The University of Sydney

Briad group actions on spin chains and supersymmetry

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University of Sydney
NSW 2006
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Most is joint work with Ruibin Zhang (Sydney), some with P. Deligne (Princeton) or H. Andersen (Aarhus)

Background

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In particular: U acts on $T^r(V) = V^{\otimes r}$ for all r

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FFT: η_r is surjective; SFT: $\ker(\eta_r)$ is generated by the idempotent in $\mathbb{C}\mathrm{Sym}_{n+1}$ corresponding to the alternating representation. (All due to Schur, 1901).

Further cases where the FFT and SFT are known

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In the orthogonal case, E_n is explicitly described in terms of diagrams, all of which have coefft ± 1.

The case of Lie superalgebras

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It is a fact that Ω is alternating for $\mathrm{OSp}(V)$, and is a polynomial function of degree $m(2n+1)$ on V.

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This will provide a source of unitarisable braid group actions on tensor space, where the braid generators satisfy polynomial equations of arbitrarily high order.

Endomorphisms of tensor powers–the quantum case

$\mathfrak{g}$ as above: a finite dimensional reductive complex Lie algebra; $\mathrm{U}(\mathfrak{g}) = \mathrm{U}(\mathfrak{g})$ its universal enveloping algebra, and $\mathrm{U}_q = \mathrm{U}_q(\mathfrak{g})$ its Drinfeld-Jimbo quantisation over $\mathcal{K} := \mathbb{C}(q)$, q an indeterminate.

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Corresponding to $\mathfrak{g}$ there is a root system $\Phi \subset \mathfrak{h}^*$, where $\mathfrak{h}$ is a Cartan subalgebra, and we assume chosen a set $\{\alpha_1, \dots, \alpha_r\} \subset \Phi$ of simple roots. There is a canonical bilinear form $(-,-)$ on $\mathfrak{h}^*$ such that $(\alpha, \alpha) = 2$ for short roots α.

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As usual, we write X for the lattice of weights of $\mathfrak{g}$; $X = \{\lambda \in \mathfrak{h}^* \mid (\alpha, \lambda) \in \mathbb{Z}\}$ for all $\alpha \in \Phi$.

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If V is a representation of U_q, $v \in V$ is a weight vector of weight $\lambda \in X$ if $k_i v = q^{(\lambda, \alpha_i)} v$ for all i. Say V is of type $(1, 1, \dots, 1)$ if it is a sum of weight spaces.

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(ii) $\boxed{R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \mathrm{End}(W_1 \otimes W_2 \otimes W_3)}$ (Yang-Baxter equation).

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Questions: When is β_r surjective? (FFT); What is $\ker(\beta_r)$? (SFT)

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(L-R. Zhang (Sydney), H. Zhang (Tsinghua) 2016) β_r is surjective when U_q is the quantised enveloping algebra assiciated with the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(m|2n, \mathbb{C})$ and $V = \mathbb{C}^{m|2n}$.

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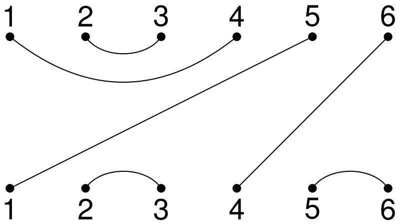

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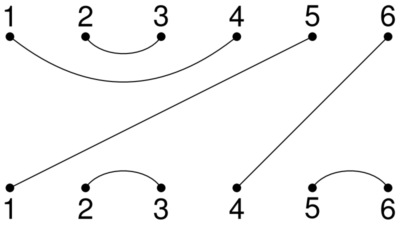

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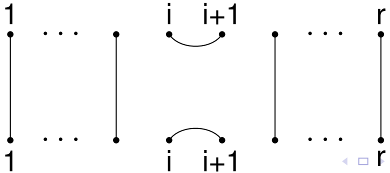

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$\mathrm{End}_{\mathrm{U}_q}(V_{\widetilde{A}}(d)^{\otimes r})$ is realised as a subalgebra (and quotient algebra) of $\mathrm{End}_{\mathrm{U}_q}(V(1)^{\otimes rd}) \simeq \mathrm{TL}_{rd}(\widetilde{A})$ as follows.

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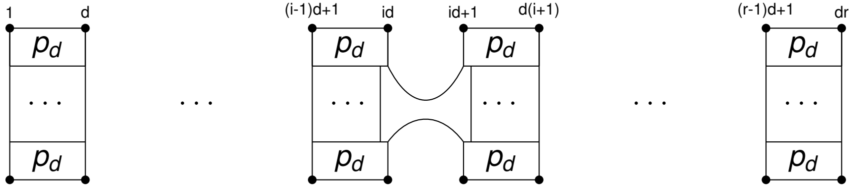

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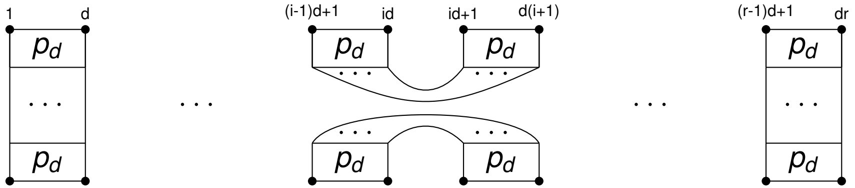

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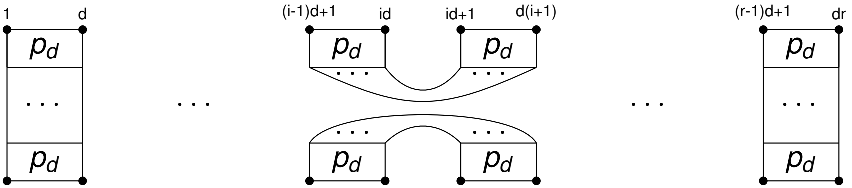

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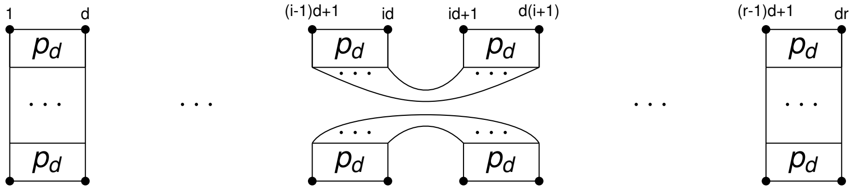

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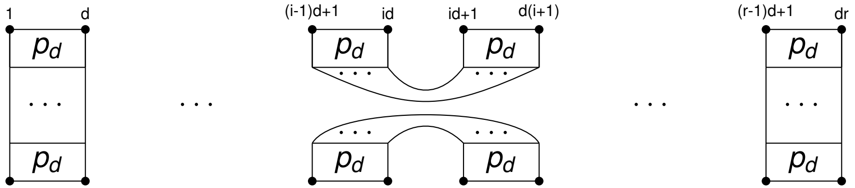

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Consequence: we can determine for which specialisations $q \mapsto \zeta$, the module $V(d)_{\zeta}^{\otimes r}$ is completely reducible.

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