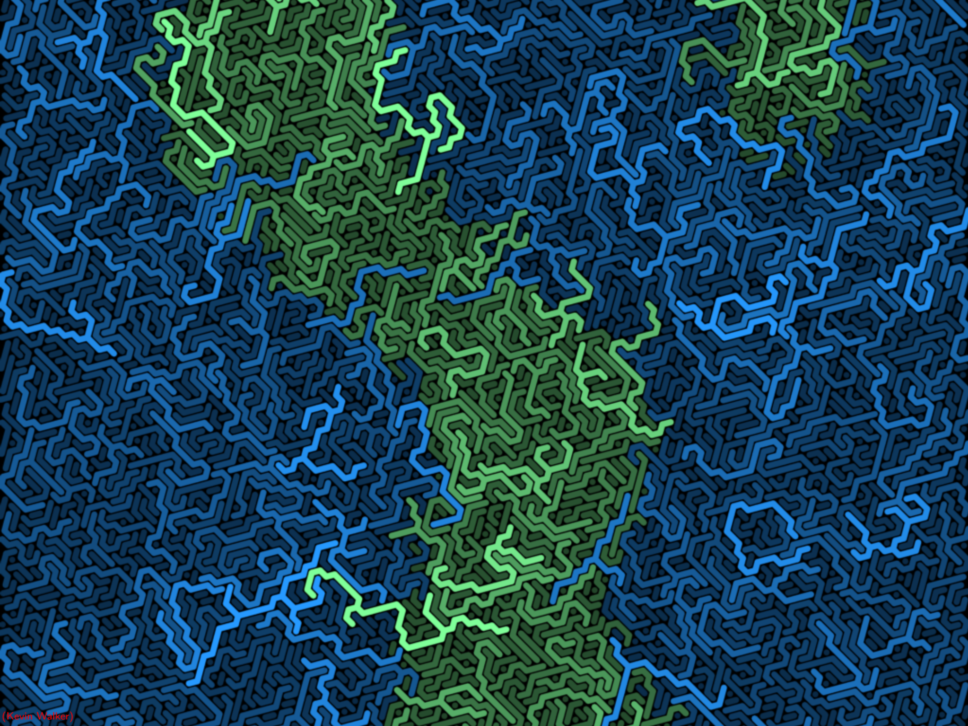
(Kevin Walker)

Topological field theories and fusion categories

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- What are 2-dimensional topological field theories?
- Local 2-d TQFTs $\Leftrightarrow$ fusion categories.
- A progress report on classification.

A 2-d topological field theory is a functor $\mathcal{F} : \mathbf{2Cob} \to \mathbf{Vec}$.

That is:

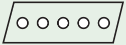
- For each surface Σ, $\mathcal{F}(\Sigma)$ is a vector space: "the Hilbert space of quantum mechanical states on Σ", and
- for each 3-manifold M, with incoming boundary Σ_{in} and outgoing boundary Σ_{out}, a linear map

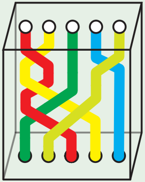
$$\mathcal{F}(M) : \mathcal{F}(\Sigma_{\text{in}}) \to \mathcal{F}(\Sigma_{\text{out}})$$

  which describes time-evolution.

The original setup for topological quantum computing fits in this framework.

We work in a punctured disc.

- The Hilbert space $\mathcal{F}\Big($  $\Big)$ is the degenerate ground state of a 2d topological phase, and we use this to encode the input qubits.
- We implement a quantum algorithm — some unitary operator — by approximating it by braiding operators

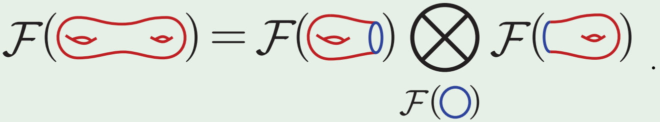
$\mathcal{F}\left(\right.$  $\left.\right).$

I'm particularly interesting in local field theories.
These additionally associate

- to each 1-manifold J (intervals and circles!), $\mathcal{F}(J)$, a category.
- and to the point $\bullet$, a tensor category $\mathcal{F}(\bullet)$.

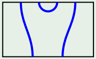
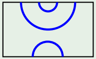
These data must satisfy gluing rules.

Suppose we can split a k-manifold (for $k = 1, 2$, or 3) into two pieces $M \cup_S N$ along some $(k-1)$-manifold S.
The value of the field theory is determined by its values on the pieces, and the way the higher algebraic object $\mathcal{F}(S)$ acts on the values of the pieces:



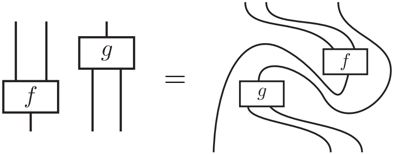
An aside: some examples of tensor categories

- **Vec**, the category of finite dimensional vector spaces.
- Rep$G$, for G a compact group. G acts on the tensor product of representations by $g(v \otimes w) = gv \otimes gw$.
- Rep$U_q\mathfrak{g}$, the representation theory of a quantised universal enveloping algebra.
- The Temperley-Lieb category, with objects $\mathbb{N}$ and

$\mathrm{Hom}(n \to m) = \mathbb{C}\left\{\right.$ , , $\left.\ldots \right\} / \bigcirc = \delta.$

- … more to come!

In a tensor category, we can draw planar diagrams representing compositions (vertical stacking) and tensor products (horizontal juxtaposition) of morphisms. Diagrams related by a planar isotopy represent the same morphism.



Planar tensor networks can be generalised to live in any tensor category. The usual notion is just planar tensor networks in Vec.

Usually, there is a d^n dimensional space of tensors that can live in an n-box, if we think of the strings as carrying $\mathbb{C}^d$. This is not true in a general (semisimple) tensor category, where $\dim \mathrm{Hom}(1 \to V^{\otimes n})$ grows as d^n, but d is not generally an integer.

The tensor category $\mathcal{F}(\bullet)$ completely determines all the higher structure.

Lurie's proof of the cobordism hypothesis tells us:

- Every *n*-dimensional local field theory is determined by its value on the point.
- That value, an *n*-category, must be fully dualizable.
- Any such fully dualizable *n*-category can be used to define a local field theory.

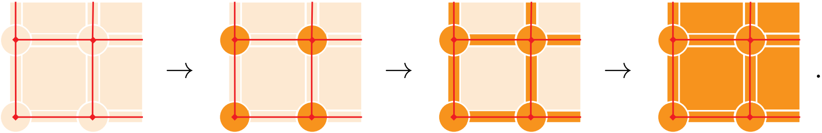
Douglas–Snyder–Schommer-Pries have recently spelled out exactly what fully dualizable means in the case of tensor categories. They are the fusion categories.

For each fusion category $\mathcal{C}$, we can write down a local Hamiltonian on a graph embedded in Σ, whose degenerate ground state is the corresponding TQFT vector space $\mathcal{F}(\Sigma)$.

This is the "Levin-Wen" model, although it goes back to Ocneanu in the subfactor literature.

$$\begin{tikzcd}
\text{local 2d TQFTs} \arrow[rr, Leftrightarrow, "\substack{\text{cobordism}\\ \text{hypothesis}}"] & & \text{fusion categories} \arrow[dl, Rightarrow] \\
& \text{local Hamiltonian} \arrow[ul, Rightarrow] &
\end{tikzcd}$$

The Levin-Wen Hamiltonian is exactly a sum of commuting projections that implement the gluing formula for a handle decomposition of the surface.



At each step we glue on more balls, and have

$$\mathcal{F}(k\text{-skeleton}) = \mathcal{F}((k-1)\text{-skeleton}) \otimes_{\mathcal{F}(\partial)} \mathcal{F}(k\text{-handles}).$$

Each of these tensor products over $\mathcal{F}(\partial)$ is a quotient of a direct sum of tensor products with matching boundary conditions. There are Hamiltonian terms for matching boundary conditions, and Hamiltonian terms for the quotient.

Definition

A fusion category is a **finitely semisimple $\otimes$-category**.

Every object can be expressed as a direct sum of 'simple objects'.
There are finitely many such simples $\{X_i\}$, and

$$\mathrm{Hom}(X_i \to X_j) = \begin{cases} \mathbb{C} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

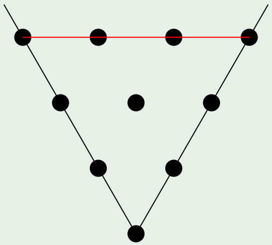
They can be encoded in a finite amount of algebraic data!

Non-examples

- RepS_n over a finite field (not semisimple).
- Rep$SU(2)$ (semisimple, but infinitely many simples).

Examples

- Vec, with just one simple object, $\mathbb{C}$.
- $\mathrm{Vec}^\omega G$, G-graded vector spaces with associator $\omega \in H^3(G)$.
- $\mathrm{Rep} G$, G a finite group (simple objects are the irreducible representations, as many as conjugacy classes).
- $\mathrm{Rep} U_q\mathfrak{g}$ at q a root of unity (not immediately semisimple, but the quotient by the negligible ideal is finitely semisimple).



$SU(3)_q$ at a 12-th root of unity

- ... what else?

What do we know about fusion categories?

Constructions of new from old:

- $\mathcal{C} \boxtimes \mathcal{D}$,
- taking a full subcategory,
- if a finite group acts $G \circlearrowright \mathcal{C}$, 'equivariantise' to obtain $\mathcal{C}^G$,
- if $\mathcal{C} \supset \mathrm{Rep}\, G$ we can (often) 'de-equivariantise' to obtain $\mathcal{C}//G$,
- there is a beautiful homotopy theoretic classification of fusion categories graded by G with specified neutral piece $\mathcal{C}_e$:

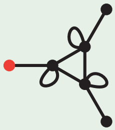
$$\mathcal{C} \cong \oplus_g \mathcal{C}_g$$
$$\otimes : \mathcal{C}_g \times \mathcal{C}_h \to \mathcal{C}_{gh}$$

But we have no structure theory that would build all fusion categories from some simpler class. (As, for example, finite groups are built from simple groups.)

Do the representation theoretic examples $\mathrm{Rep}\, G$, $\mathrm{Rep}\, U_q\mathfrak{g}$ 'generate' everything?

No: The Haagerup subfactor $N \subset M$ gives (as the $N - N$ bimodules generated by M) a fusion category $\mathcal{H}_1$ which is genuinely new.

The fusion category $\mathcal{H}_1$ has an object M such that $\dim \mathrm{Hom}(1 \to M^n)$ is the number of based loops of length n on



.

Asymptotically this grows as $\left(\frac{3+\sqrt{13}}{2}\right)^n$.

Theorem (Morrison-Snyder '10)

Rep$G$ *and* Rep$U_q\mathfrak{g}$ *can be defined over a cyclotomic field, but* $\mathcal{H}_1$ *cannot!*

We now recognise $\mathcal{H}_1$ as a quadratic category, for which

$$|\mathrm{Inv}\,\mathcal{C}\backslash\mathrm{Irr}\,\mathcal{C}/\mathrm{Inv}\,\mathcal{C}| = 2$$

and there has been considerable progress (Izumi, Evans-Gannon, Grossman-Snyder, ...) understanding these.

Partial classifications (many coauthors)

- Fusion categories with no subcategories, up to global dimension 37.5.
- Fusion categories $\otimes$-generated by a s.s.d. object with dimension $\leq 2.29\ldots$
- Fusion categories with rank at most 4 (incomplete).

Present status (as of late '15, many contributors)

Every known fusion category can be built out of $\mathrm{Rep}\, G$, $\mathrm{Rep}\, U_q\mathfrak{g}$, and quadratic categories by the standard constructions, with 2 or 3 closely related exceptions,

$$\mathcal{EH}_1, \mathcal{EH}_2, \mathcal{EH}_3(?)$$

coming from the extended Haagerup subfactor.

Modular categories are braided pivotal fusion categories such that

$S = \Bigg($ 

$\Bigg)_{ij}$

is an invertible matrix.
Viewed narrowly, these are a subset of the fusion categories.

The relationship is perhaps the other way round!

- The Drinfeld centre $Z(\mathcal{C})$ of a pivotal fusion category $\mathcal{C}$ is modular.
- $Z(\mathcal{C})$ is the value $\mathcal{F}(S^1)$ of the associated TQFT on the circle.
- The 3-2-1d part of the TQFT is determined by $Z(\mathcal{C})$.
- In fact all 'extended' TQFTs $\mathcal{F}$ (i.e. defined on 3-2-1d manifolds) are determined by the modular category $\mathcal{F}(S^1)$.
- Several fusion categories can have the same Drinfeld centre (if and only if they are categorically Morita equivalent).

We have

$$\begin{tikzcd}
\text{extended field theories} \arrow[r, Leftrightarrow, "1{:}1"] & \text{modular categories} \\
\text{local field theories} \arrow[u, "\text{finite:1}"] \arrow[r, Leftrightarrow, "1{:}1"'] & \text{fusion categories} \arrow[u, "\text{finite:1}"']
\end{tikzcd}$$

We have better tools, mostly originating in the subfactor literature, for constructing 'exotic' fusion categories. Taking the centre then gives a new modular tensor category.

Question

Is the ***Witt group***

$$\{\textit{unitary MTCs under } \boxtimes\}/\{Z(\mathcal{C}), \mathcal{C} \textit{ fusion}\}$$

generated by $\mathrm{Rep} U_q\mathfrak{g}$ *examples?*

Example

With Gannon, we've recently computed the S and T matrices for $Z(\mathcal{EH}_\bullet)$, an MTC with 22 simple objects, as well as all potential character vectors for a possible CFT.