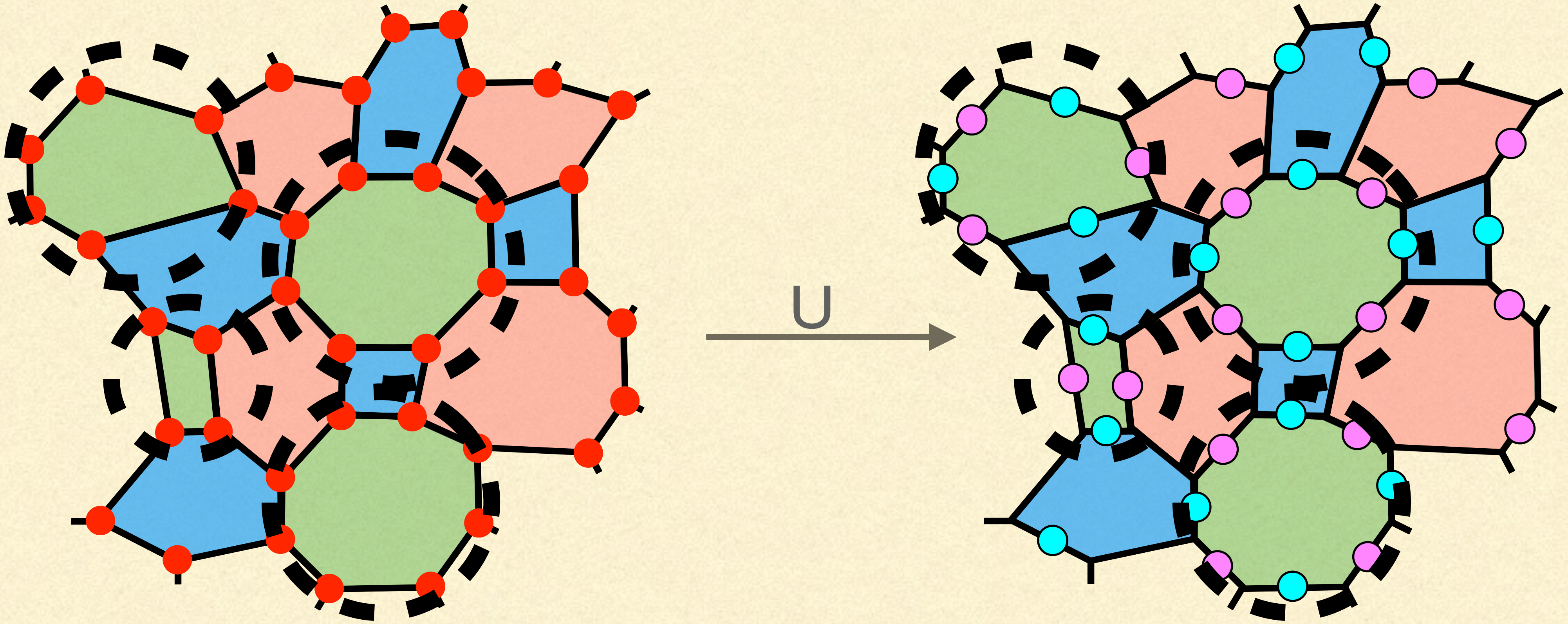
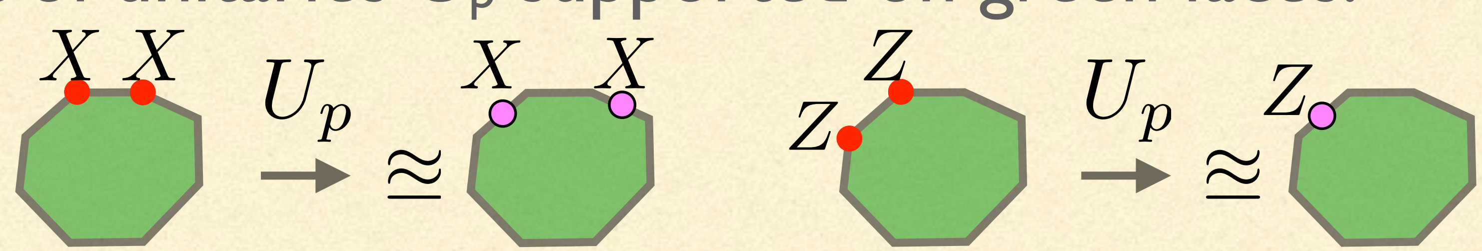
TRANSFORMATION IN 2D

- Every qubit belongs to exactly one green plaquette.



- Unitary U = tensor product of unitaries U_P supported on green faces.
- Desired transformation:



- X/Z-plaquette transforms into an operator either on pink or blue qubits.

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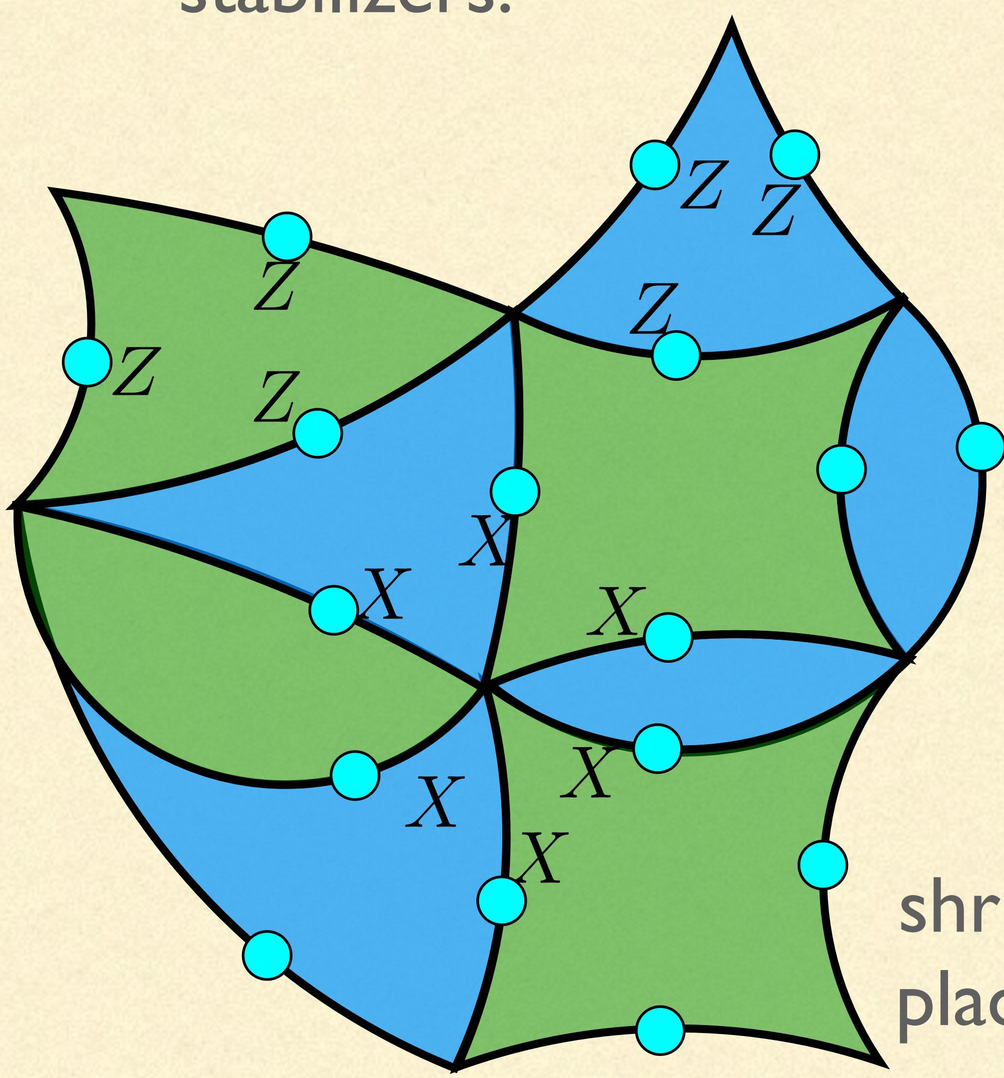
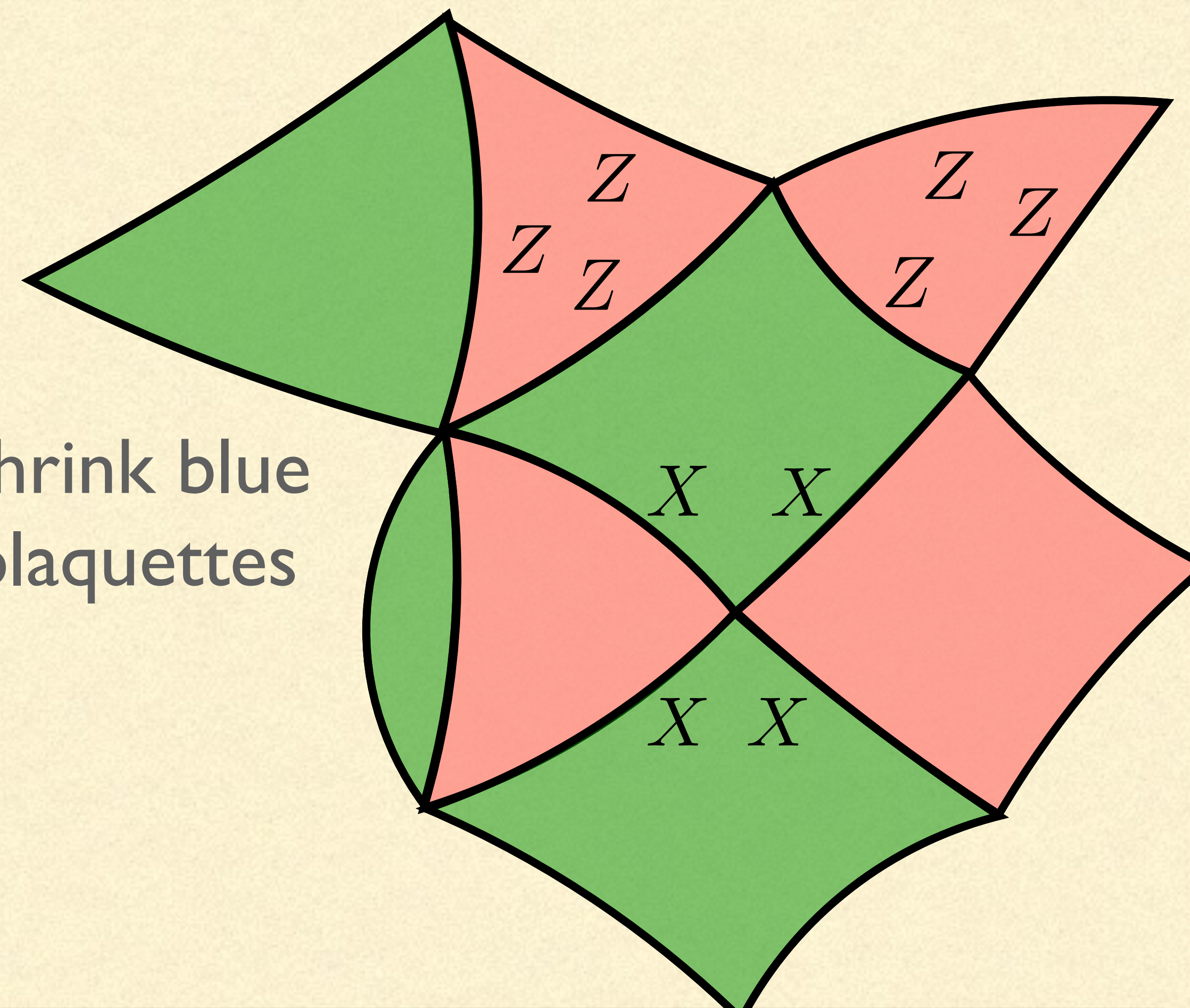
EQUIVALENCE IN 2D

- Stabilizers of color code mapped to new stabilizers - either pink or blue.
- Two decoupled codes = 2 x toric code w/ X-vertex and Z-plaquette stabilizers.



shrink red plaquettes

shrink blue plaquettes



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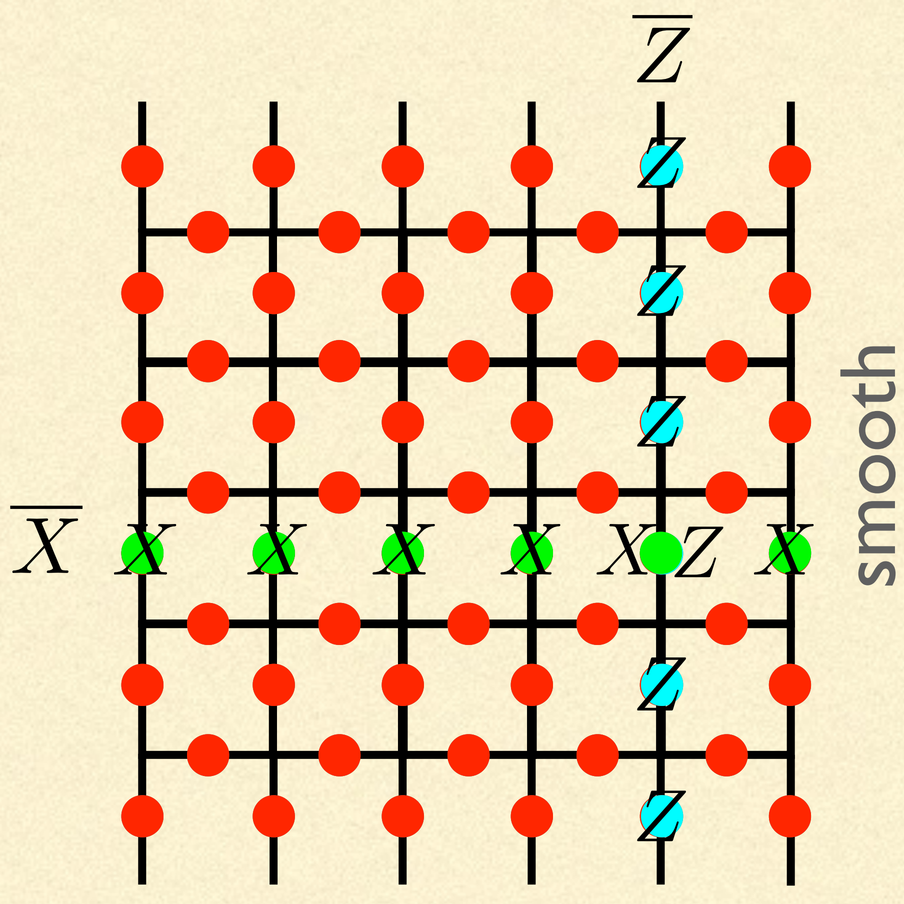
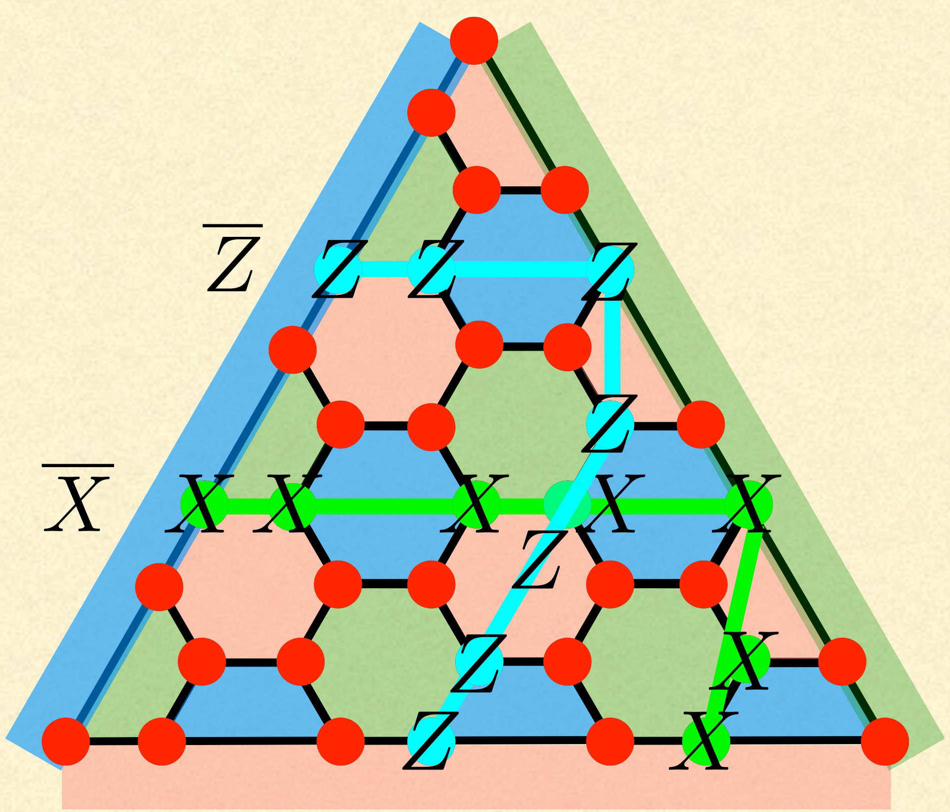
CODES WITH BOUNDARIES

- 2 dim toric code w/ boundaries: rough and smooth
- 2 dim color code w/ boundaries: red, green and blue
- anyons (excitations) can condense on boundaries!





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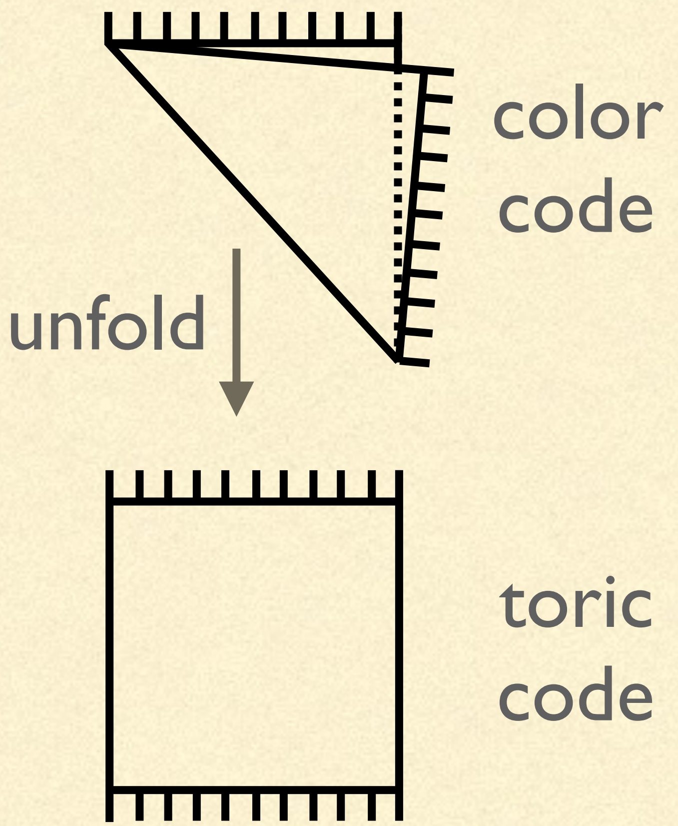
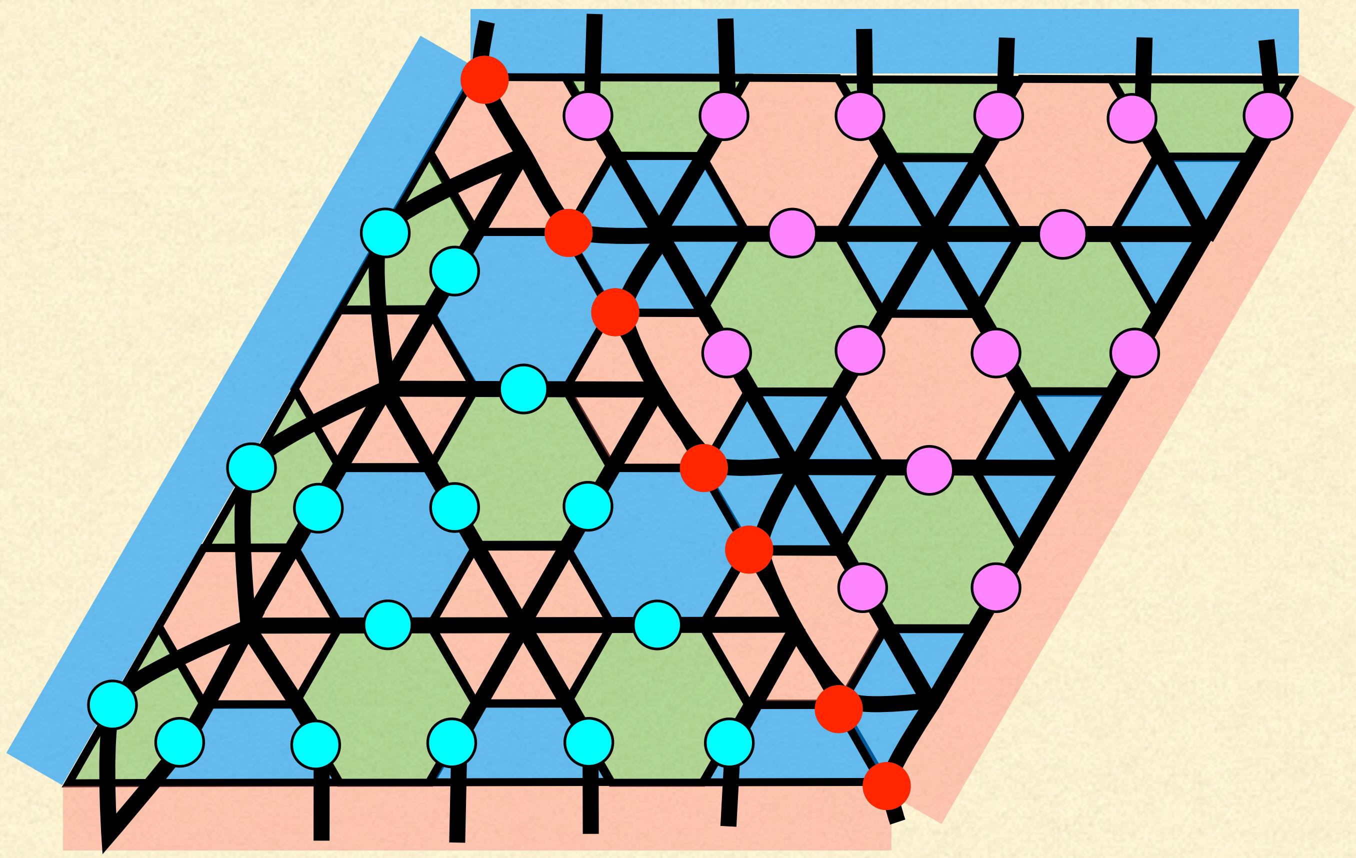
COLOR CODE UNFOLDED





- local unitaries on green plaquettes
- two copies are not decoupled along the green boundary

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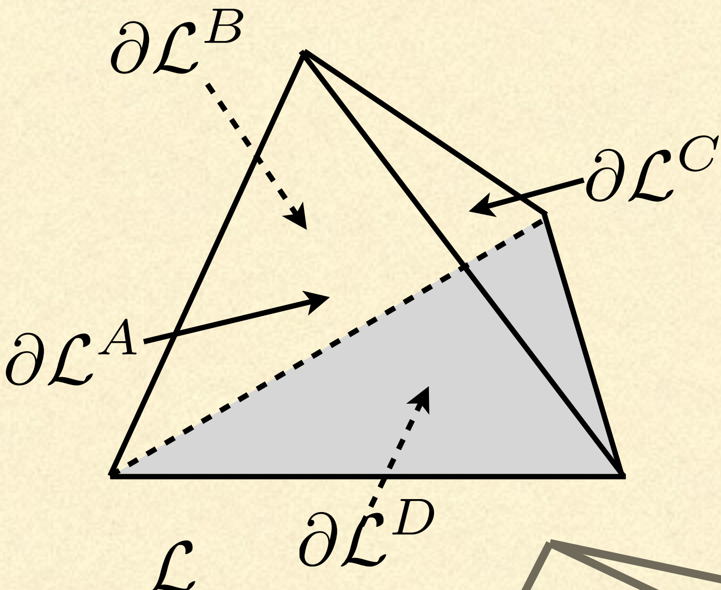
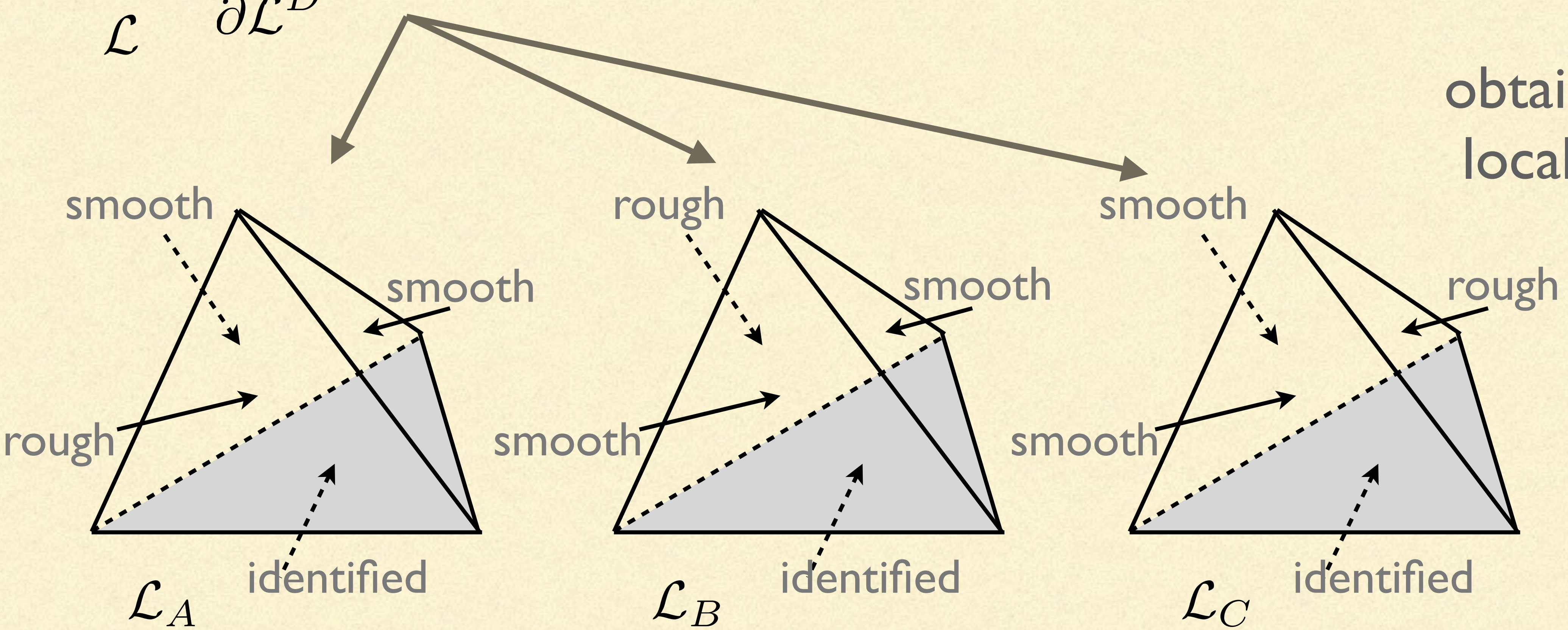
BOUNDARIES IN 3D (OR MORE)



Theorem: color code with point-like excitations in d dim with d+1 boundaries is equivalent to d copies of toric code attached along (d-1)-dim boundary.

$$U\left[CC(\mathcal{L})\right]U^\dagger = TC(\#_{i=1}^{d}\mathcal{L}_i)$$

obtained from $\mathcal{L}$ by local deformations



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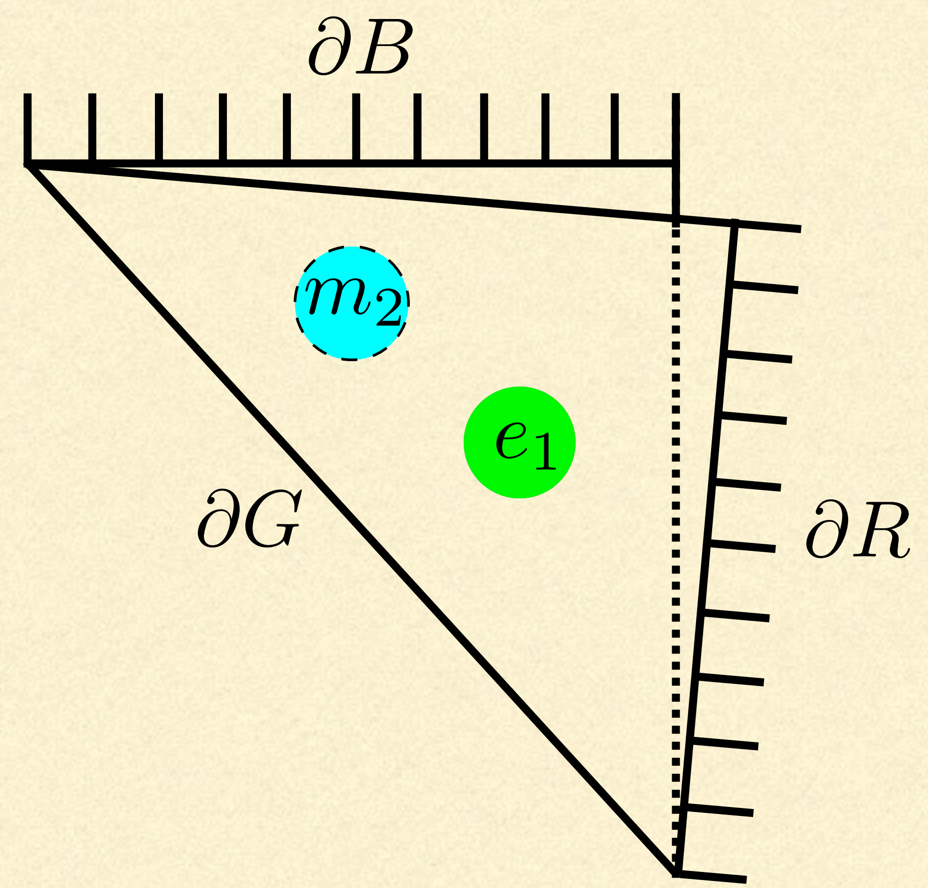
ANYONS AND CONDENSATION

- **Fact**: anyons condensing into a gapped boundary have mutually trivial statistics.
- **Toric code:** e - rough, m - smooth.
- **Folded toric code:**

$\partial R = \{1, e_1, m_2, e_1 m_2\}$ rough/smooth

$\partial B = \{1, e_2, m_1, e_2 m_1\}$ smooth/rough

$\partial G = \{1, e_1 e_2, m_1 m_2, \epsilon_1 \epsilon_2\}$ fold



- Correspondence between anyonic excitations in toric code and color code:

$e_1 \equiv R_X \qquad e_2 \equiv B_X$

$m_2 \equiv R_Z \qquad m_1 \equiv B_Z$

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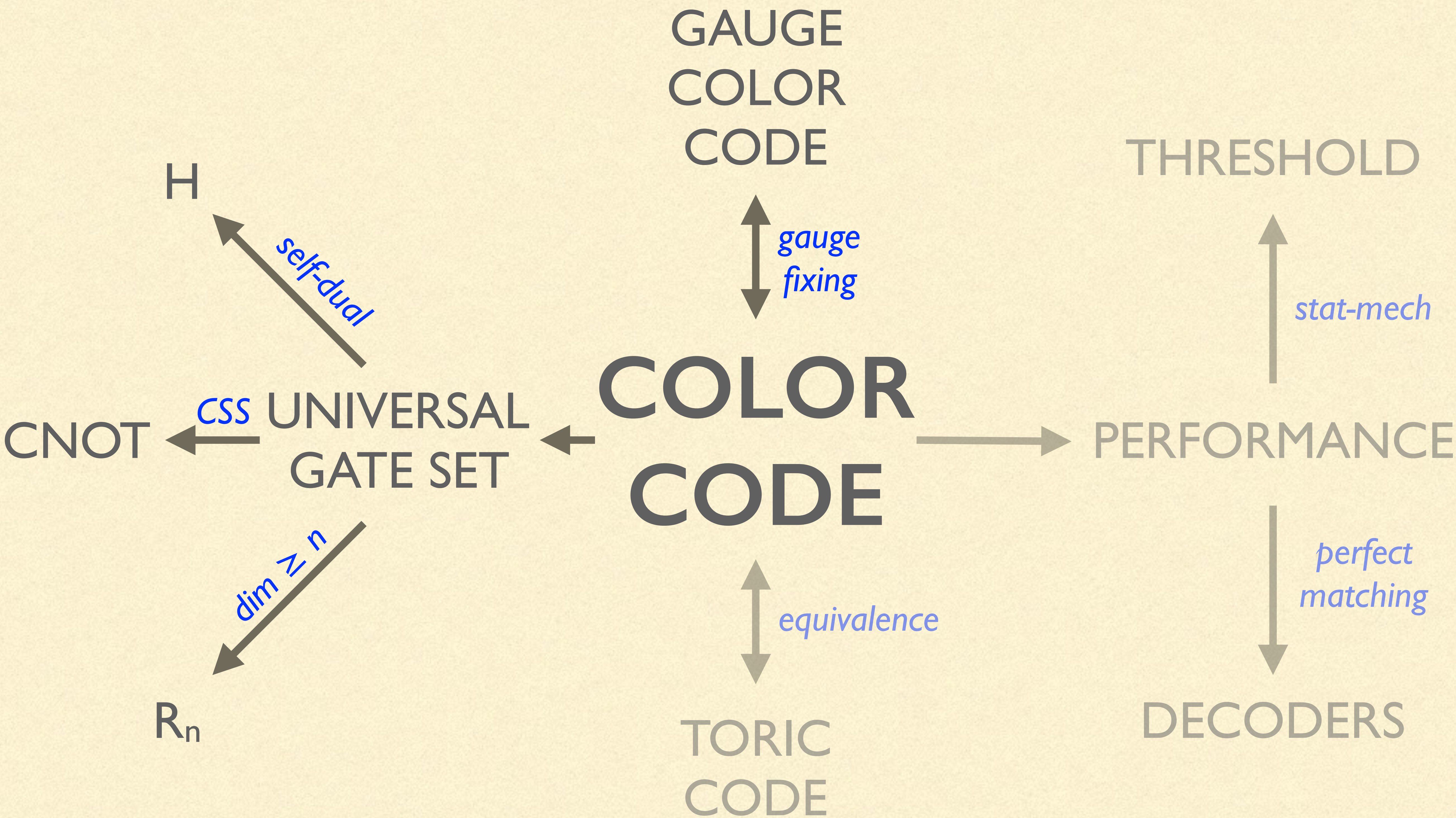

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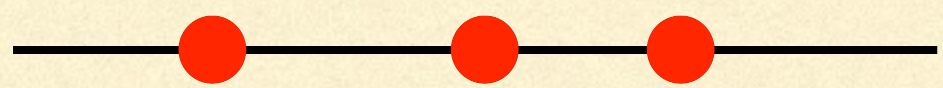
PERFORMING COMPUTATION

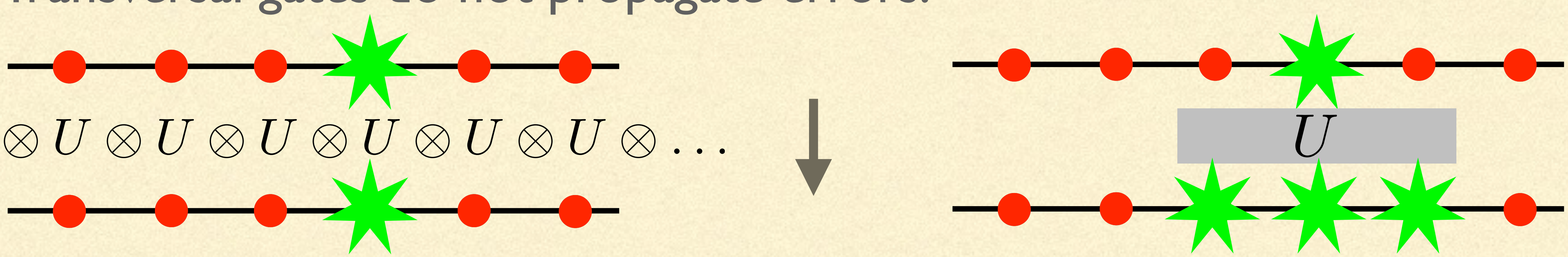
- When performing computation, we do not want to spread errors and corrupt encoded information!

- **Transversal gates** - tensor product of single-qubit unitaries.



$$\overline{U} = \ldots \otimes U_{i-1} \otimes U_i \otimes U_{i+1} \otimes \ldots$$

- Transversal gates do not propagate errors.

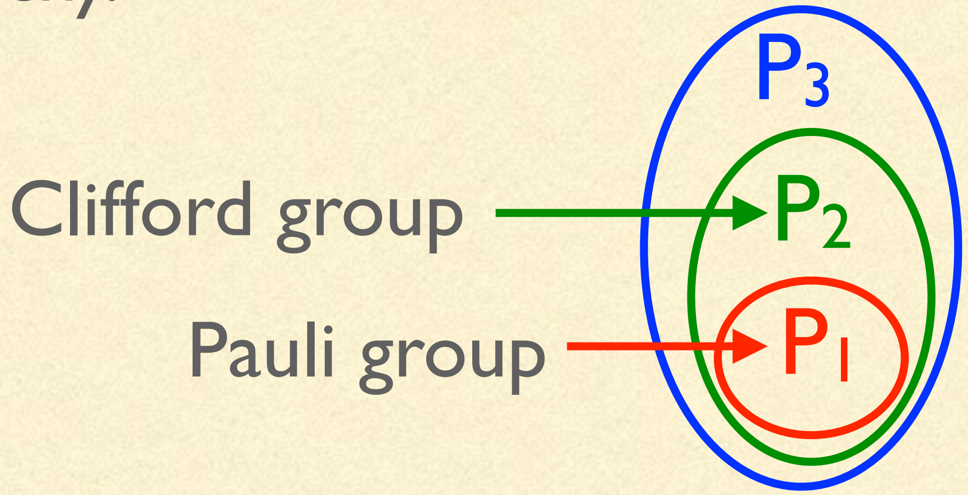


$$\ldots \otimes U \otimes U \otimes U \otimes U \otimes U \otimes U \otimes \ldots$$

NO-GO RESULTS

- **Dream**: transversal universal gate set. Can we have one?
- **Eastin&Knill'09**: for any nontrivial local-error-detecting quantum code, the set of transversal, logical unitary operators is not universal.
- **Bravyi&König'13**: for a topological stabilizer code in d dim, a constant-depth quantum circuit preserving the codespace implements an encoded gate from the d^{th} level of the Clifford hierarchy.
- Clifford hierarchy defined recursively.

$$\mathcal{P}_j = \{\text{unitary } U | UPU^\dagger \in \mathcal{P}_{j-1}\ \forall P \in \mathcal{P}_1\}$$

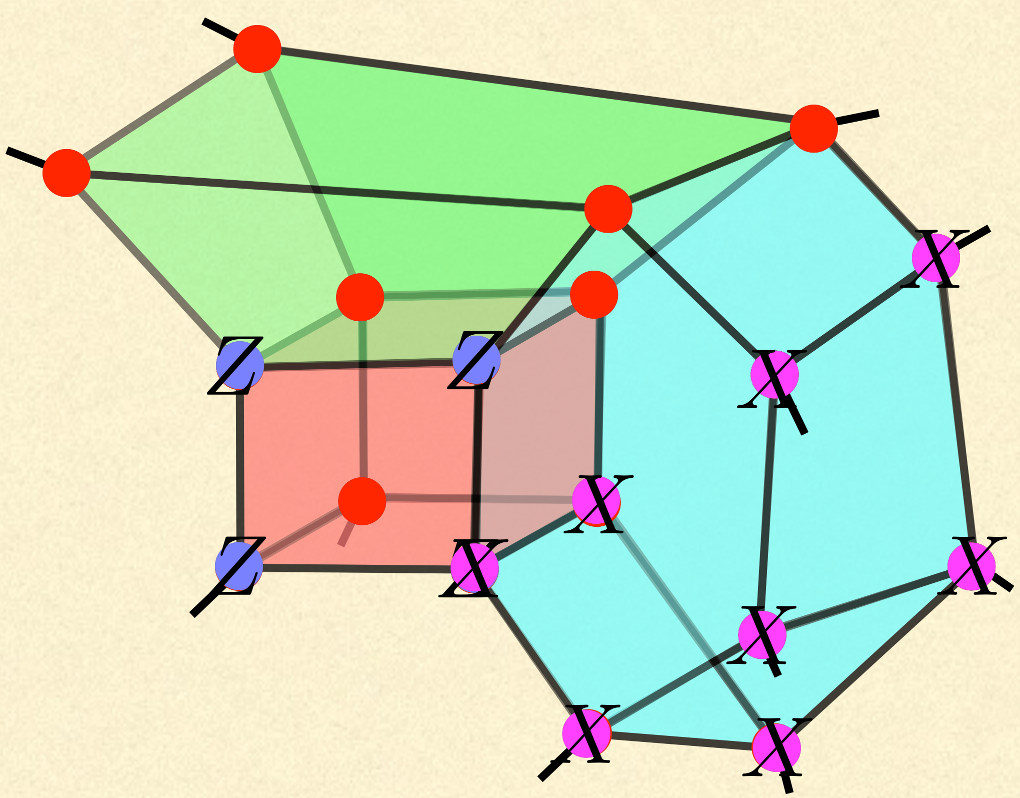


- Gates in d^{th} level of Clifford hierarchy

$$R_d|x\rangle = e^{2\pi i x/2^d}|x\rangle \qquad C^{d-1}Z|x_1,\ldots,x_d\rangle = (-1)^{x_1\ldots x_d}|x_1,\ldots,x_d\rangle$$

SUBSYSTEM COLOR CODES

- Stabilizer code - specified by a stabilizer group S (Abelian).
- Stabilizers w/ big weight - difficult to engineer and measure.
- Subsystem code - specified by a gauge group $\mathcal{G}$ (non-Abelian).



- Subsystem color code **CC_d(x,z)**:
  - X gauge generators: x-cells
  - Z gauge generators: z-cells
- x,z ≥ 2 and x+z ≤ d+2
- Stabilizer group (center):
  - X stabilizers: (d+2-z)-cells
  - Z stabilizers: (d+2-x)-cells

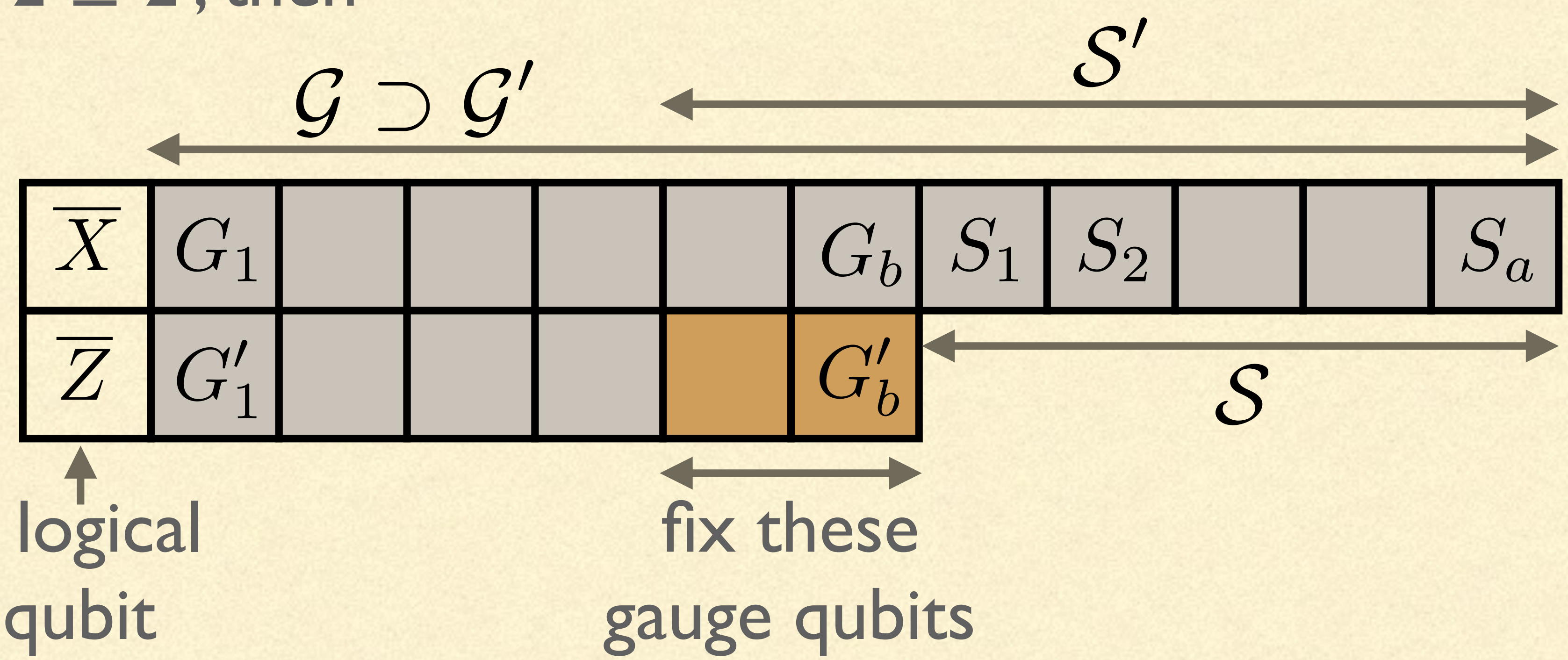
arXiv:**1410.0069**

CODE SWITCHING BETWEEN COLOR CODES

- d-simplex-like lattice w/ boundaries: C = $CC_d(x,z)$ and C' = $CC_d(x',z')$ are two (subsystem) color codes with 1 logical qubit.

  $\mathcal{G} = \langle x\text{−cell}, z\text{−cell}\rangle$ $\qquad\qquad$ $\mathcal{S} = \langle (d+2-z)\text{−cell}, (d+2-x)\text{−cell}\rangle$

- If x ≤ x' and z ≤ z', then
  - $\mathcal{G} \supset \mathcal{G}'$
  - $\mathcal{S} \subset \mathcal{S}'$



- Switching C' ⟶ C - trivial.
- Switching C ⟶ C' - fix certain gauge qubits in +1 state.

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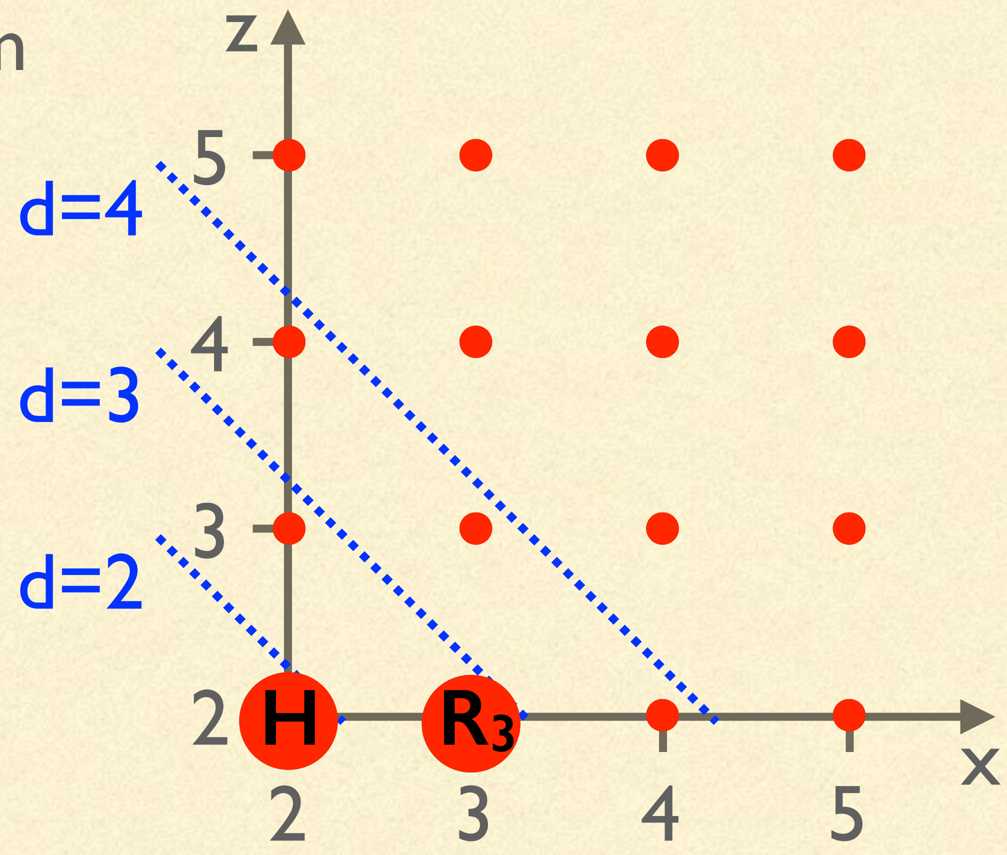
UNIVERSAL GATE SET W/ COLOR CODES IN 3D

- Stabilizer and subsystem color codes in d dim defined by $x,z \geq 2$ such that $x+z \leq d+2$.
- d-simplex-like lattice - 1 logical qubit.
- CNOT - transversal in CSS codes.
- R_3 - transversal in $C' = CC_3(3,2)$.
- H - transversal in $C = CC_3(2,2)$ (self-dual).
- Can switch between $C = CC_d(x=2,z=2)$ and $C' = CC_d(x'=3,z'=2)$ since $x \leq x'$ and $z \leq z'$ - universal gate set in 3D!
- Circumventing no-go result by Eastin and Knill.



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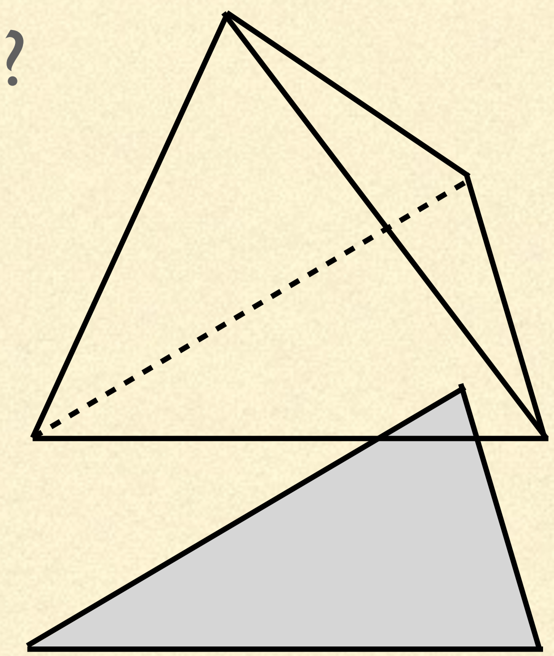
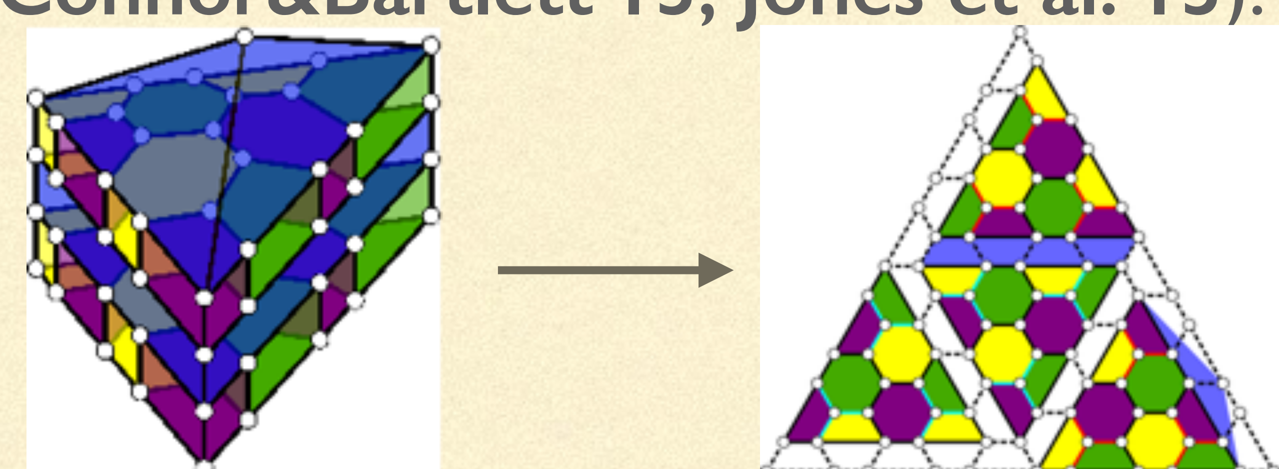
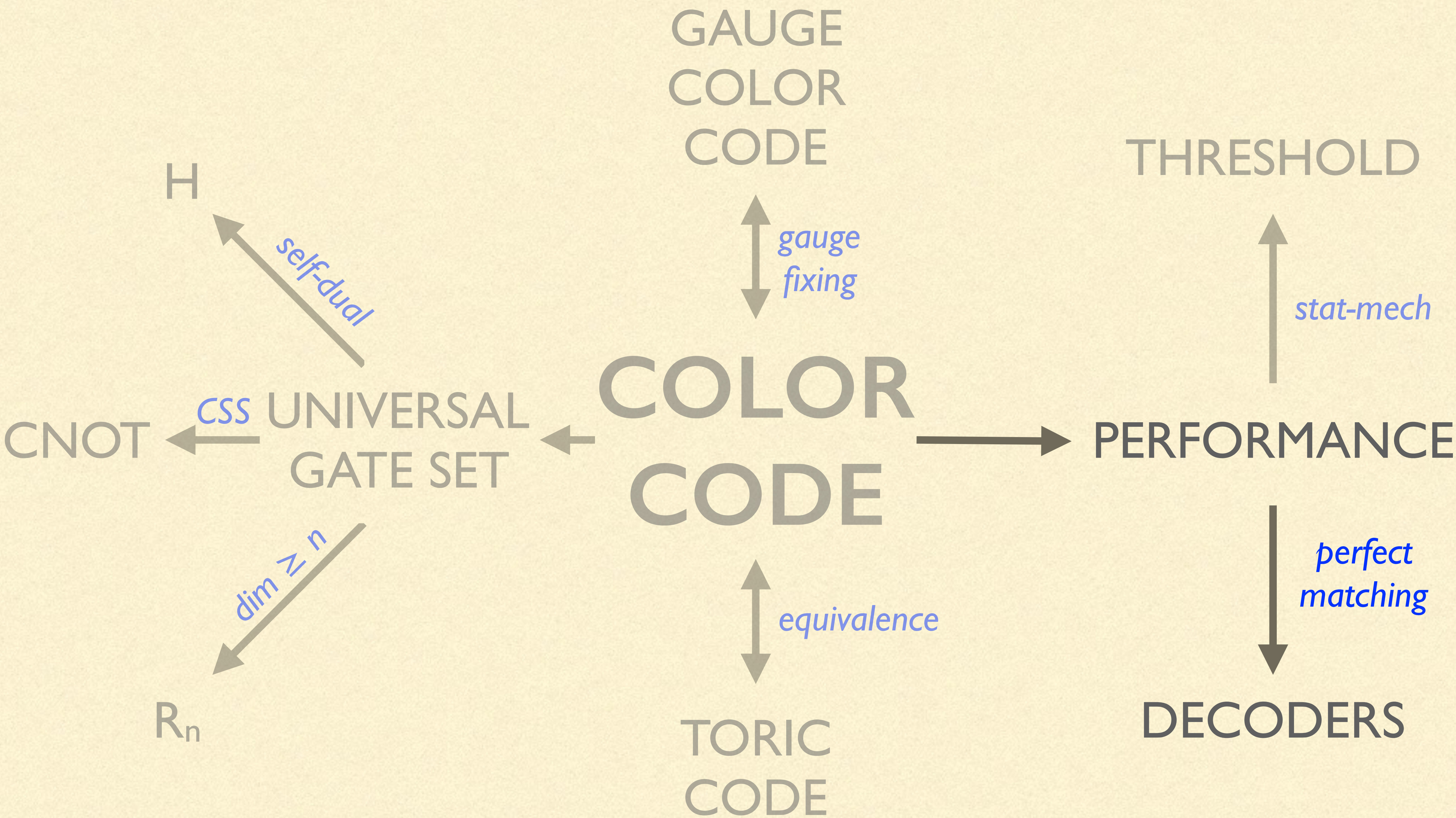
UNIVERSAL GATE SET W/ COLOR CODES, CONTINUED



- 3D architecture challenging! 2D possible to build (soon)?
- 2D color code - transversal logical Clifford group. When logical R_3 needed, switch to 3D color code - dimensional jump w/ gauge fixing (**Bombin'14**).
- Code switching in 3D or dimensional jump - overhead associated with gauge fixing.
- Alternative approach - "flatten" 3D gauge color code to become 2D (**Bravyi&Cross'15, Jochym-O'Connor&Bartlett'15, Jones et al.'15**).
- Issues:
  - non-locality,
  - no threshold,
  - macroscopic stabilizers.

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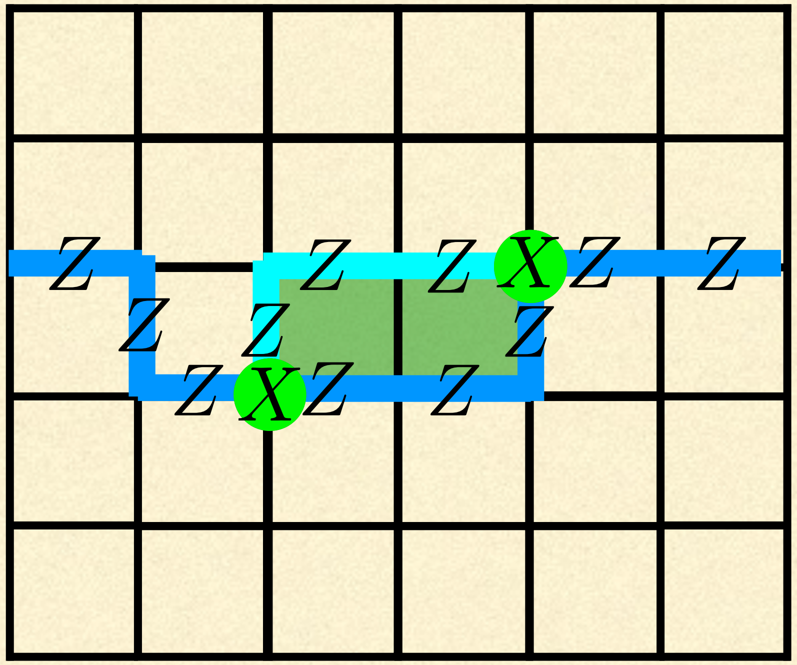
DECODING TORIC CODE



- Toric/color code - conceivable quantum computer architectures. We need to **efficiently** decode them!
- CSS codes - can decode X and Z separately (might reduce threshold!).
- Error happens, e = locations of Z-errors.
- **Input:** s = violated X-star stabilizers.
- **Output:** e' = Z-errors which might have happened.
- **Success:** e and e' differ only by stabilizer b (product of Z-plaquettes).

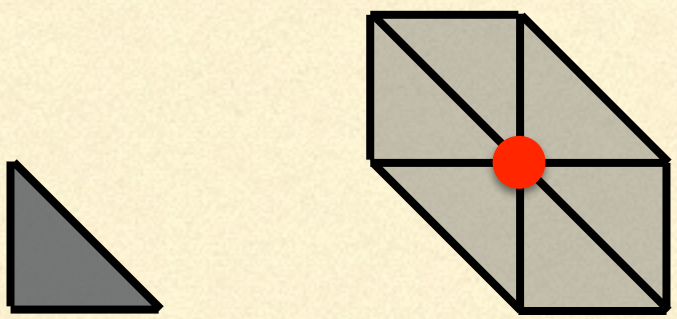
work w/ **N. Delfosse**

DECODING TORIC CODE

- **Best strategy**: if syndrome s observed, pick the most probable equivalence class of errors consistent with s (errors e ~ e' iff differ by stabilizer b).
- Min weight perfect matching - efficient and almost optimum.
- Qudit toric code for d≥3 - min weight hypergraph matching is in general NP-hard!
- Other decoders, e.g. RG (**Duclois-Cianci&Poulin'10, Bravyi&Haah'11, Anwar et al.'13**).

work w/ **N. Delfosse**

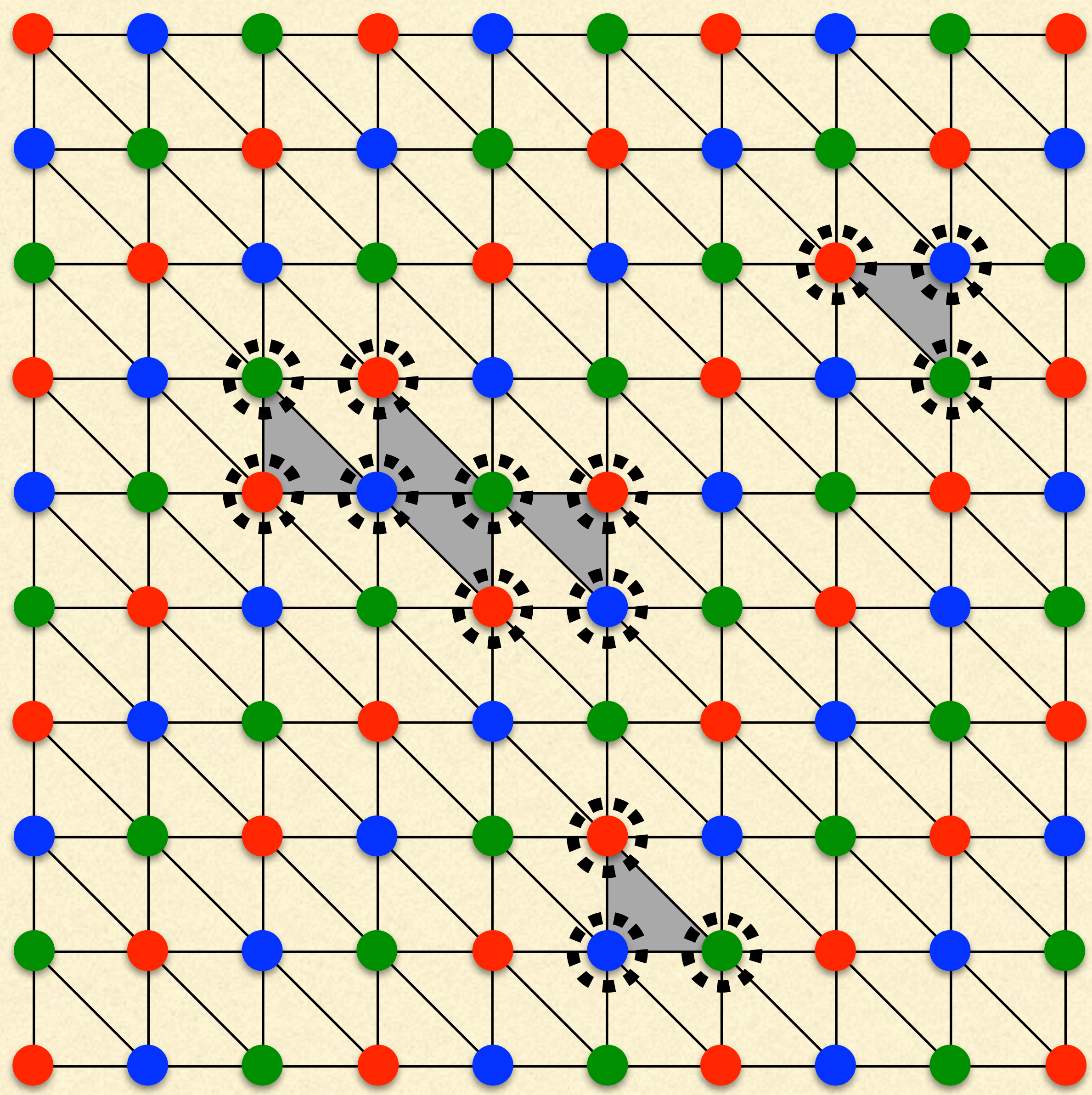
DECODING COLOR CODE

- Can we use toric code decoders?
- Dual lattice L=(V, E, F) - triangulation w/ 3-colorable vertices.
- 

  qubit stabilizer
- **Goal:** error syndrome

$$\begin{tikzcd} 2D & & 0D \arrow[ll] \arrow[dl] \\ & 1D \arrow[ul] & \end{tikzcd}$$

- **Delfosse'13:** project onto 3 sublattices and use toric code decoder!



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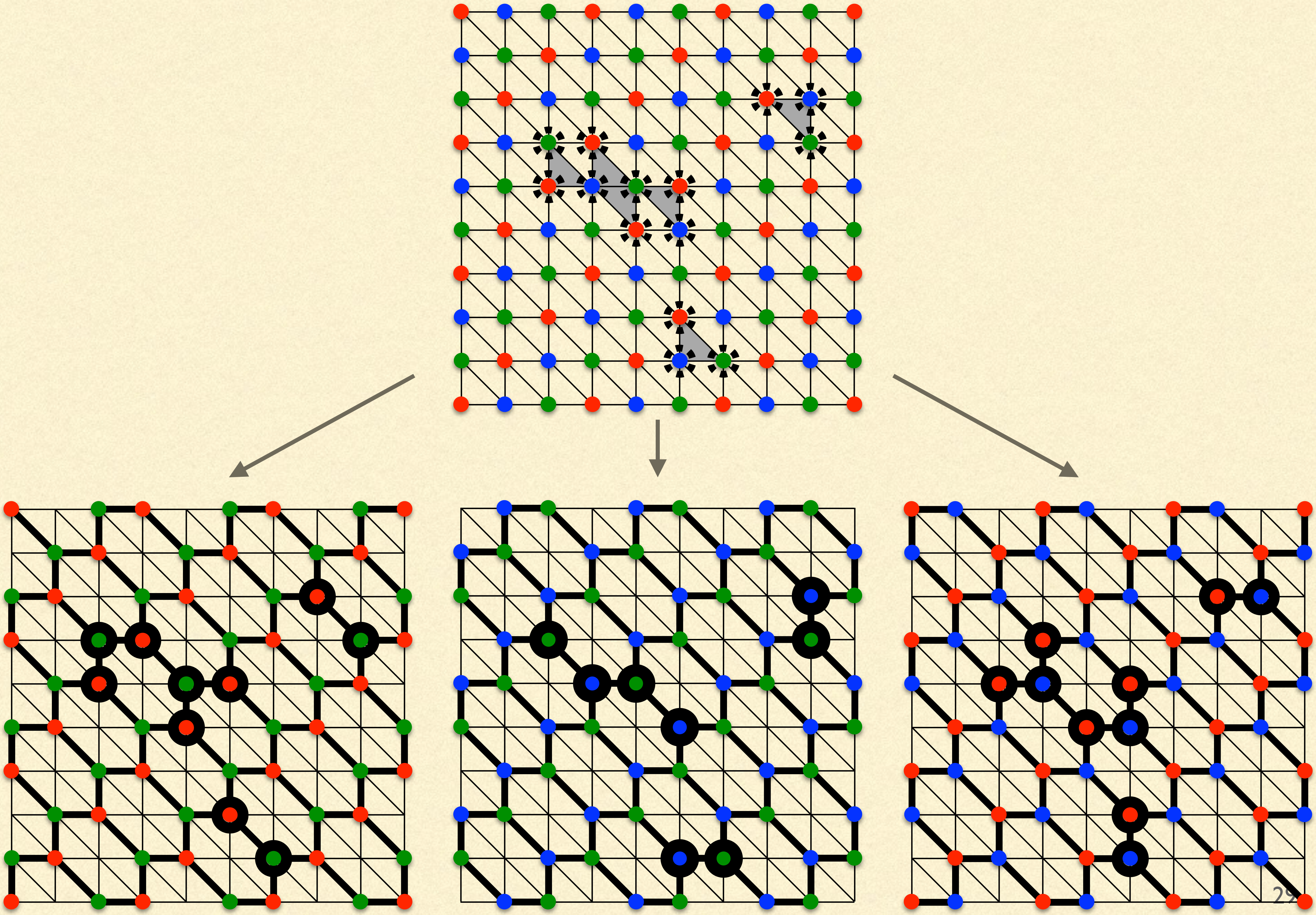

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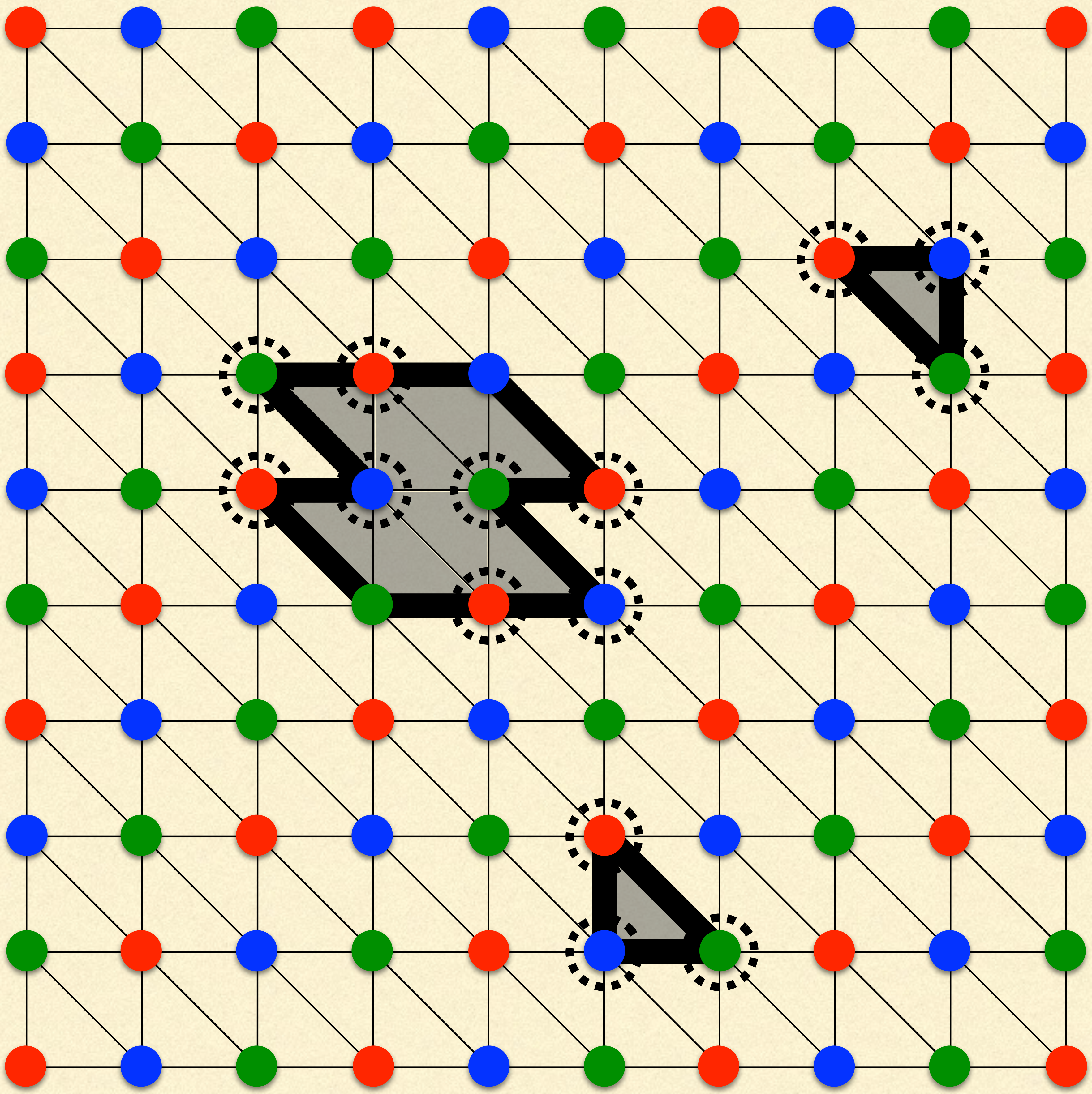
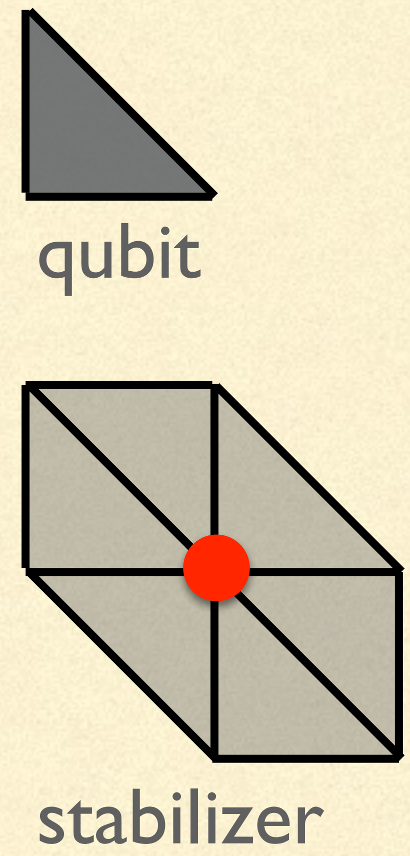
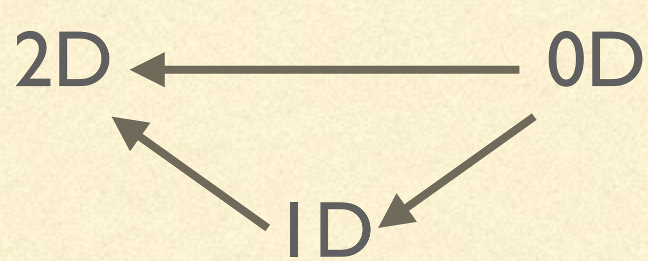






work w/ **N. Delfosse**

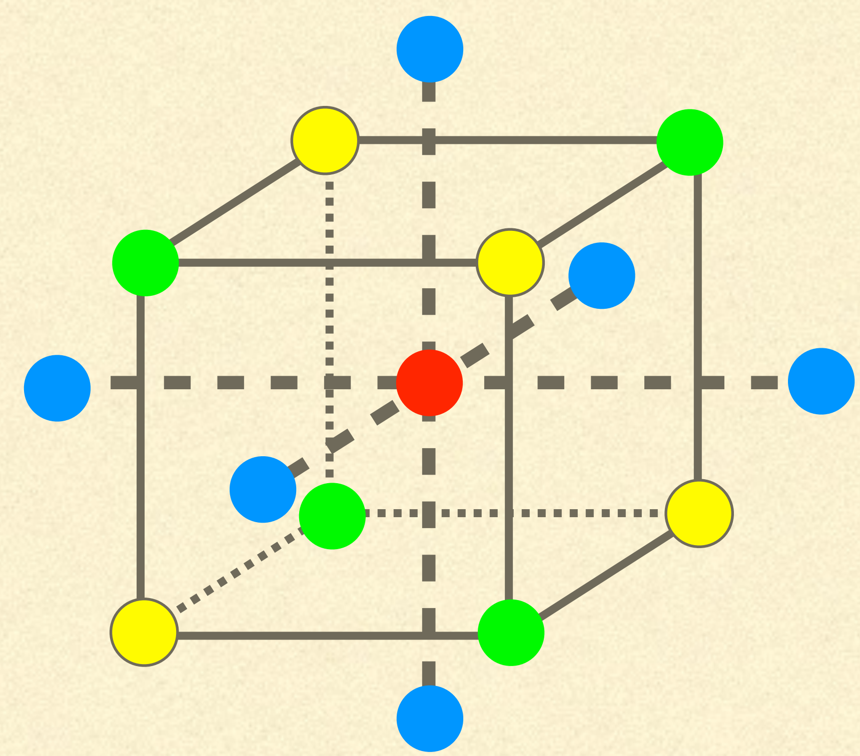
DECODING BY PROJECTION

- Decoding of color code not harder - can use toric code decoders!
- Lower bound on threshold p_{CC} of color code decoder, $p_{CC} \geq f(p_{TC})$.
- Threshold p_{CC} matches threshold of other decoders - RG and unitary mapping (**Bombin et al.'11**).
- Still room for improvement!
- Idea can be generalized to 3D (or more).

work w/ **N. Delfosse**

DECODING IN 3D (OR MORE)

- Dual lattice L=(V, E, F, C) consists of tetrahedra w/ 4-colorable vertices.



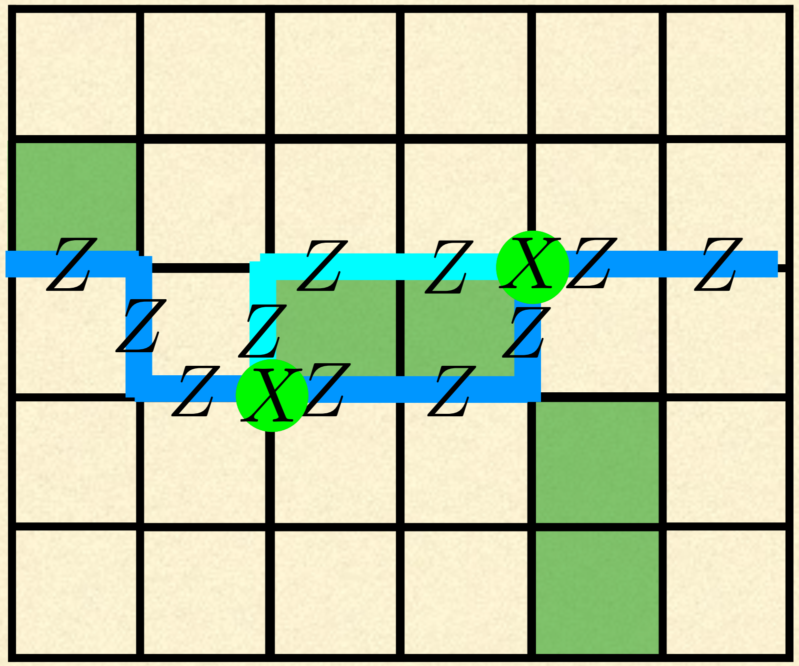
- Qubits = tetrahedra, stabilizers = vertices/edges.
- **Goal:** error $\longleftarrow$ syndrome

  3D $\nwarrow$ 2D $\longleftarrow$ 1D $\swarrow$ 0D; 1D $\swarrow$ 2D

- **Idea:** project onto 6 or 4 sublattices (0D and 1D syndromes, respectively). Use toric code decoder!
- Beyond perfect matching - finding min area k-chain w/ given boundary, i.e. (k-1)-chain. Efficient for (n-1)-chains in n dim (**Sullivan PhD thesis'94**).

work w/ **N. Delfosse**

CHAIN COMPLEX CONSTRUCTION



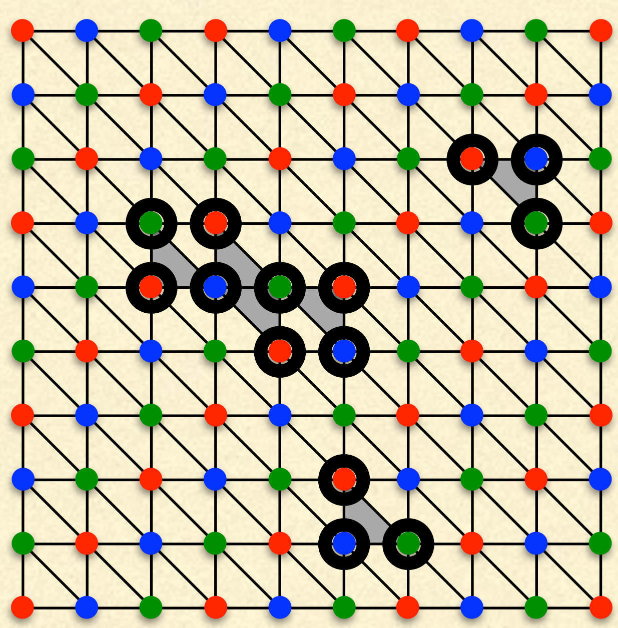
- Lattice (tiling of a manifold) L = (V, E, F).
- $\mathbb{Z}_2$-vector spaces C_i and boundary operators ∂_i

$$C_2 = \bigoplus_{f\in F} \mathbb{Z}_2 f,\ C_1 = \bigoplus_{\epsilon\in E} \mathbb{Z}_2 \epsilon,\ C_0 = \bigoplus_{v\in V} \mathbb{Z}_2 v$$

- Toric code decoding: Chain complex $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$

stabilizer b error e syndrome s

- Correction succeeds iff e+e'=∂_2b, where ∂_1e=∂_1e'=s.
- Color code decoding - chain complex on L' = (V, E', F').
- Projection works - morphism of chain complexes!



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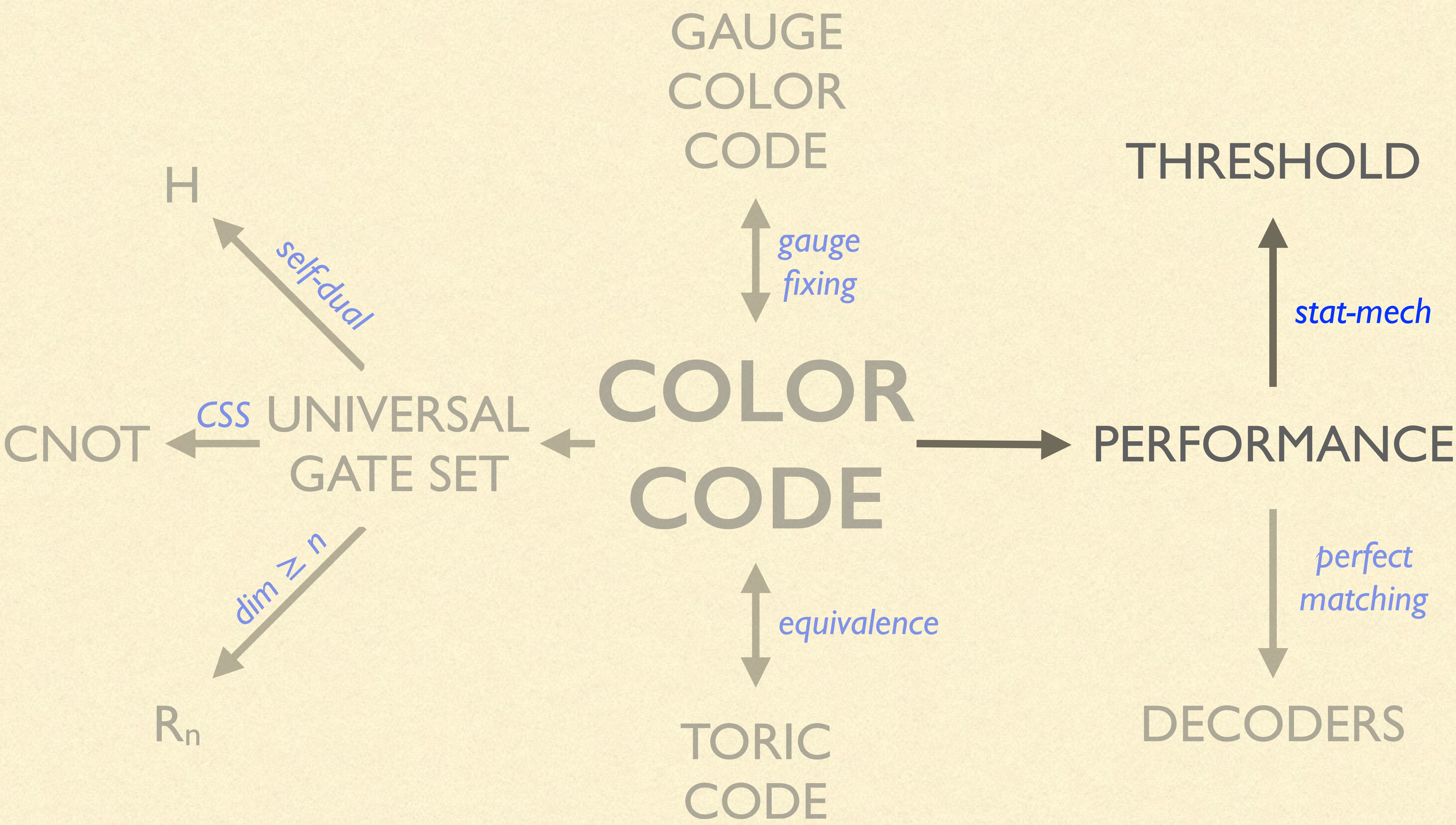
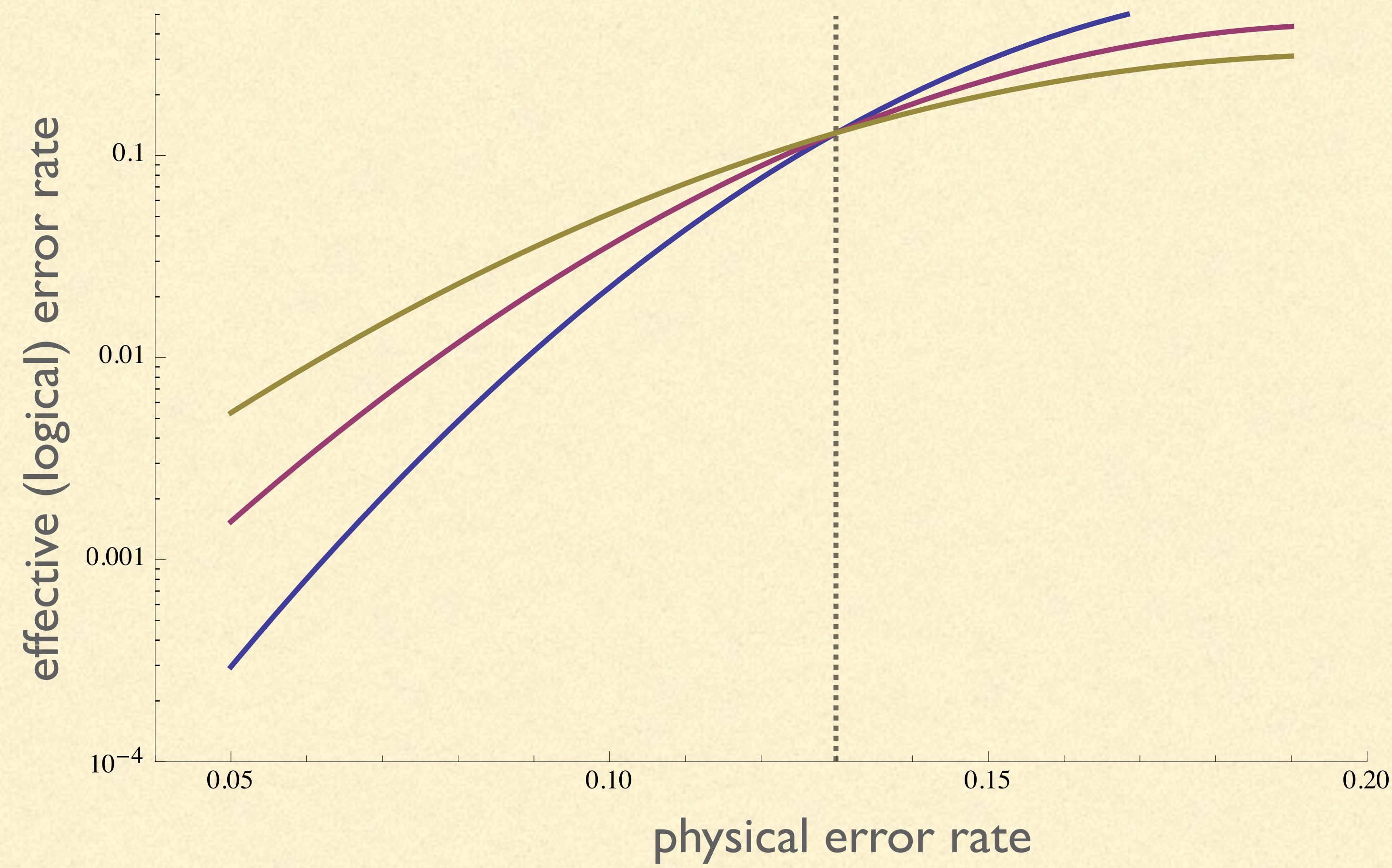


GAUGE COLOR CODE
gauge fixing
H
self-dual
THRESHOLD
stat-mech
CNOT
CSS
UNIVERSAL GATE SET
COLOR CODE
PERFORMANCE
dim ≥ n
perfect matching
equivalence
Rn
TORIC CODE
DECODERS

THRESHOLDS

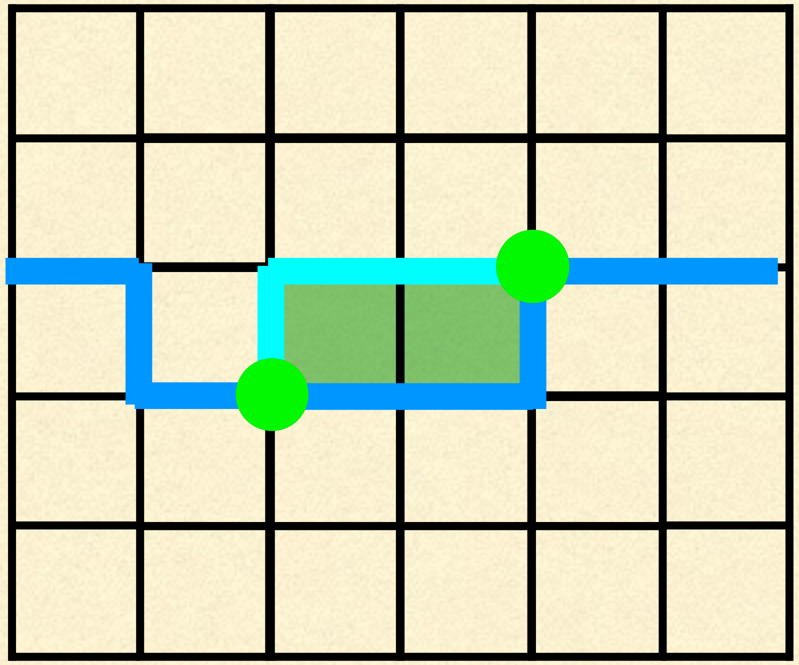


- Threshold p_{th} - maximum error rate the code can tolerate.
- If $p < p_{th}$ - can decode perfectly (for increasing system sizes).
- Relevant for: guiding the experiment (surface code), benchmarking decoders (**Brown et al.'15**), resource analysis (magic state distillation).
- Often, evaluated for specific models (depends on the lattice, noise model, decoder, etc.).
- Stat-mech models can give insight into "decoder-independent" thresholds.

DECODING OF TORIC CODE REVISITED

- Assumption - errors happen independently with probability p, i.e. $\mathrm{pr}(e) = (1-p)^{N-|e|}p^{|e|}$
- Success iff e and e' differ only by some stabilizer b (product of Z-plaquettes) - same equivalence class.

$$\mathrm{pr}(\bar{e}) = \sum_{b \in C_2} \mathrm{pr}(e + \partial_2 b) = (1-p)^N \sum_{b \in C_2} (\frac{p}{1-p})^{|e+\partial_2 b|}$$

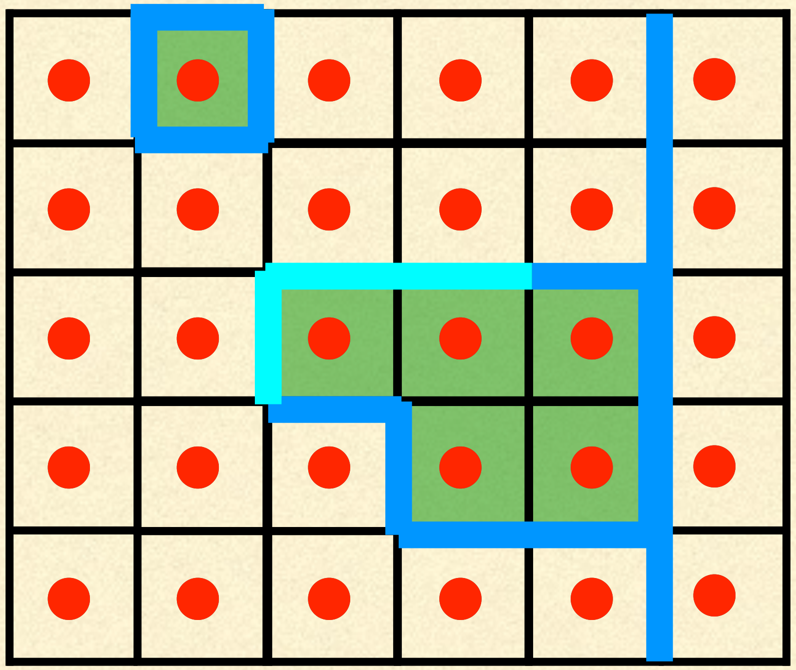


- **Best strategy**: if syndrome s observed, pick the most probable equivalence class consistent with s.
- **Dennis et al.'02**: connection with a stat-mech model!
- Perfect decoding (large L): $\mathrm{pr}(\mathrm{success}) \to 1$. Then,

$$-\sum_e \mathrm{pr}(e) \log \frac{\mathrm{pr}(\overline{e+h})}{\mathrm{pr}(\overline{e})} \to \infty$$

work w/ **F. Brandao, K. Svore**

RANDOM BOND ISING MODEL



- 2D model, spins on faces, 2-body interactions, couplings: **+1** w/ pr = 1-p and **-1** w/ pr = p.

$$H_e = -\sum_{\langle i,j\rangle} \varkappa_{ij} s_i s_j, \quad e = \{(i,j) | \varkappa_{ij} = -1\}$$

- Partition function

$$Z_e = \sum_{\{s_i\}} \exp(-\beta H_e(\{s_i\}))$$

$$= \sum_{b \in C_2} \exp(-\beta H_{e+\partial_2 b}(\{s_i = 1\})) = \sum_{b \in C_2} \exp(\beta(N - 2|e + \partial_2 b|))$$

- Free energy difference of introducing a domain wall h in the system

$$\Delta_h = \sum_e \mathrm{pr}(e)(F_{e+h} - F_e) = -\frac{1}{\beta} \sum_e \mathrm{pr}(e) \log \frac{Z_{e+h}}{Z_e}$$

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DECODING VS. STAT-MECH

- We want to relate successful decoding to phase transition.

$$-\sum_e \mathrm{pr}(e)\log\frac{\mathrm{pr}(\overline{e+h})}{\mathrm{pr}(\overline{e})} \quad \text{vs.} \quad -\frac{1}{\beta}\sum_e \mathrm{pr}(e)\log\frac{Z_{e+h}}{Z_e}$$

- Partition function

$$Z_e = \sum_{b\in C_2}\exp(\beta(N-2|e+\partial_2 b|))$$

- Equivalence class of error

$$\mathrm{pr}(\overline{e}) = (1-p)^N \sum_{b\in C_2}\left(\frac{p}{1-p}\right)^{|e+\partial_2 b|}$$

- Expressions match if $\dfrac{p}{1-p} = \exp(-2\beta)$

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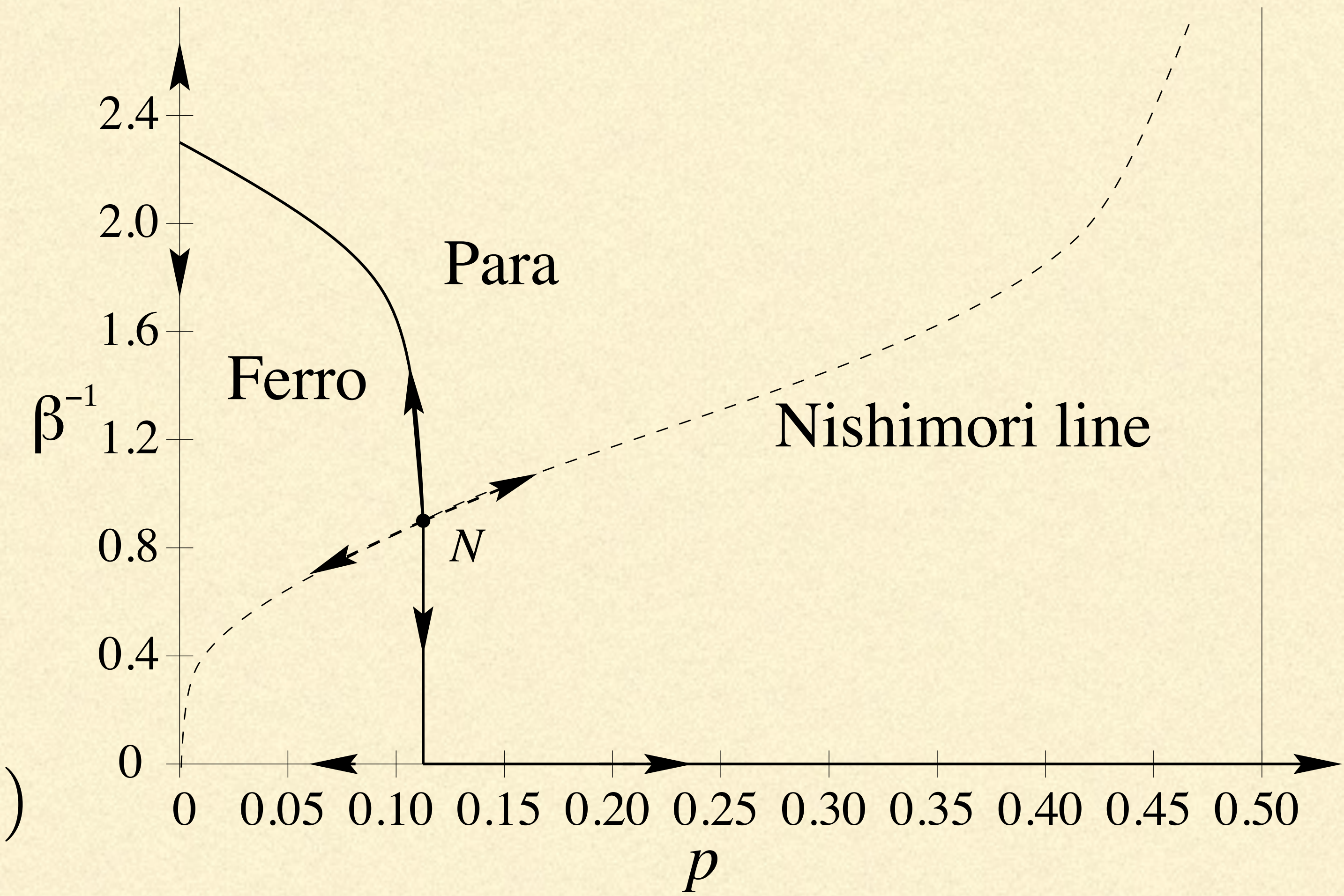
THRESHOLD FROM PHASE DIAGRAM



- Phase diagram - ordered and disordered phases.
- Parameter space: temperature T and disorder p.
- Nishimori line $\frac{p}{1-p} = \exp(-2\beta)$
- Perfect decoding = ordered phase.
- Threshold p_{th} - critical point along Nishimori line.
- **Honecker et al.'00** - 2D toric code threshold p_{th} = .1094(2)

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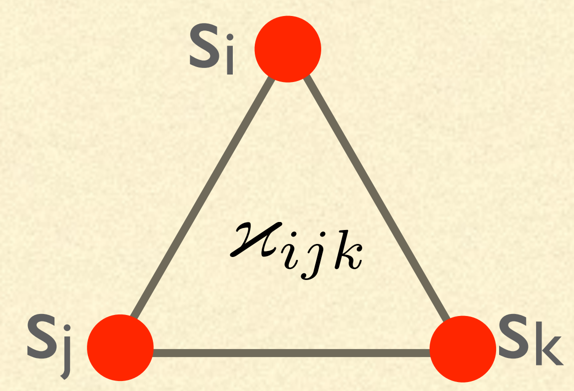
2D COLOR CODE VS. ISING MODEL

- Model: 3-body random bond Ising model in two dimensions.

$$H = -\sum \varkappa_{ijk} s_i s_j s_k$$

$\varkappa_{ijk}$: -1 with pr = p; +1 with pr = 1-p



- Previous work - extensive analysis (measurement errors, different lattices), e.g. **Bombin et al.'08, Katzgraber et al.'09, Andrist PhD thesis'12.**

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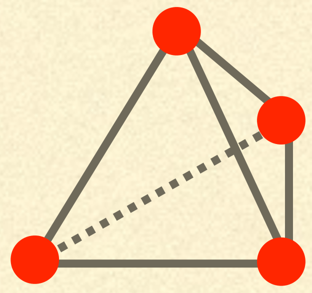
3D COLOR CODE VS. ISING MODEL

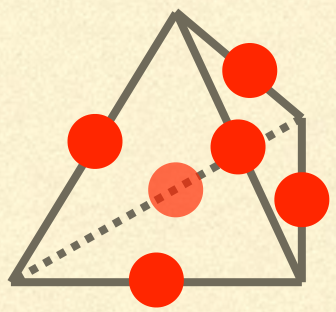
- Color code in 3 dim
  - stabilizer CC: stabilizers on 2-cells (A) and 3-cells (B),
  - subsystem CC: gauge generators on 2-cells, stabilizers on 3-cells (C).
- Two models to analyze:
  - 4-body random bond Ising model (A),

$$H = -\sum \varkappa_{abcd} s_a s_b s_c s_d$$



  - local order parameter - easy to analyze!
  - 6-body random bond Ising model (B & C),

$$H = -\sum \varkappa_{abcdef} s_a s_b s_c s_d s_e s_f$$



  - challenging due to gauge symmetries (no local order parameter).

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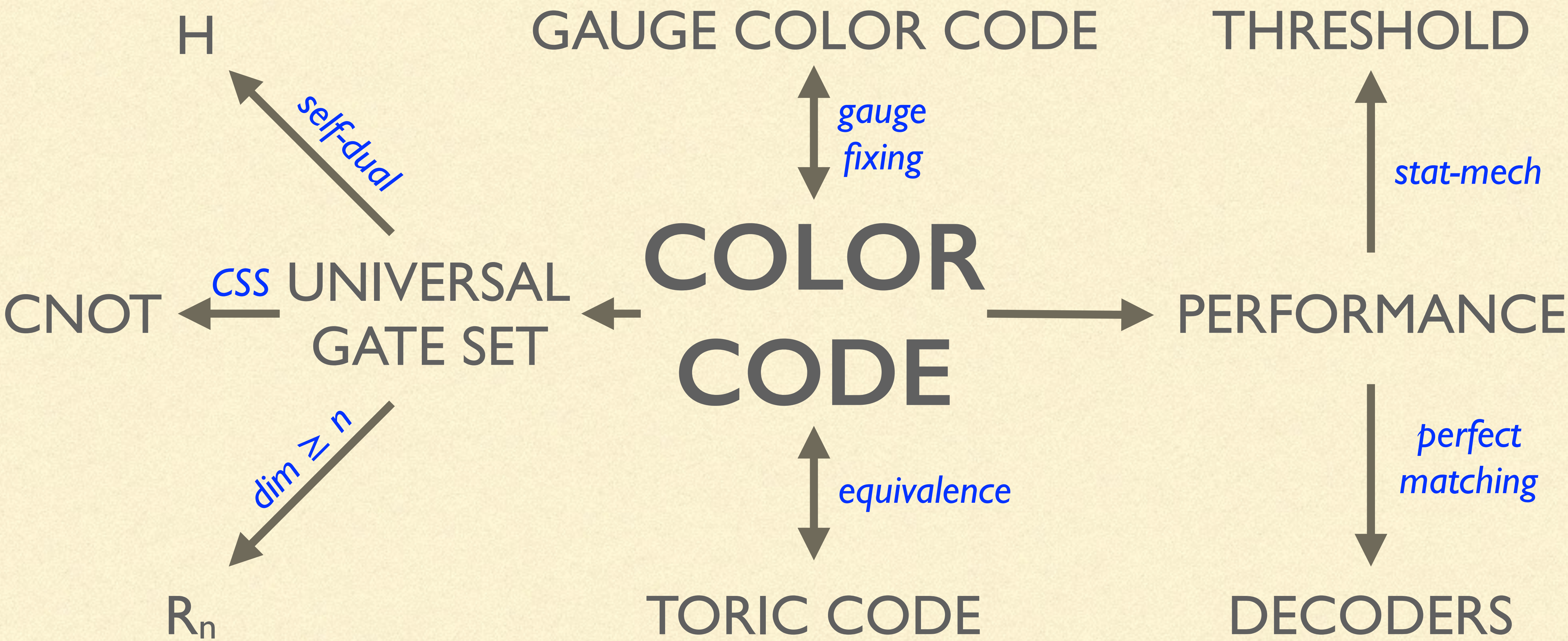
SUMMARY



Thank you!