Effective conformal field theories for tensor network states

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Quantum phases of a chain of strongly interacting anyons

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Quantum gates for the manipulation of topological qubits rely on interactions between non-Abelian anyonic quasiparticles. We study the collective behaviour of systems of anyons arising from such interactions. In particular, we study the effect of favouring different fusion channels of the screened Majorana spins appearing in the recently proposed topological Kondo effect. Based on the numerical solution of a chain of $SO(5)_2$ anyons we identify two critical phases whose low-energy behaviour is characterised by conformal field theories with central charges $c = 1$ and $c = 8/7$, respectively. Our results are complemented by exact results for special values of the coupling constants which provide additional information about the corresponding phase transitions.

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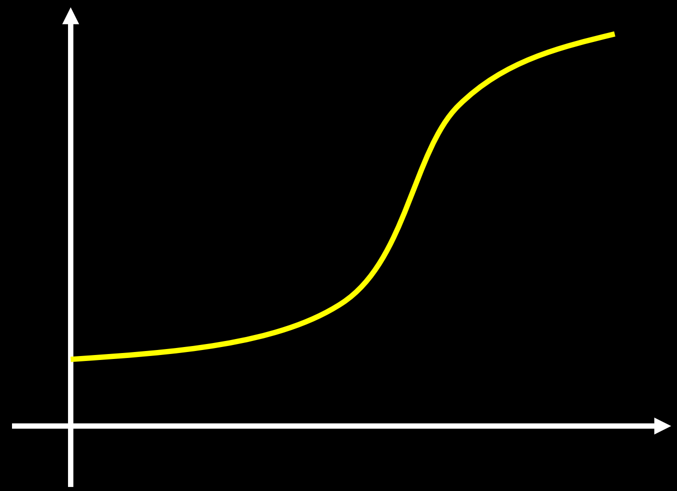
Low-dimensional quantum systems hold an irresistible and enduring fascination because they can support topological states of matter with exotic quasiparticles, *anyons*, exhibiting unusual braiding statistics [1]. While initially a curiosity, anyons generated considerable excitement when it was realized that the fractional quantum Hall effect [2] — and later nanowires [3, 4] and the p_x+ip_y time history in order to discuss their dynamics. While the non-interacting case is now becoming well understood (see, e.g., [10]) the classification of phases for systems of *interacting* anyons has progressed much slower. An additional complication is that the description of the dynamics of a highly entangled $SO(M)$ Majorana spin in the topological Kondo model, and the collective behaviour

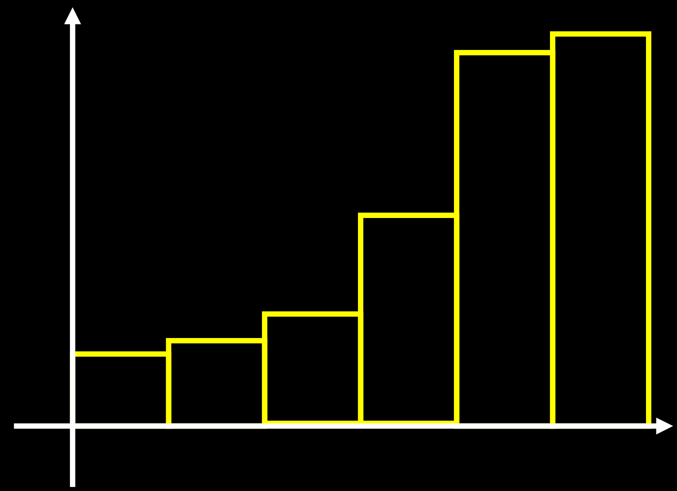
P. E. Finch, H. Frahm, M. Lewerenz, A. Milsted, TJO, Phys. Rev. B 90, 081111

"… we identify two critical phases whose low-energy behaviour is characterised by conformal field theories with central charges $c = 1$ and $c = 8/7$, respectively."

?



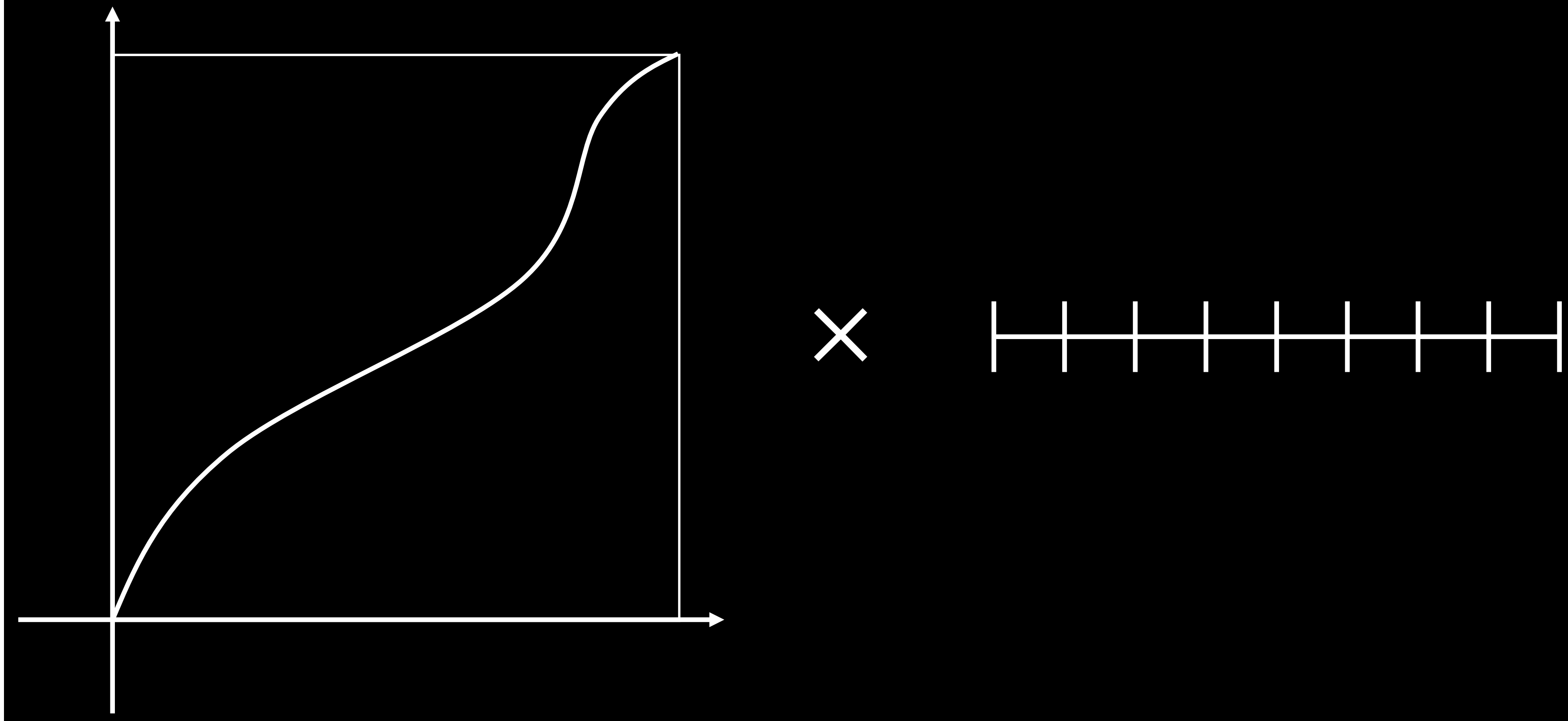
$\overset{?}{=}$



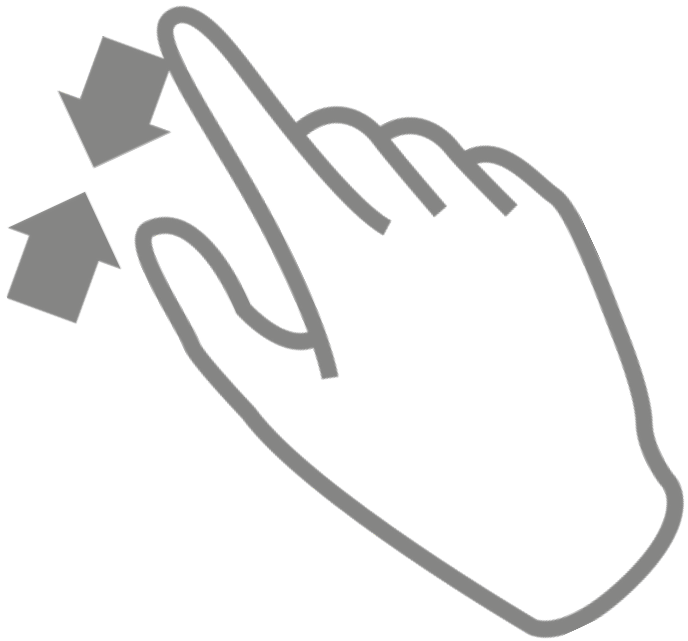

CFT Dream:
find unitary action of conformal group on low-energy space

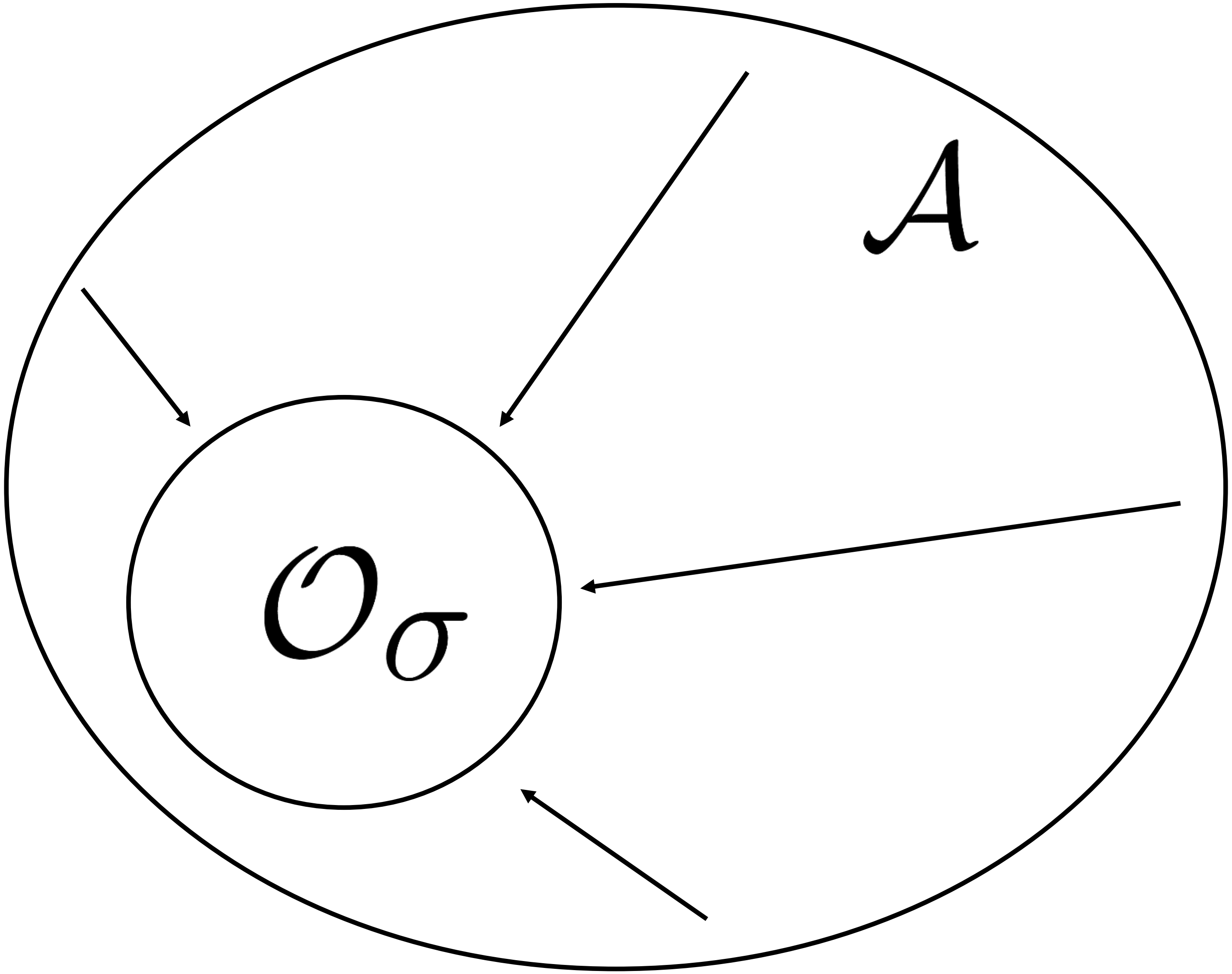
Conformal group:

$$\mathrm{conf}(\mathbb{R}^{1,1}) \cong \mathrm{diff}_+(S^1)\times\mathrm{diff}_+(S^1)$$



Zoom out
= fewer observables

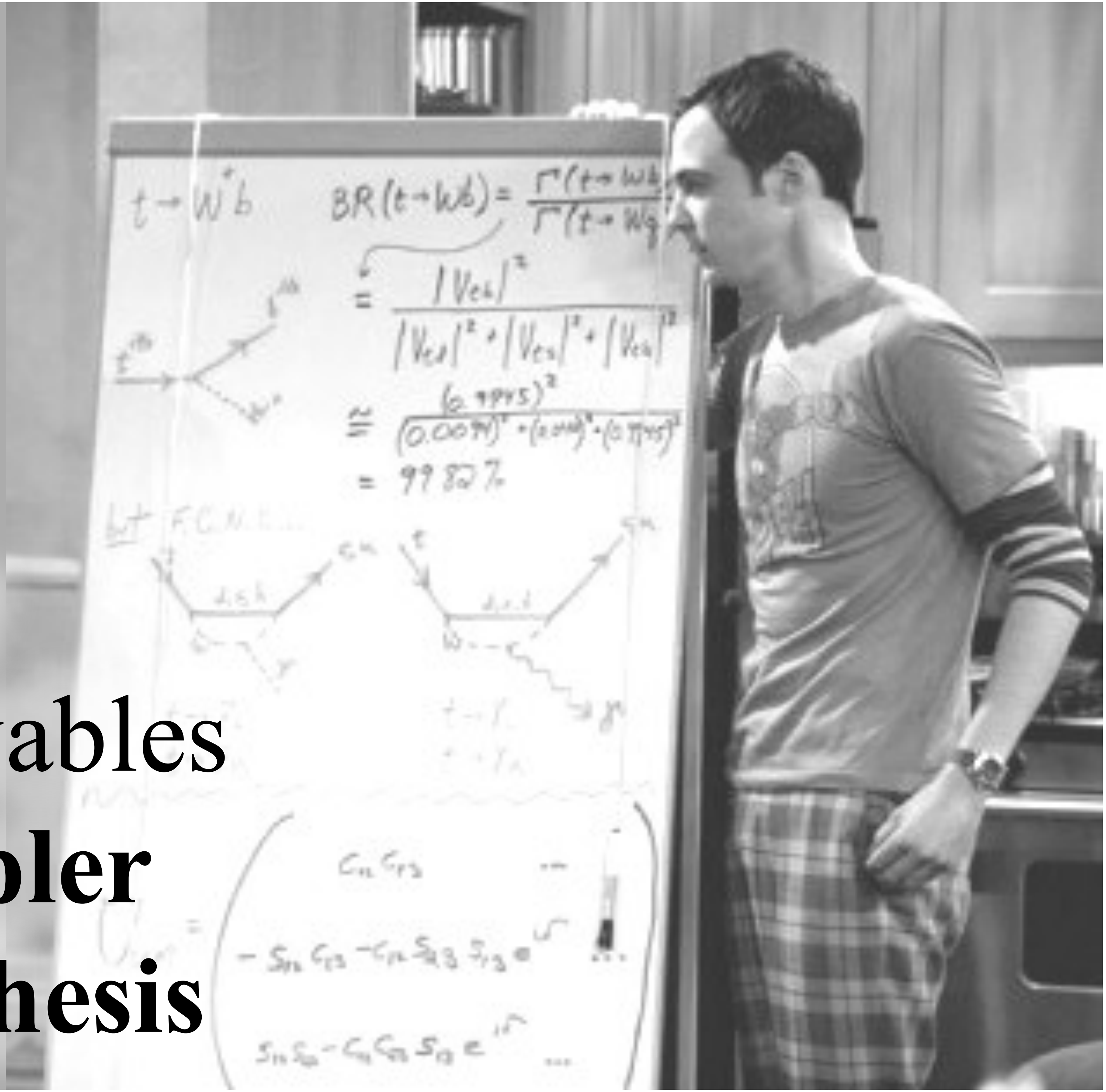




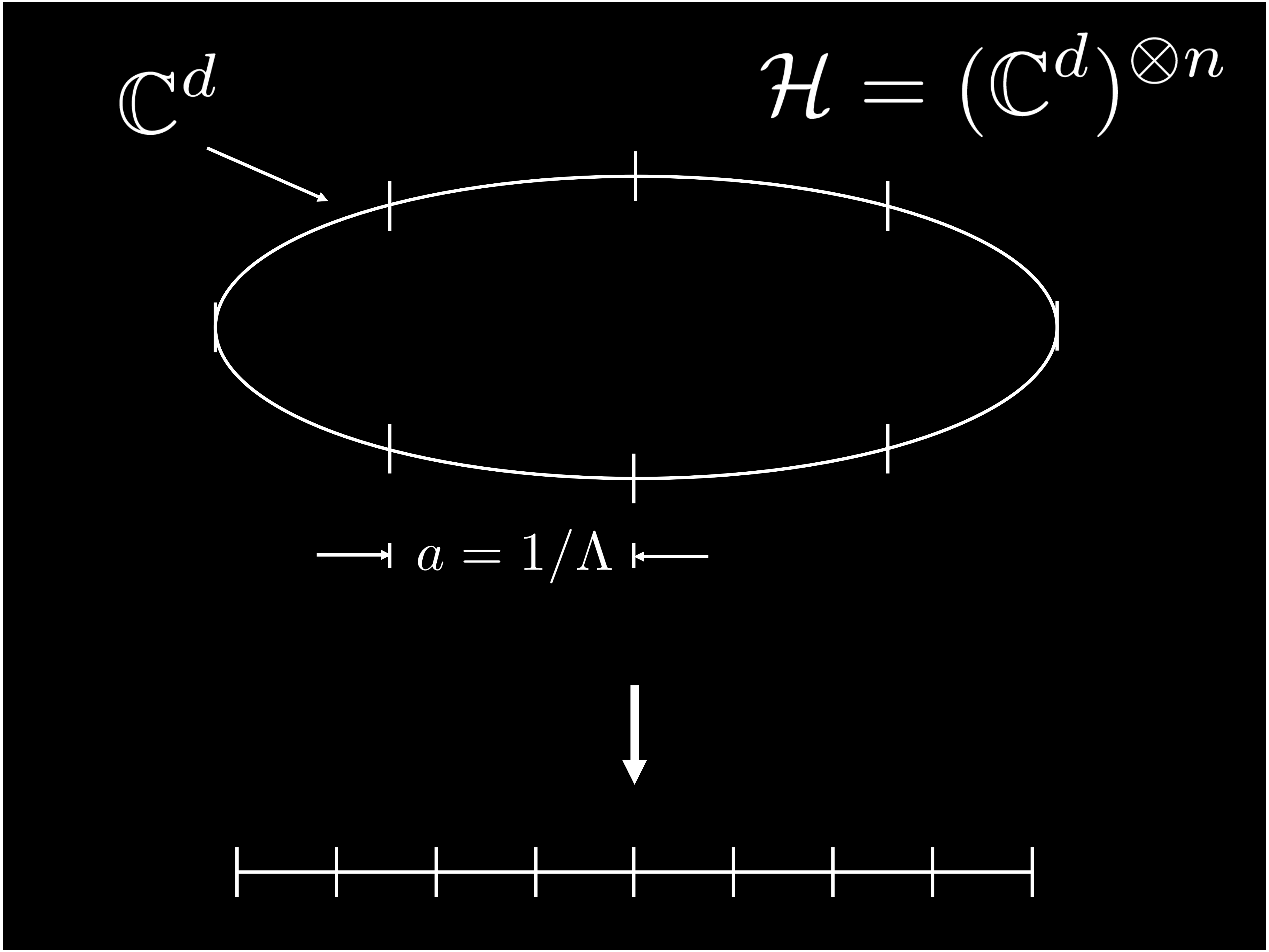
$\mathcal{A}$
$\mathcal{O}_\sigma$

Fewer observables

= simpler hypothesis



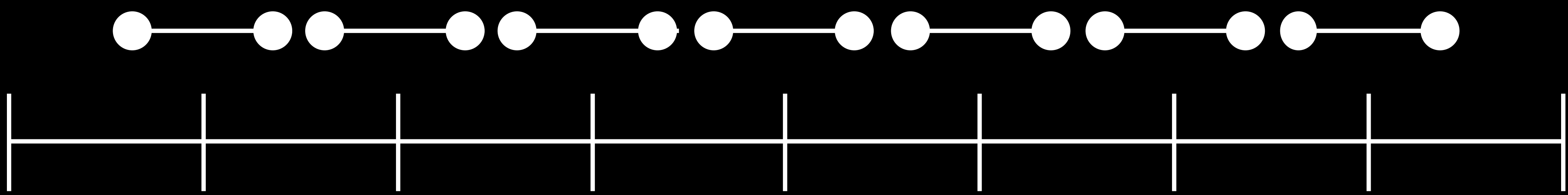
Simpler = **calculus**


$\mathbb{C}^d$
$\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$
$a = 1/\Lambda$

This is important!

$$H(\Lambda) = \sum_j h_{j,j+1}(\Lambda)$$

$$\cdots \quad h_{j-1,j} \quad h_{j,j+1} \quad \cdots$$



$H(\Lambda) \to$ a **sequence** of ground states $|\Omega_\Lambda\rangle$

WILSON:

QFT is low-energy effective theory of (2^{nd}-order) quantum phase transition

TNS Translation:

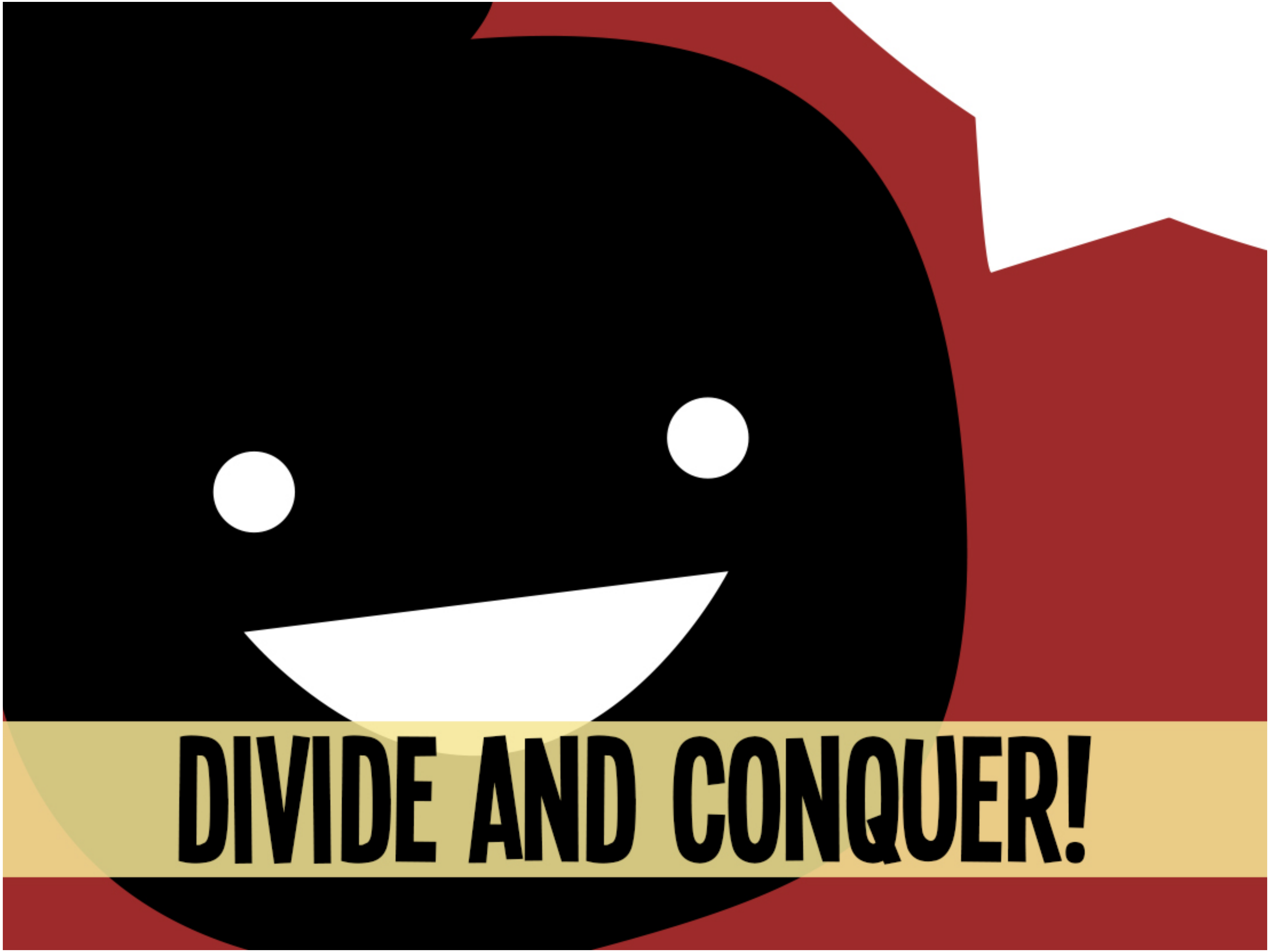
QFT is effective theory of **sequences** of TNS with increasing correlation length

MAIN TASK 1:

find a TNS subspaces for

low energy & large scale

excitations

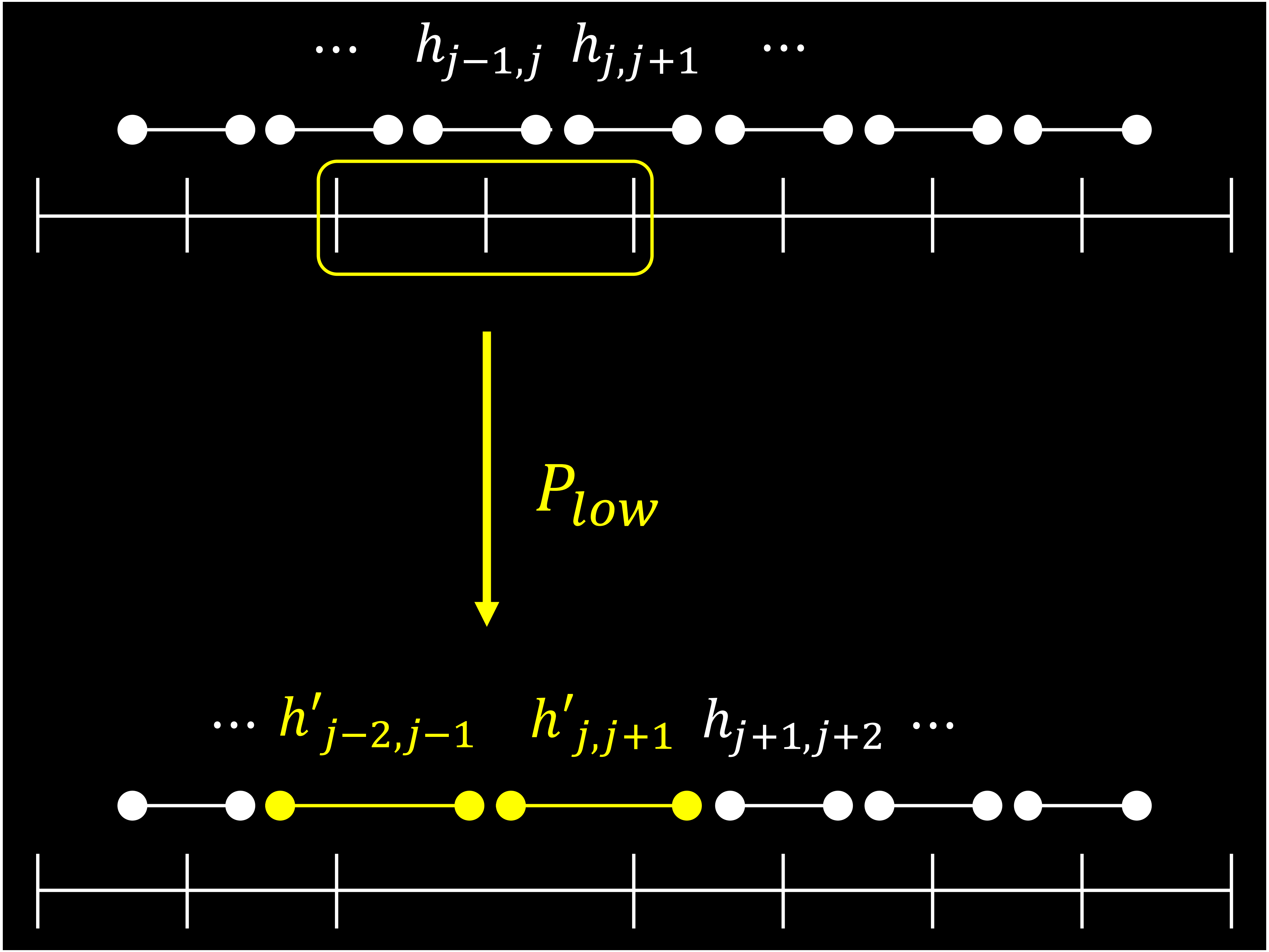

DIVIDE AND CONQUER!

Nonuniform Kadanoff RG (good for MBL and TI)

$$h_{j,j+1} = \sum_{\alpha=1}^{d^2} |\psi_\alpha\rangle\langle\psi_\alpha|$$

$$\downarrow$$

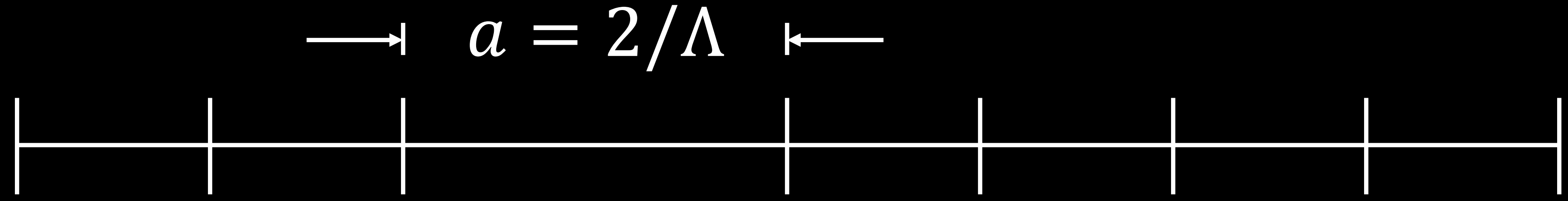
$$P_{low} = \sum_{\alpha=1}^{d} |\psi_\alpha\rangle\langle\psi_\alpha|$$

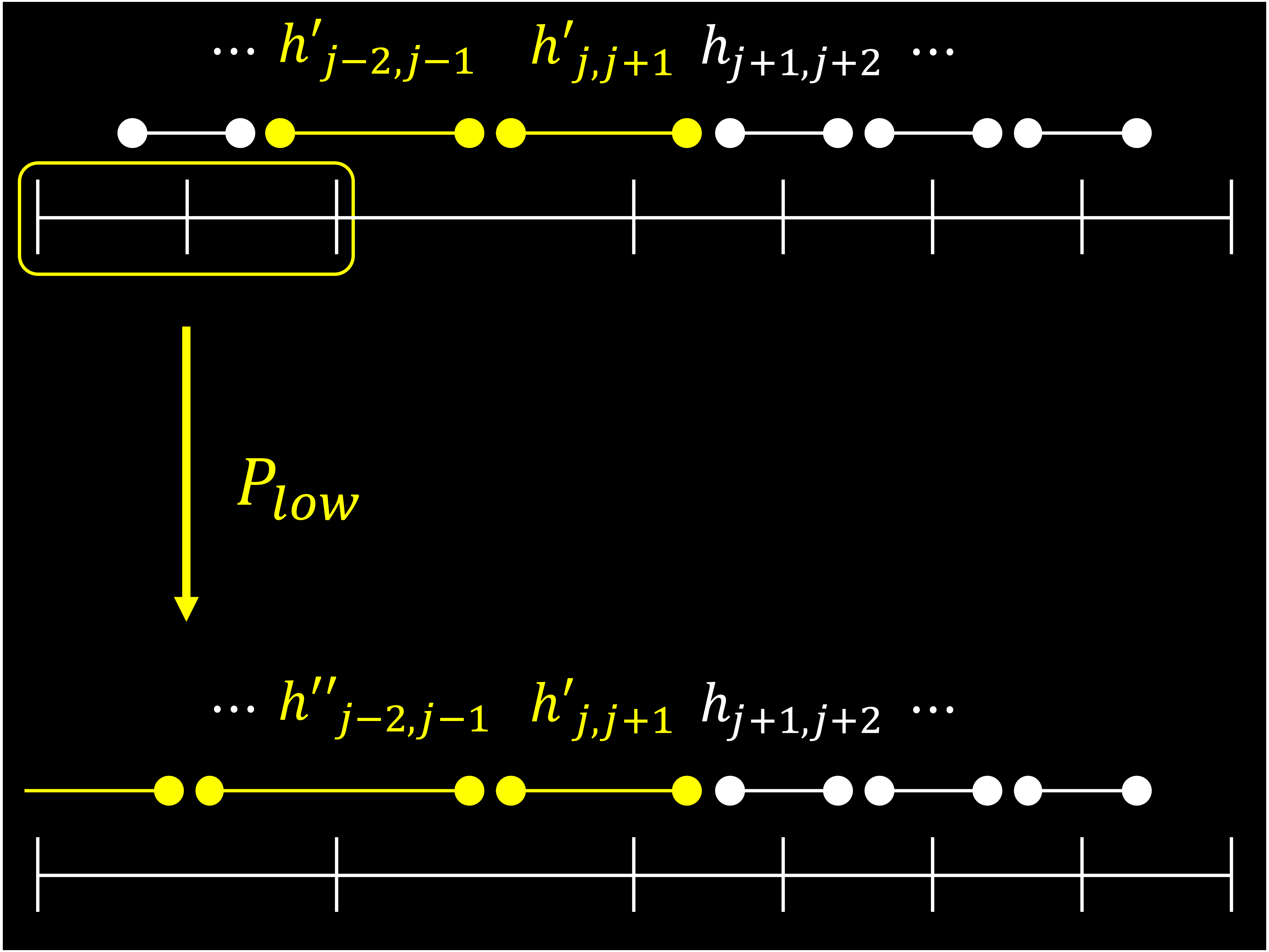


$\cdots \quad h_{j-1,j} \quad h_{j,j+1} \quad \cdots$
P_{low}
$\cdots \quad h'_{j-2,j-1} \quad h'_{j,j+1} \quad h_{j+1,j+2} \quad \cdots$

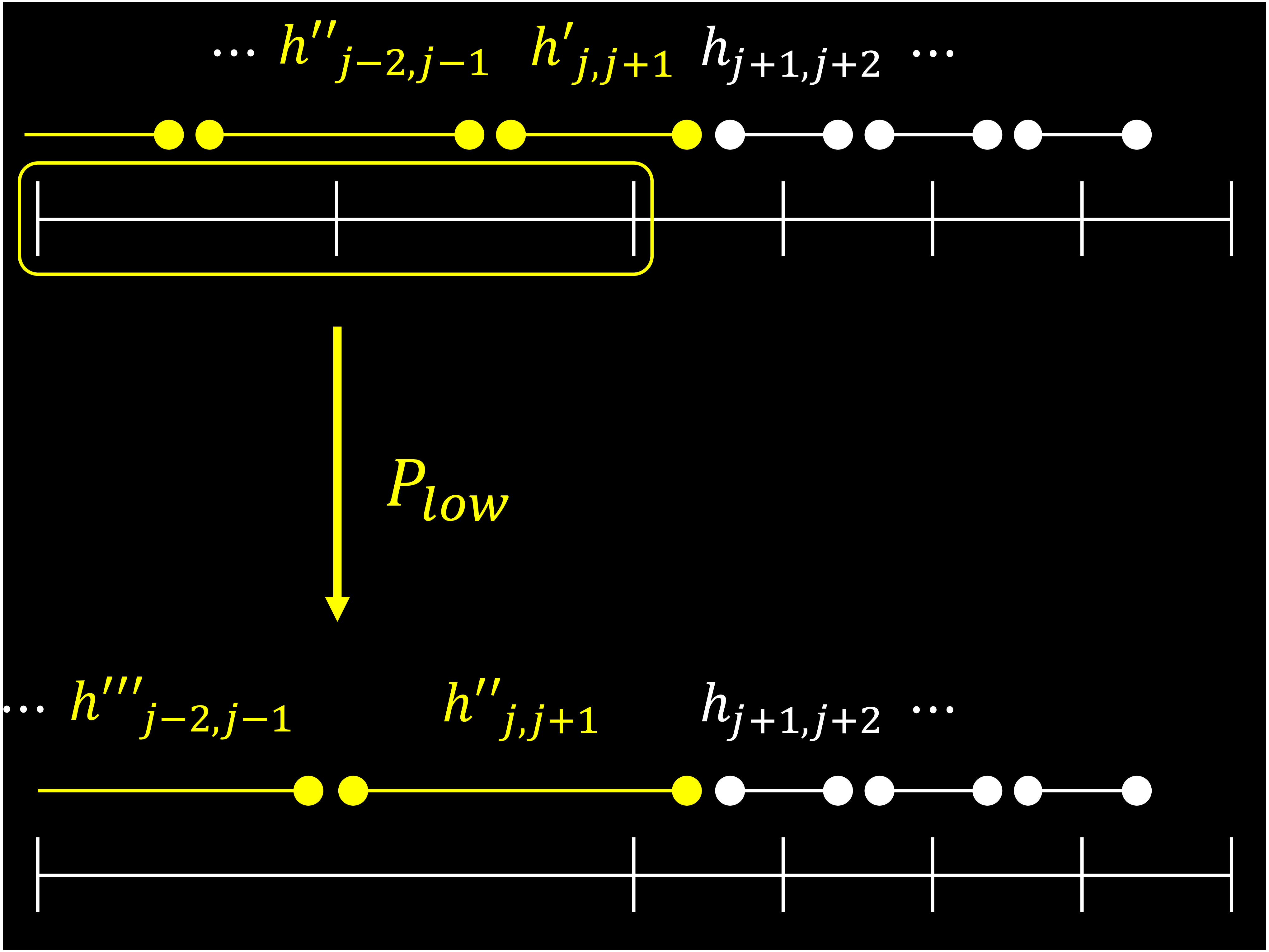
$$H_{\mathrm{eff}} = \sum_{k \neq j-2, j-1, j} h_{k,k+1}$$
$$+ P_{\mathrm{low}} \left(h_{j-2,j-1} + h_{j-1,j} + h_{j,j+1} \right) P_{\mathrm{low}}$$

$$\mathcal{H}_{\mathrm{eff}} = P_{\mathrm{low}} \mathcal{H} \cong \left(\mathbb{C}^d \right)^{\otimes n-1}$$

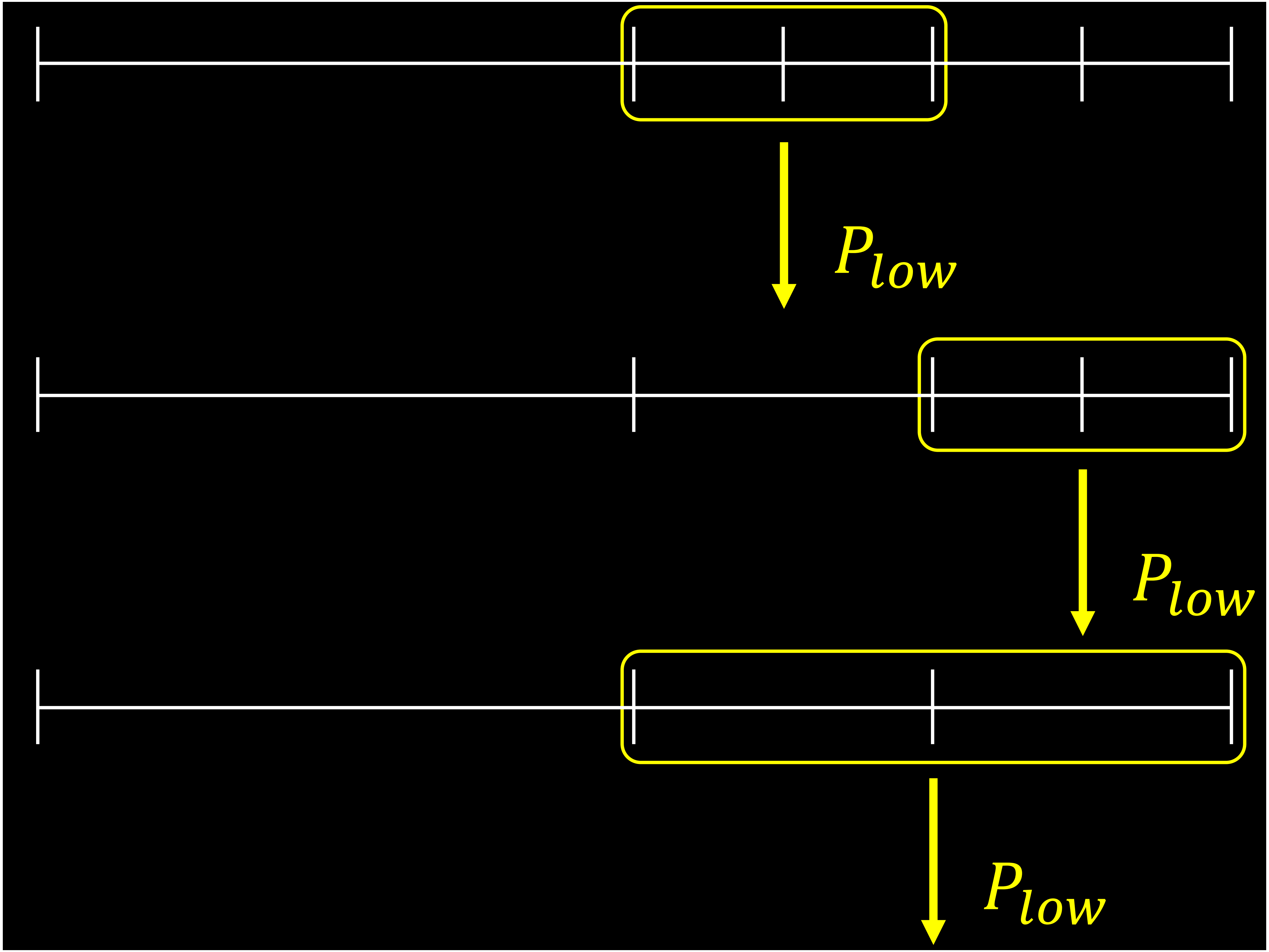




$\cdots\ h'_{j-2,j-1}\quad h'_{j,j+1}\quad h_{j+1,j+2}\ \cdots$
P_{low}
$\cdots\ h''_{j-2,j-1}\quad h'_{j,j+1}\quad h_{j+1,j+2}\ \cdots$



$\cdots\ h''_{j-2,j-1}$ $h'_{j,j+1}$ $h_{j+1,j+2}\ \cdots$
P_{low}
$\cdots\ h'''_{j-2,j-1}$ $h''_{j,j+1}$ $h_{j+1,j+2}\ \cdots$


P_{low}
P_{low}
P_{low}

$$a = 2^n / \Lambda$$

$$\mathcal{H}_{\mathrm{eff}}^{(n)} = P_{\mathrm{low}} \cdots P_{\mathrm{low}} \mathcal{H} \cong \mathbb{C}^d$$

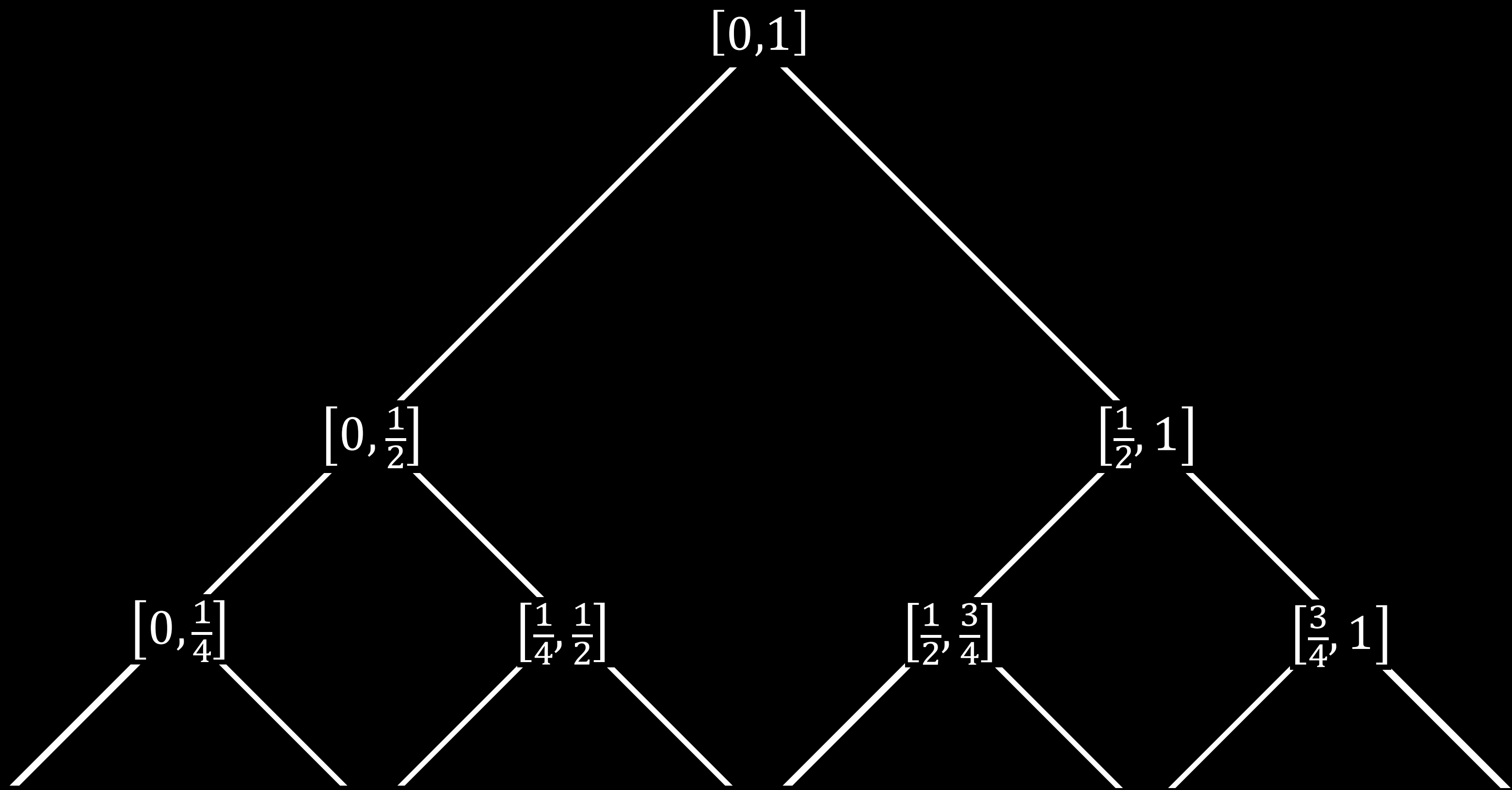
$$H_{\mathrm{eff}}^{(n)} = h^{(n)}$$

Intermediate
lattice systems are
coarser partitions
of circle

No rescaling is applied!

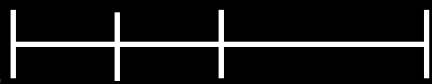
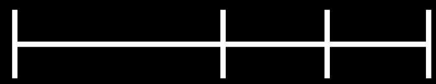
Standard dyadic interval:

interval of form $\left[\frac{a}{2^n}, \frac{a+1}{2^n}\right]$:



Standard dyadic partitions:

partitions into std. dyadic intervals

$\mathcal{P} = \Bigg\{ \; , \ldots ,$  $,$  $, \ldots \Bigg\}$

If $P, Q \in \mathcal{P}$ say

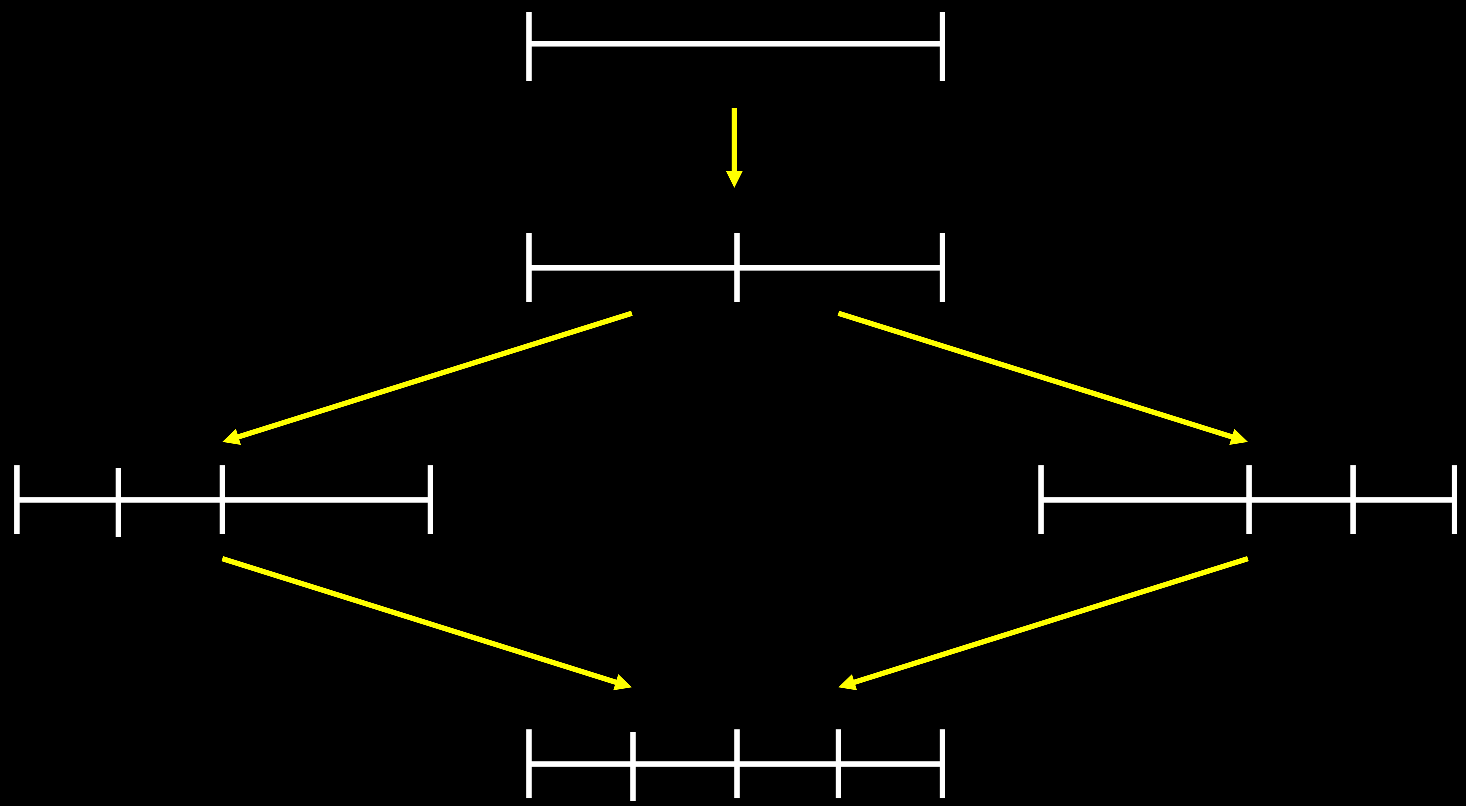
"$P \leq Q$" to mean partition

Q is a **refinement** of P

(Q has more cells)

Standard dyadic partition:

partition into std. dyadic intervals

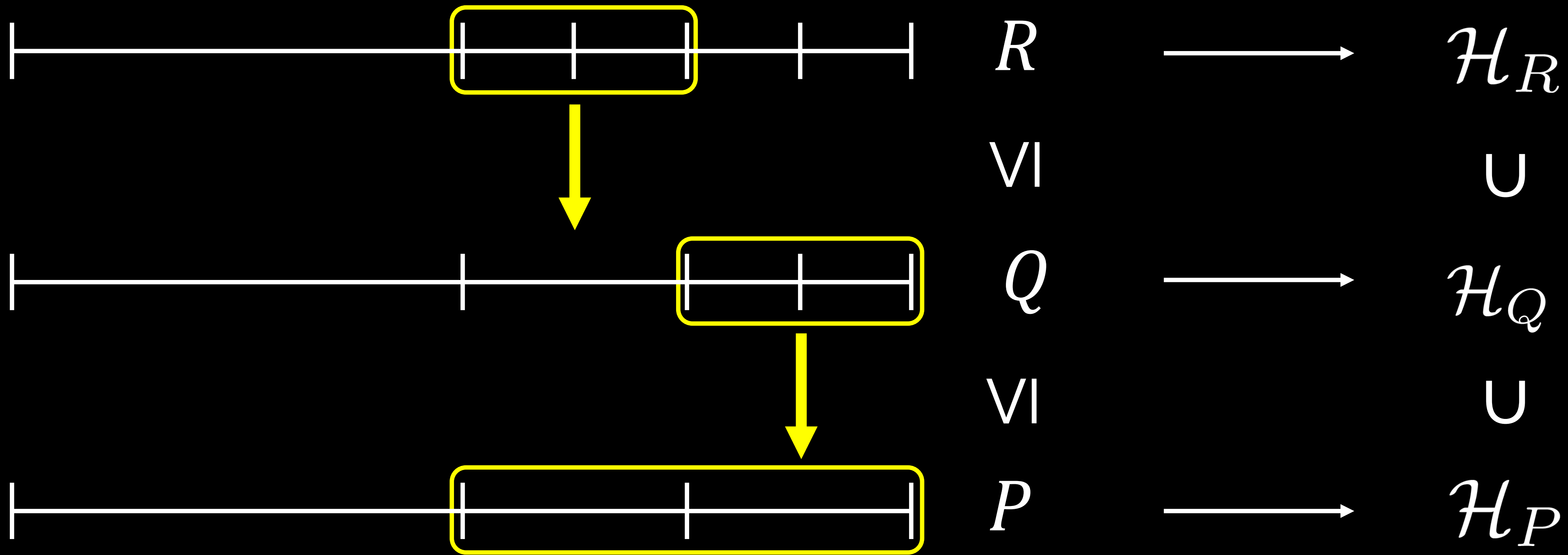


they form a "Verband" (mathematical lattice): $(\mathcal{P}, \vee, \wedge)$

Kadanoff RG:

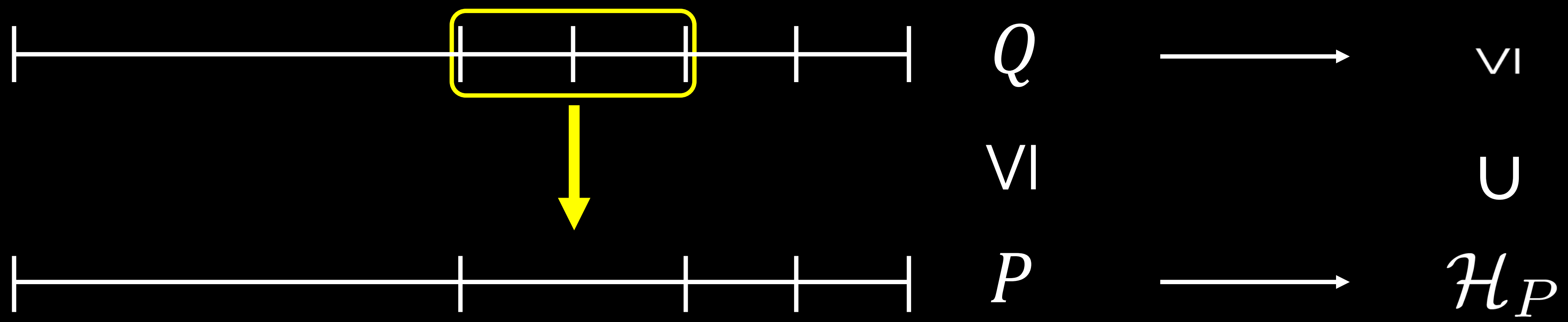
a **sequence** (net) of effective hilbert spaces & hamiltonians

(for standard dyadic partitions)

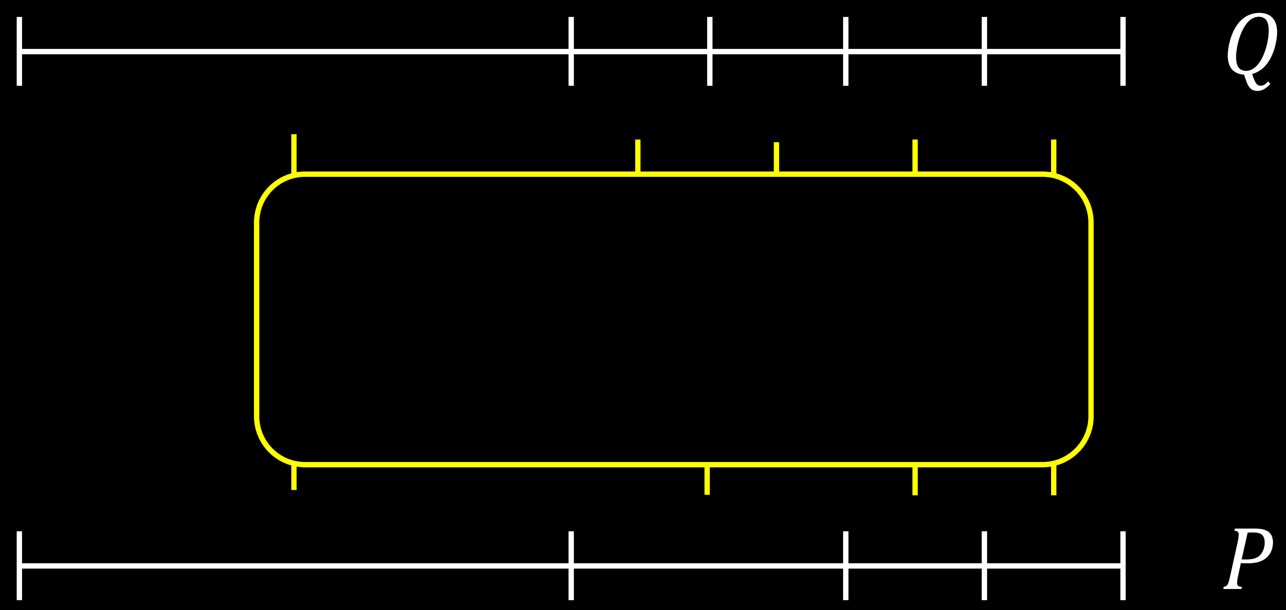

R
$\mathcal{H}_R$
Q
$\mathcal{H}_Q$
P
$\mathcal{H}_P$

If $P \leq Q$ identify $\mathcal{H}_P \subset \mathcal{H}_Q$ via isometry:

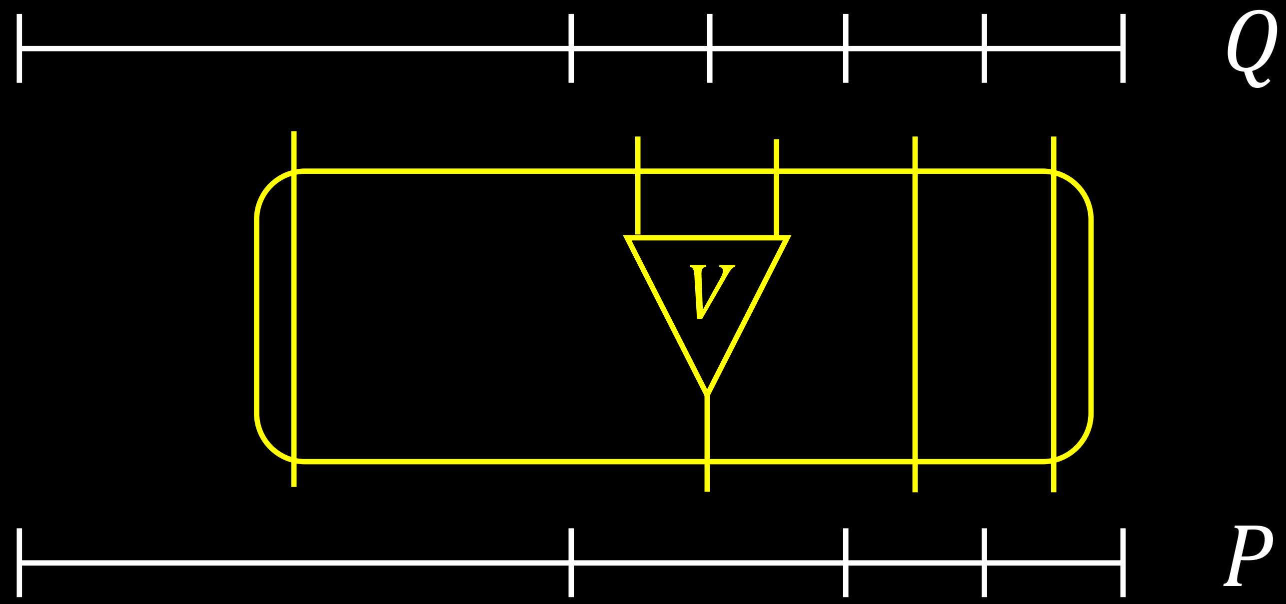
$$T_Q^P : \mathcal{H}_P \to \mathcal{H}_Q$$



$$T_Q^P =$$



$$T^P_Q =$$

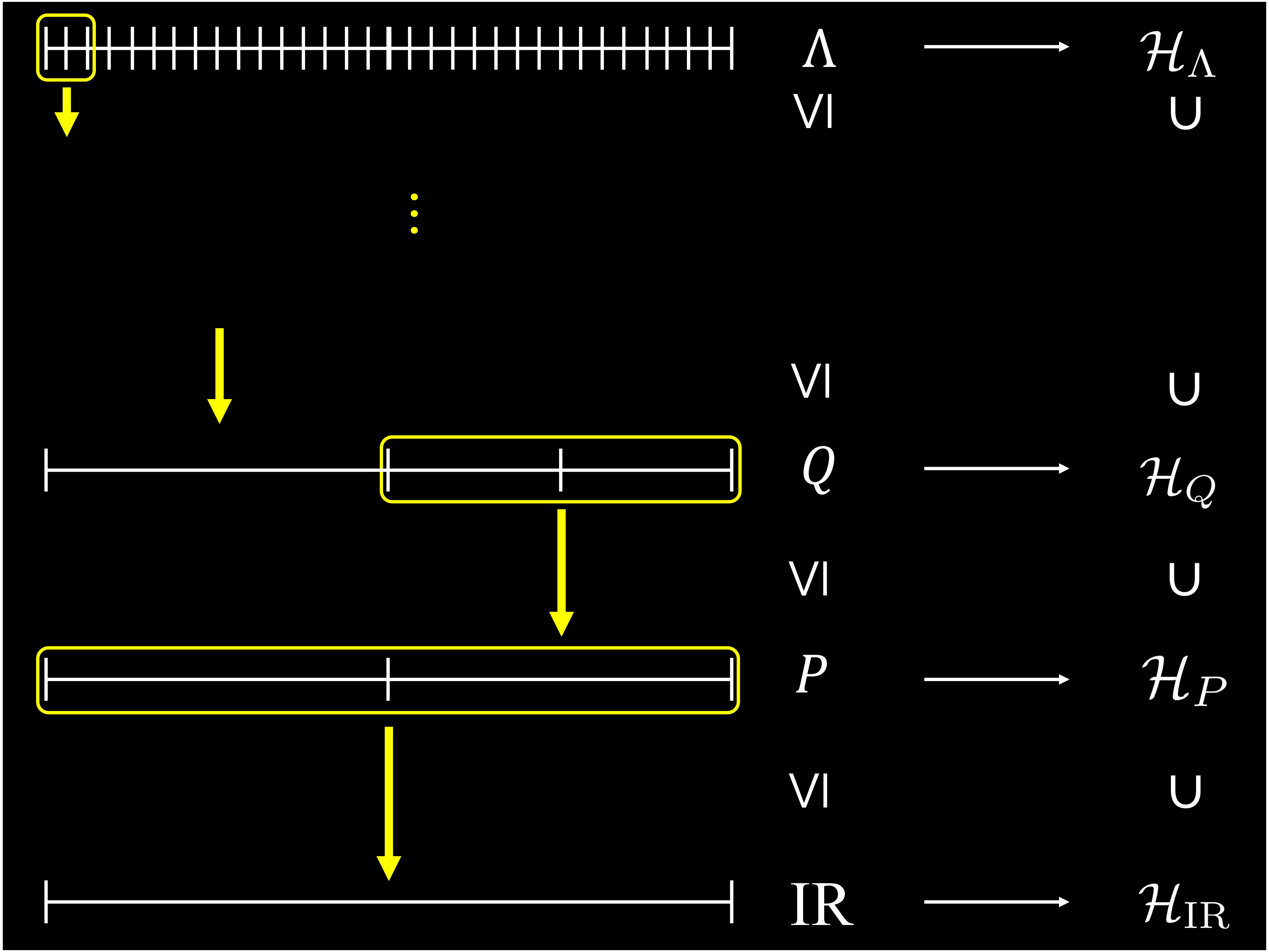


$$T_Q^P : \mathcal{H}_P \to \mathcal{H}_Q$$

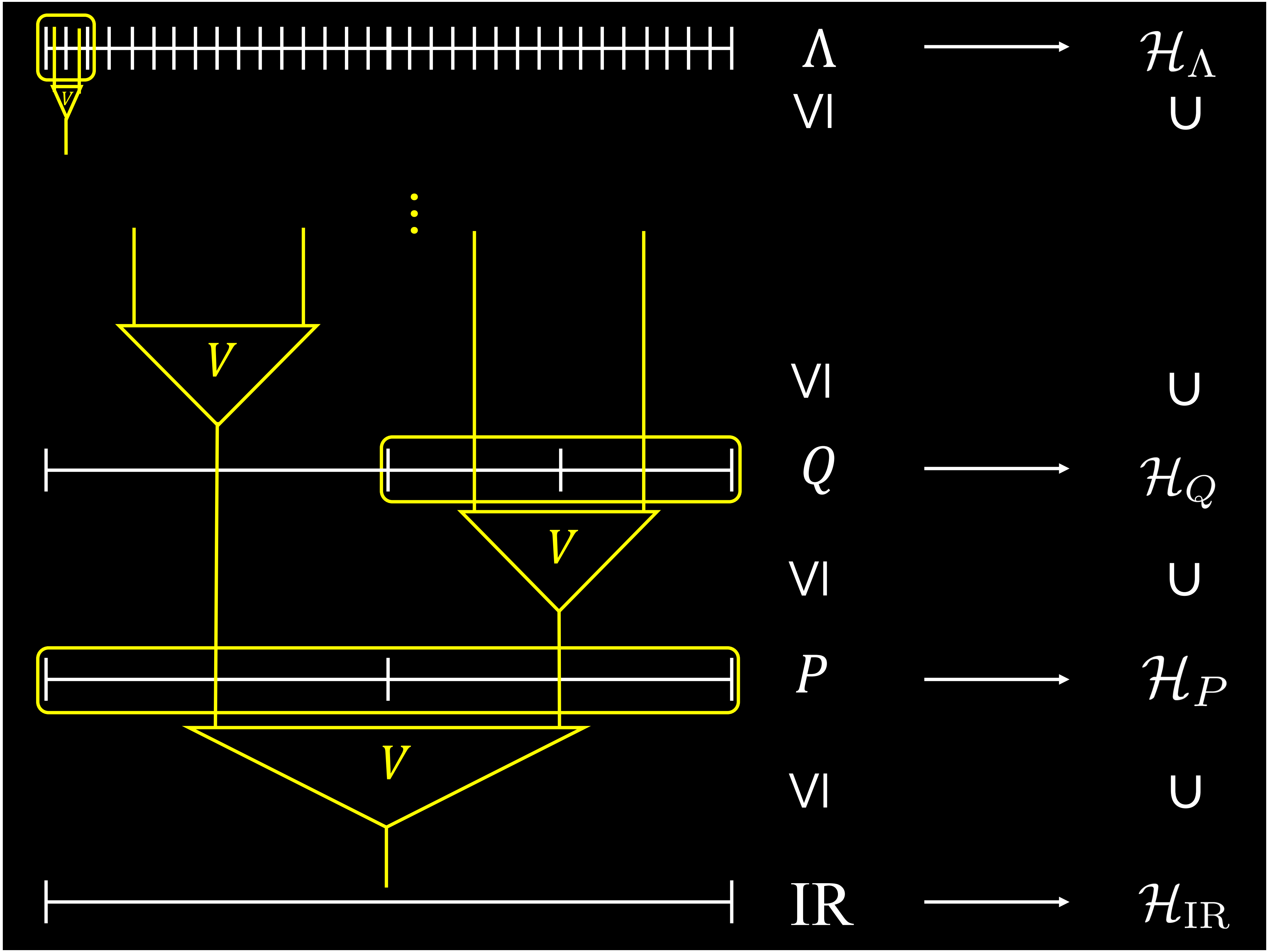
every low energy state in $\mathcal{H}_P$ is identified with a **precursor** in $\mathcal{H}_Q$

$$T_Q^P : \mathcal{H}_P \to \mathcal{H}_Q$$

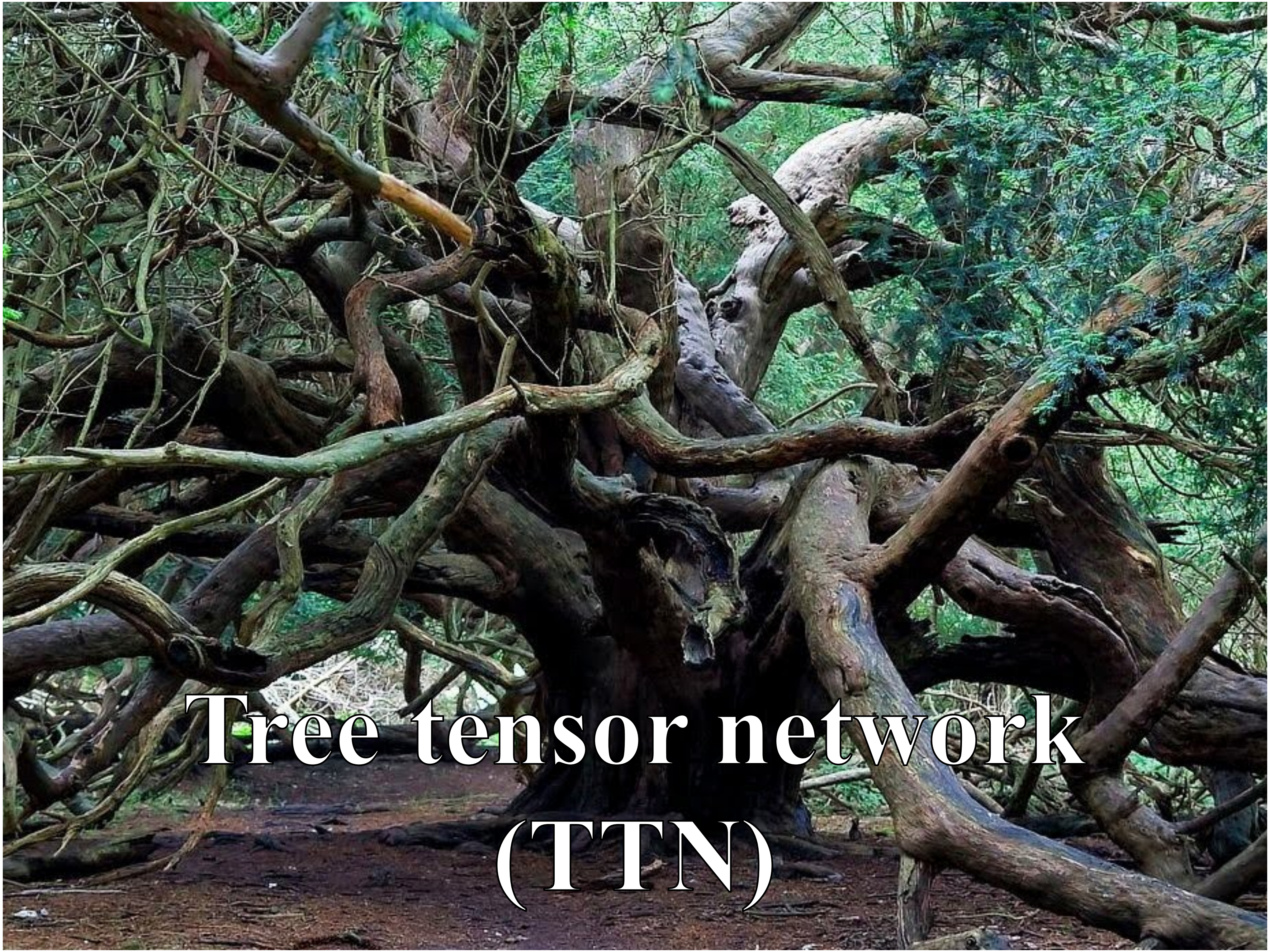
isometry T_Q^P
is an **inverse** to the
Kadanoff block RG
from P to Q



$\Lambda \longrightarrow \mathcal{H}_\Lambda$
$\vee\!|$ $\cup$
$\vee\!|$ $\cup$
$Q \longrightarrow \mathcal{H}_Q$
$\vee\!|$ $\cup$
$P \longrightarrow \mathcal{H}_P$
$\vee\!|$ $\cup$
$\mathrm{IR} \longrightarrow \mathcal{H}_{\mathrm{IR}}$



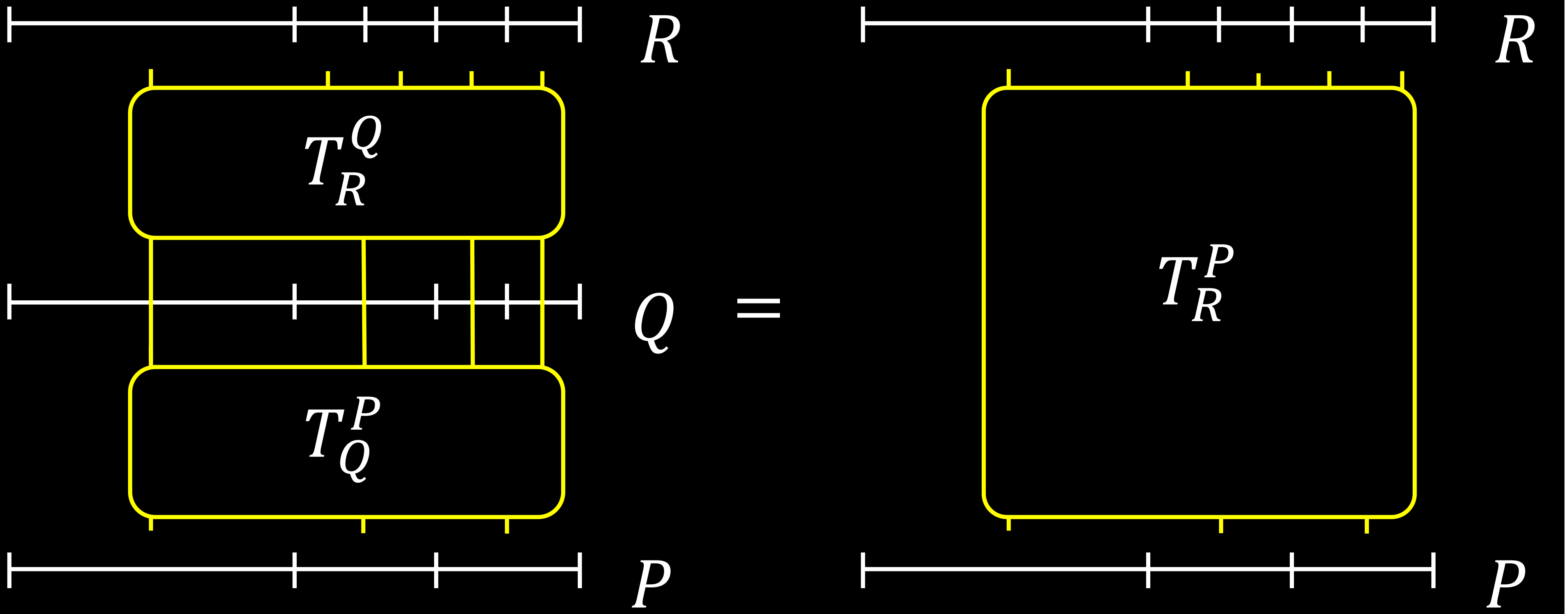
$\Lambda \longrightarrow \mathcal{H}_\Lambda$
$\vee\!\mid$ $\cup$
$\vee\!\mid$ $\cup$
$Q \longrightarrow \mathcal{H}_Q$
$\vee\!\mid$ $\cup$
$P \longrightarrow \mathcal{H}_P$
$\vee\!\mid$ $\cup$
$\mathrm{IR} \longrightarrow \mathcal{H}_{\mathrm{IR}}$
V
V
V
V



Tree tensor network (TTN)

Demand WLOG

$$T_R^Q T_Q^P = T_R^P, \quad \forall P \leq Q \leq R$$

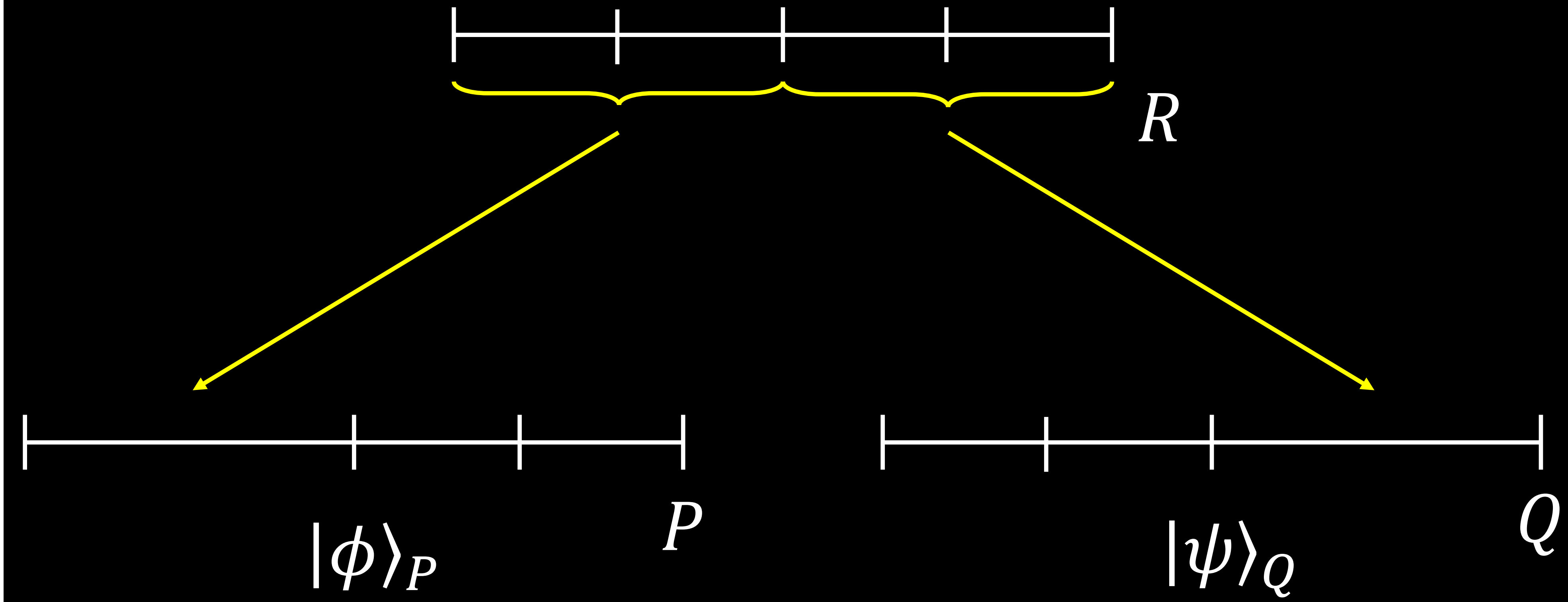


T_R^Q
T_Q^P
R
Q
P
$=$
T_R^P
R
P

Equivalence: $|\phi\rangle_P \sim |\psi\rangle_Q$

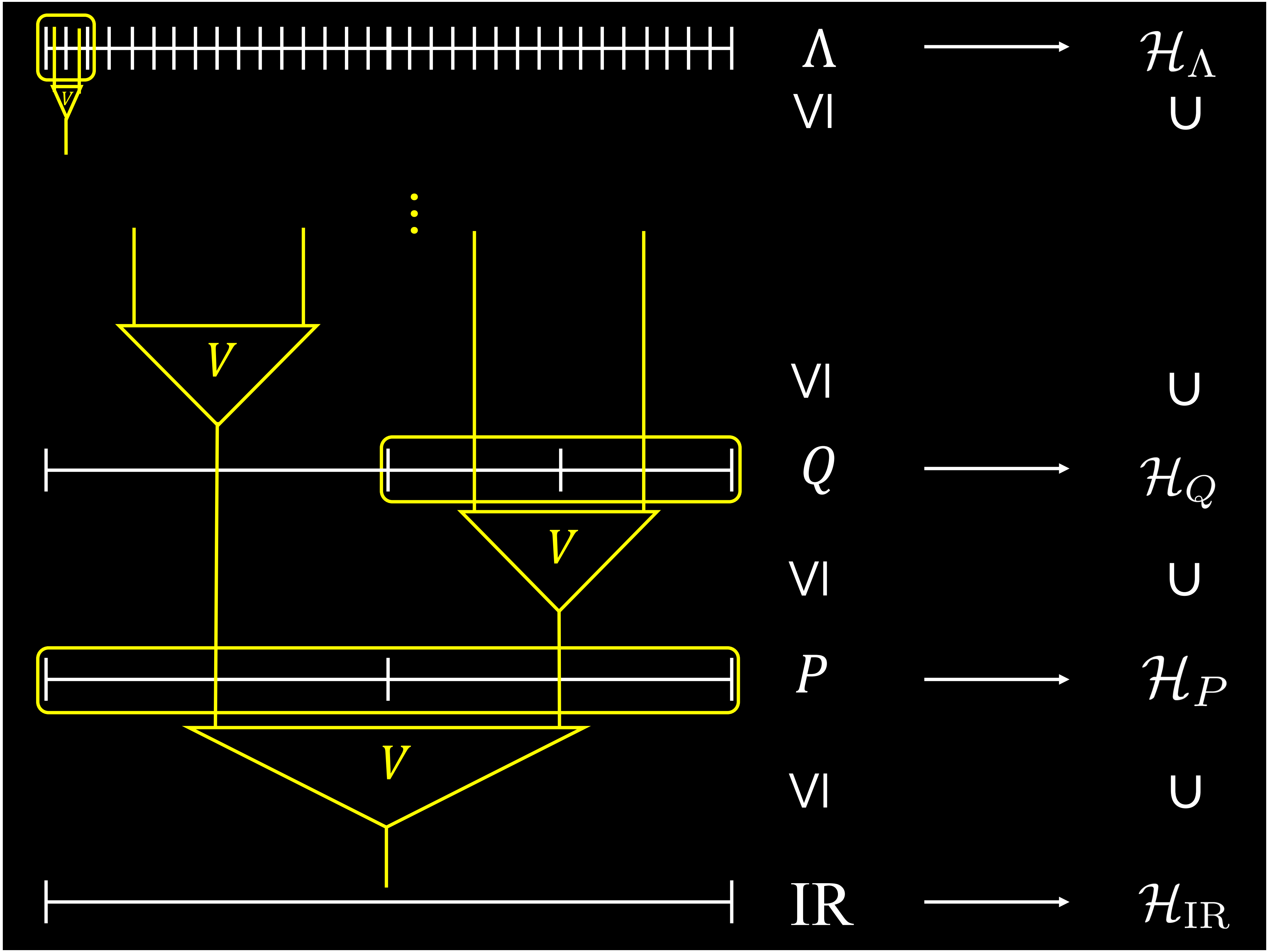
if $\exists R$

$$T_R^P|\phi\rangle_P = T_R^Q|\psi\rangle_Q$$

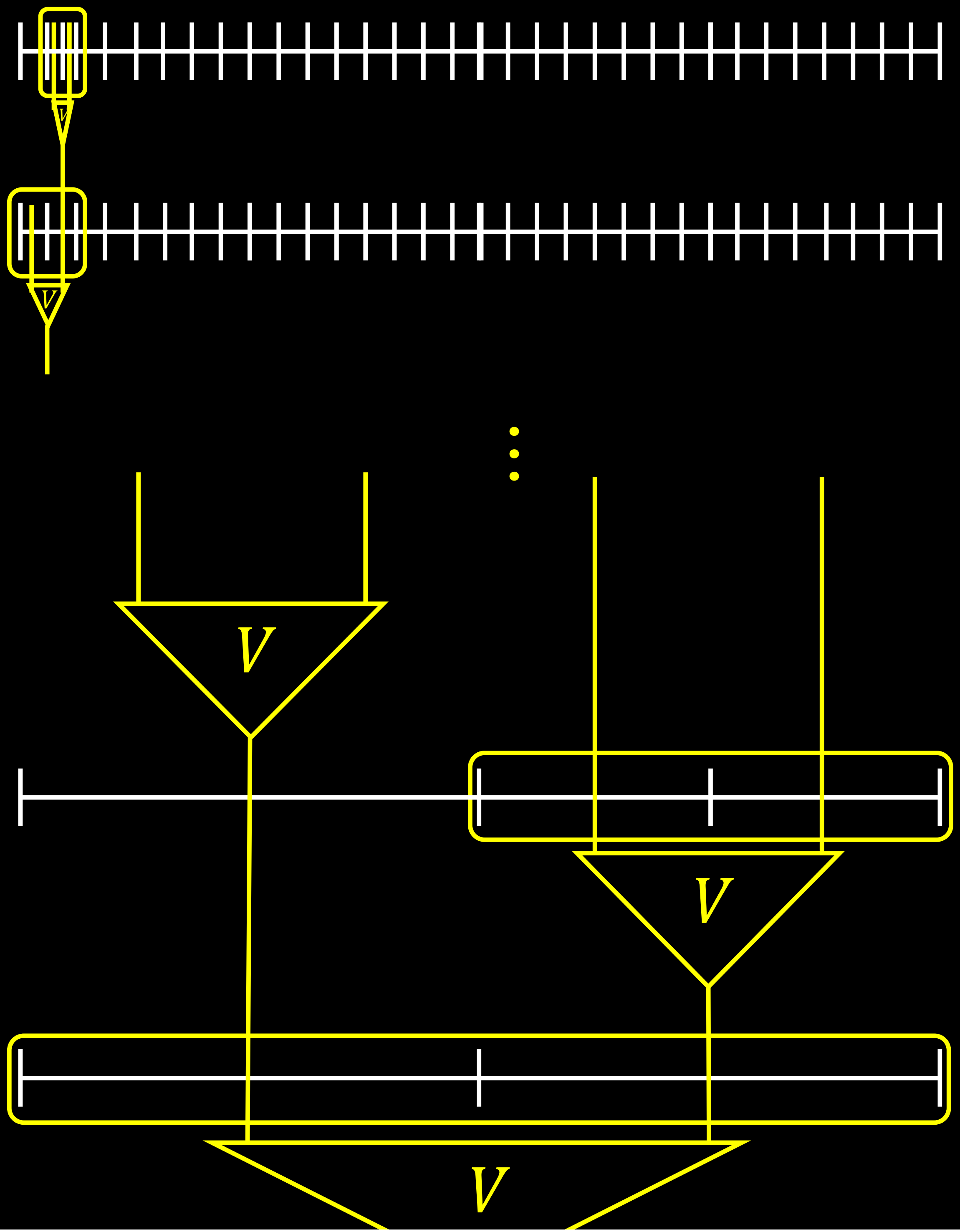


"Continuous limit": Extrapolate!

T_Q^P embeds into arbitrarily fine (std. dyadic) lattices



Λ
$\mathcal{H}_\Lambda$
V
V
Q
$\mathcal{H}_Q$
V
P
$\mathcal{H}_P$
V
IR
$\mathcal{H}_{\mathrm{IR}}$



$\Lambda \longrightarrow \mathcal{H}_\Lambda$

VI U

VI U

$Q \longrightarrow \mathcal{H}_Q$

VI U

$P \longrightarrow \mathcal{H}_P$

VI U

Definition: let $(\mathcal{P}, \leq)$ be a directed set. Let a hilbert space $\mathcal{H}_P$ be given for each $P \in \mathcal{P}$
For all $P \leq Q$ let $T_Q^P : \mathcal{H}_P \to \mathcal{H}_Q$ be an isometry such that:

(1) T_P^P is the identity
(2) $T_R^Q T_Q^P = T_R^P, \quad \forall P \leq Q \leq R$

Then $(\mathcal{H}_P, T_Q^P)$ is a **directed system**.

Precontinuous limit:

$$\widehat{\mathcal{H}} \equiv \varinjlim_{P \in \mathcal{P}} \mathcal{H}_P$$

$$= \left\{ \begin{array}{l} \text{the disjoint union of } \mathcal{H}_P \text{ over all } P \in \mathcal{P} \\ \text{modulo the equivalence relation } |\phi\rangle_P \sim |\psi\rangle_Q \\ \text{if there is } R \geq P \text{ and } R \geq Q \text{ such that} \\ \qquad\qquad T_R^P\,|\phi\rangle_P = T_R^Q\,|\psi\rangle_Q \end{array} \right\}$$

1. any book on algebra
2. R. F. Werner, unpublished (1993)
3. V. F. R. Jones, arXiv:1412.7740 (2014)

Residents of $\widehat{\mathcal{H}}$:

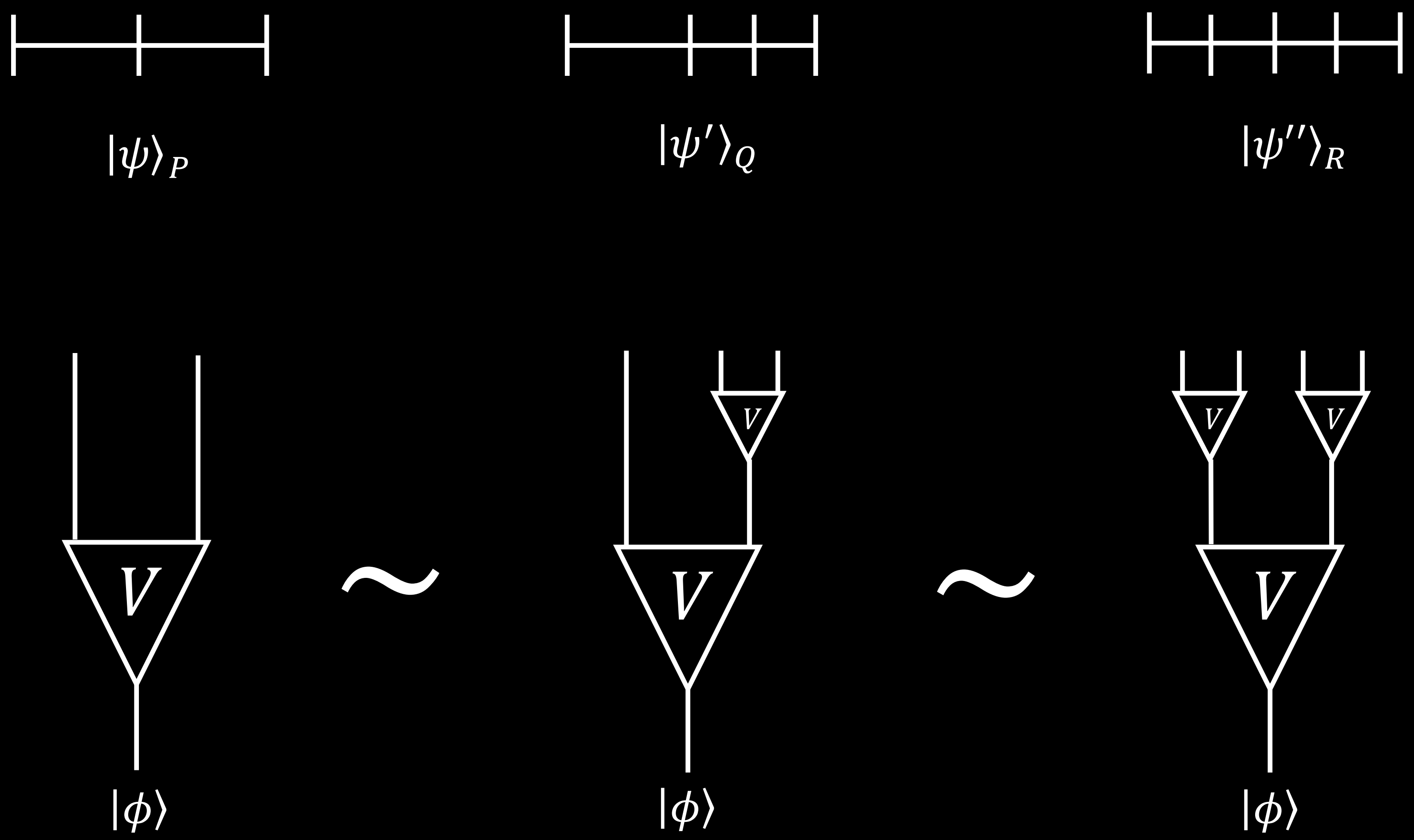
$$[|\psi\rangle_P] \equiv \{|\phi\rangle_Q = T_Q^P |\psi\rangle_P\}$$

Each hilbert space $\mathcal{H}_P$ is a natural subspace of $\widehat{\mathcal{H}}$:

$$\mathcal{H}_P \hookrightarrow \widehat{\mathcal{H}}$$

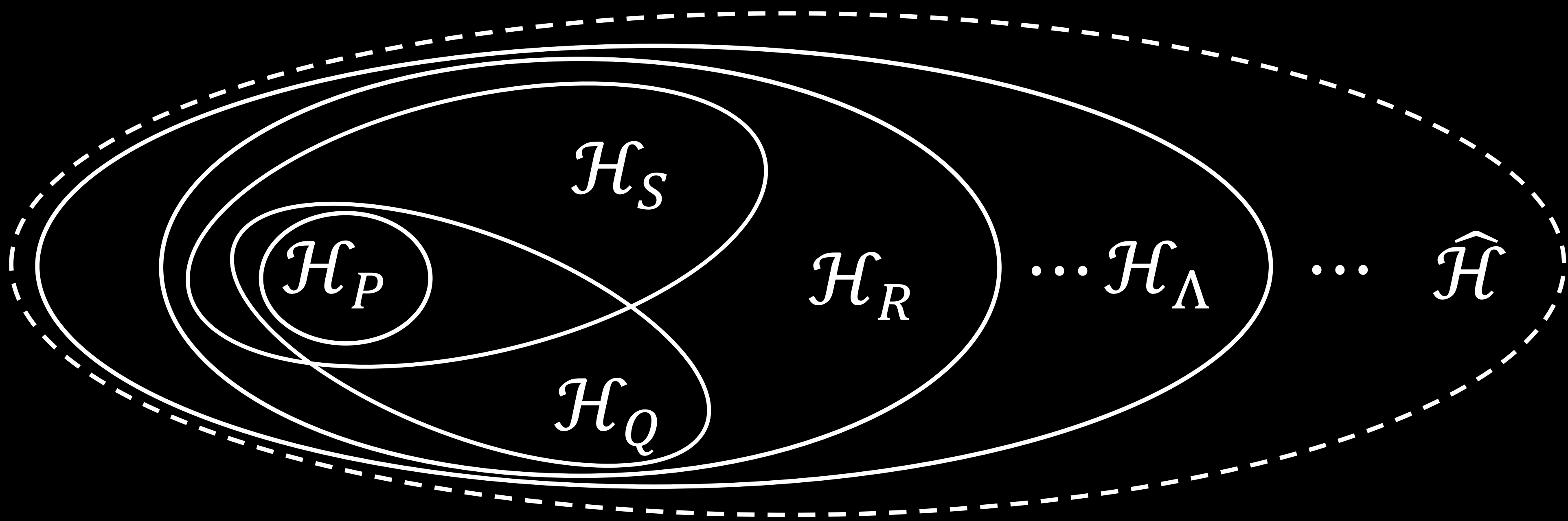
via

$$|\psi\rangle_P \longmapsto [|\psi\rangle_P]$$


$|\psi\rangle_P$
$|\psi'\rangle_Q$
$|\psi''\rangle_R$
V
V
V
V
V
V
$|\phi\rangle$
$|\phi\rangle$
$|\phi\rangle$

What did we really need?

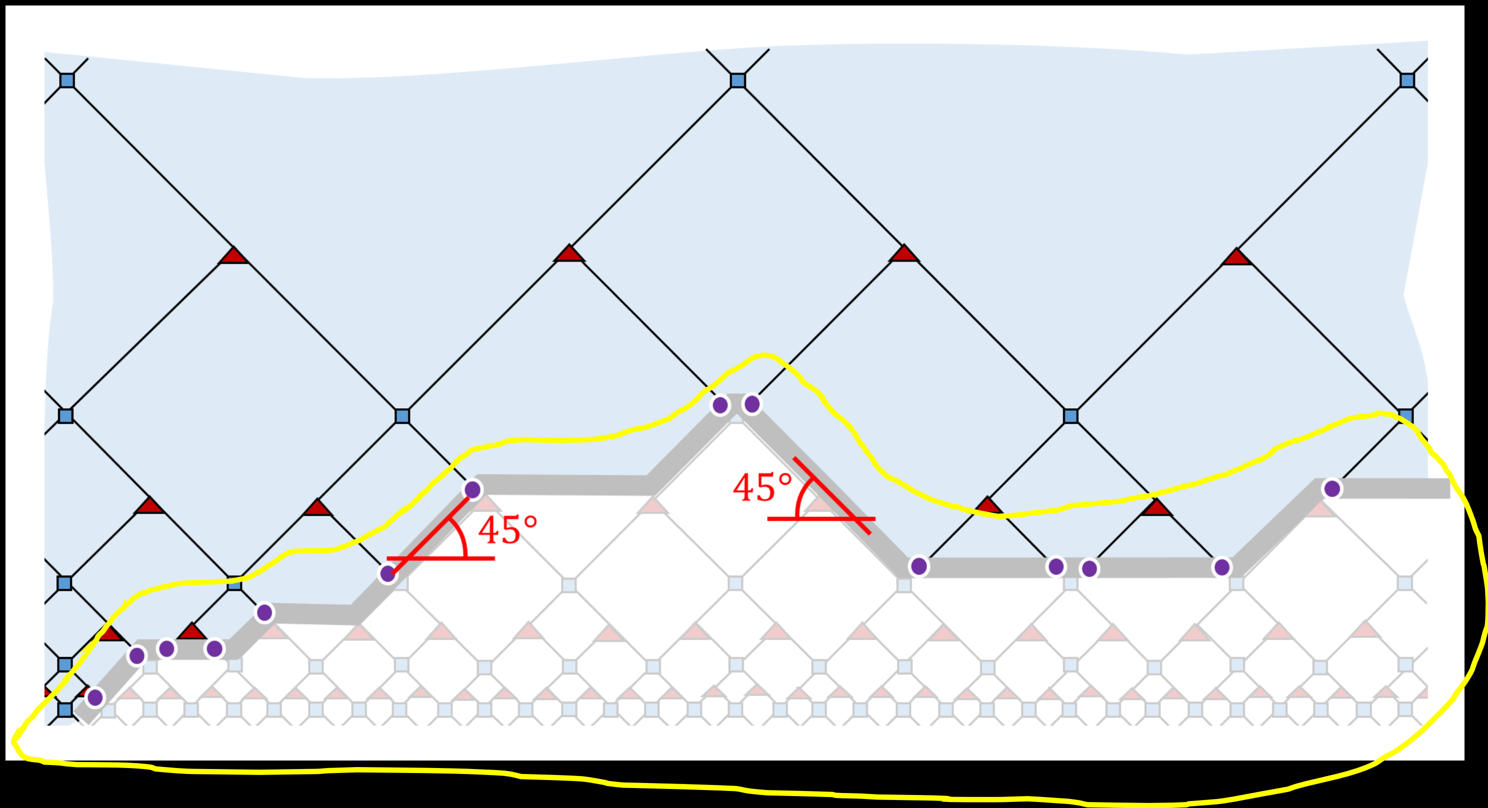
1. Hilbert spaces $\mathcal{H}_P$ for partitions P (parametrised by TN)
2. Isometries $T_R^Q T_Q^P = T_R^P$ for each $P \leq Q \leq R$



Examples

1. Tree tensor networks
2. (Injective) MPS
3. (Injective) PEPS
4. MERA

$$T_Q^P =$$



B. Czech, G. Evenbly, L. Lamprou, S. McCandlish, X.-L. Qi, J. Sully, and G. Vidal, arXiv:1510.07637 (2015).

Unitary actions of homeomorphisms (AKA “fields”)

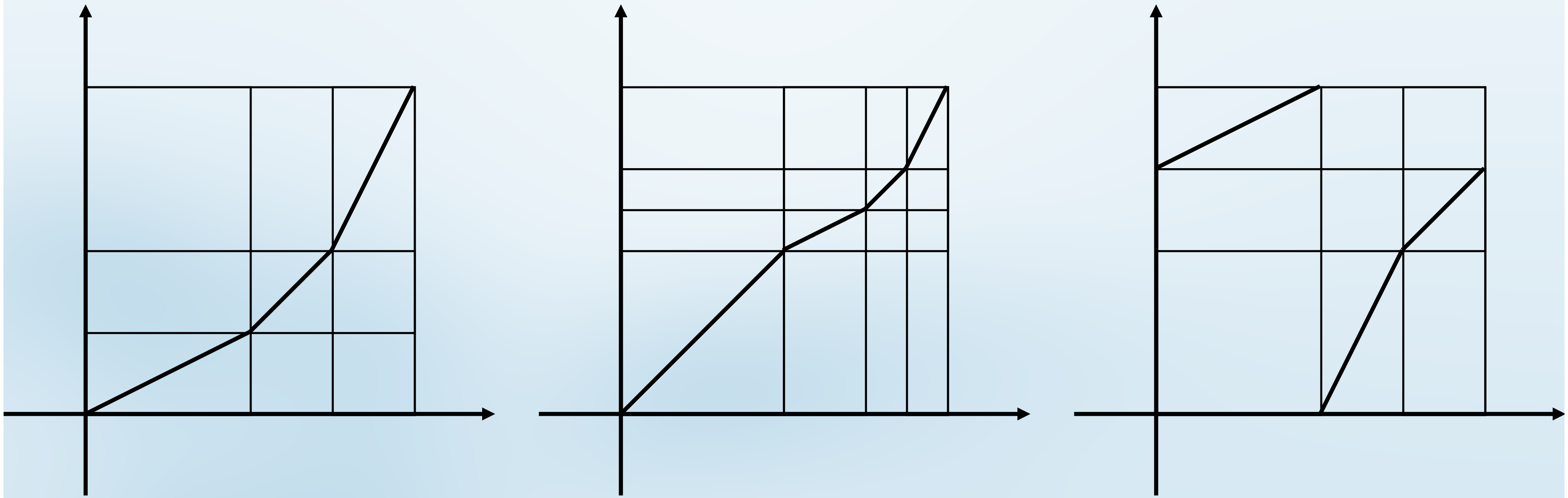
CFT Dream: find a unitary action of $\mathrm{conf}(\mathbb{R}^{1,1})$ on $\widehat{\mathcal{H}}$

$$\mathrm{conf}(\mathbb{R}^{1,1}) \cong \mathrm{diff}_+(S^1) \times \mathrm{diff}_+(S^1)$$

Problem: $\mathrm{diff}_+(S^1)$ is incompatible with std. dyadic partitions

Strategy: study “discrete” version of conformal group; Thompson’s group T

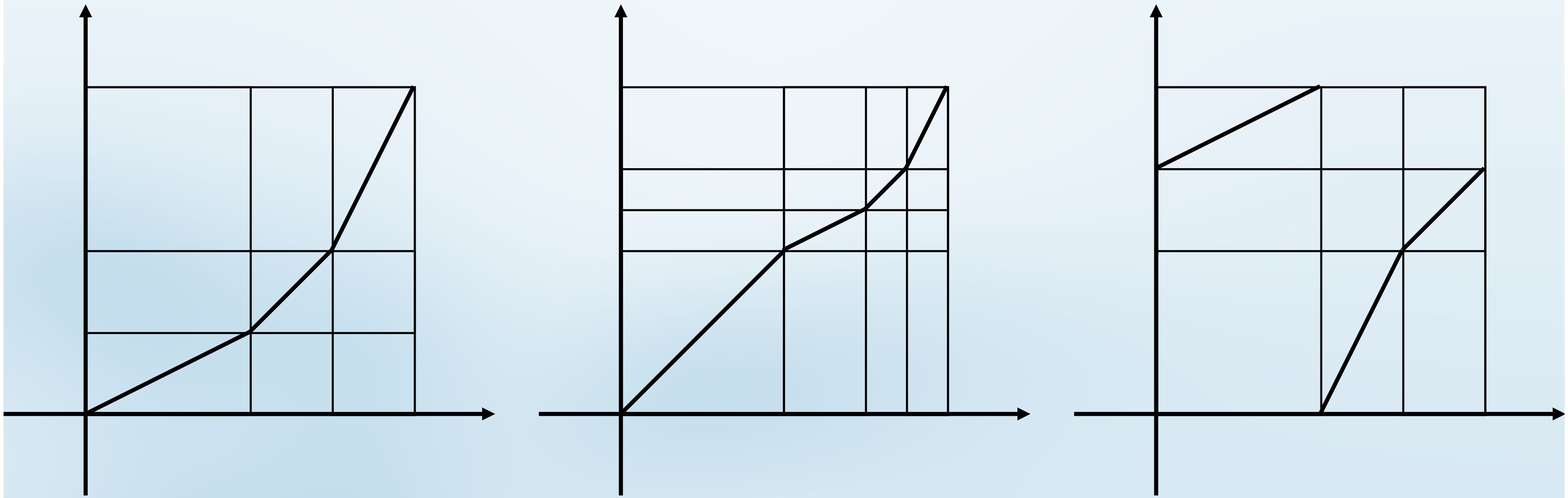
Thompson's group T: generated by $A(x)$, $B(x)$, and $C(x)$ under composition



$A(x)$ $B(x)$ $C(x)$

J. W. Cannon, W. J. Floyd, and W. R. Parry, *Enseign. Math.*, vol. 42, no. 3–4, pp. 215 – 256,

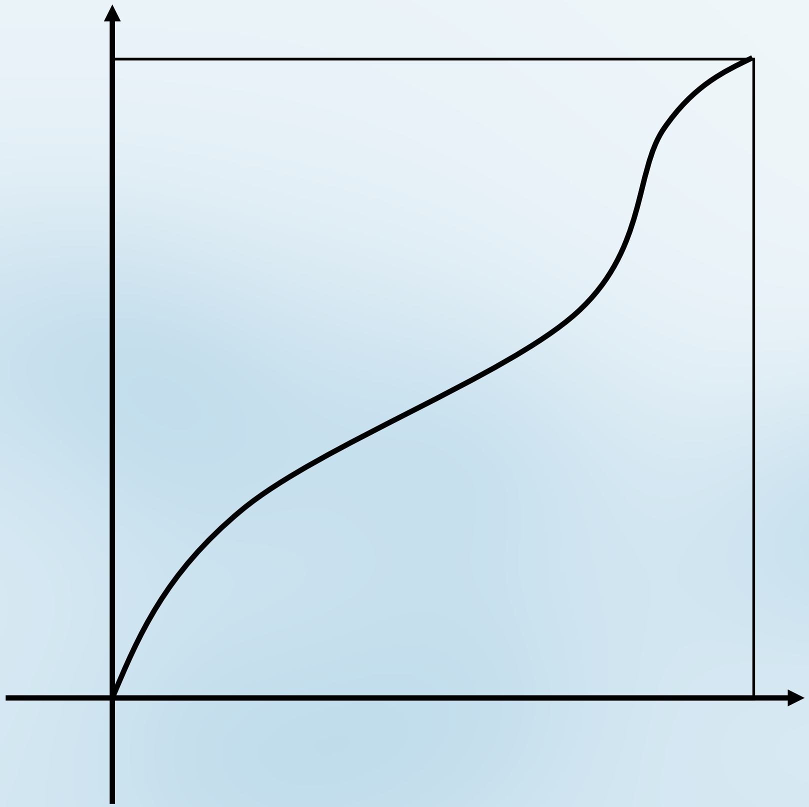
Thompson's group *T*: *compatible* with Verband of dyadic rational partitions



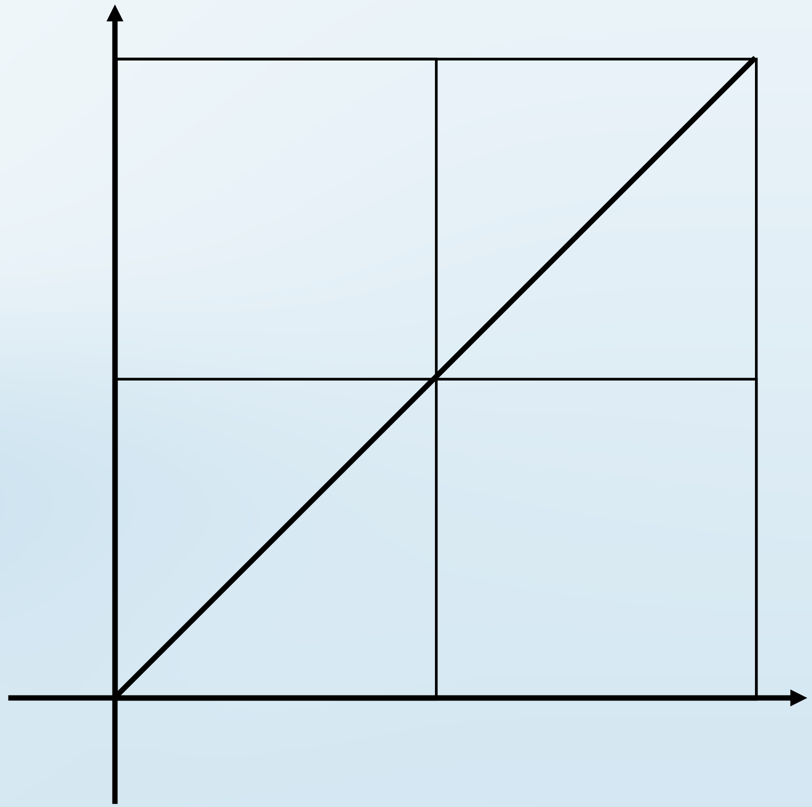
$A(x)$ $B(x)$ $C(x)$

J. W. Cannon, W. J. Floyd, and W. R. Parry, *Enseign. Math.*, vol. 42, no. 3–4, pp. 215 – 256,

Proposition ("well known"): let $f \in \mathrm{diff}_+(S^1)$. Then $\exists$ sequence $A_n(x) \in T$ s.t. $\|A_n - f\|_\infty \to 0$.



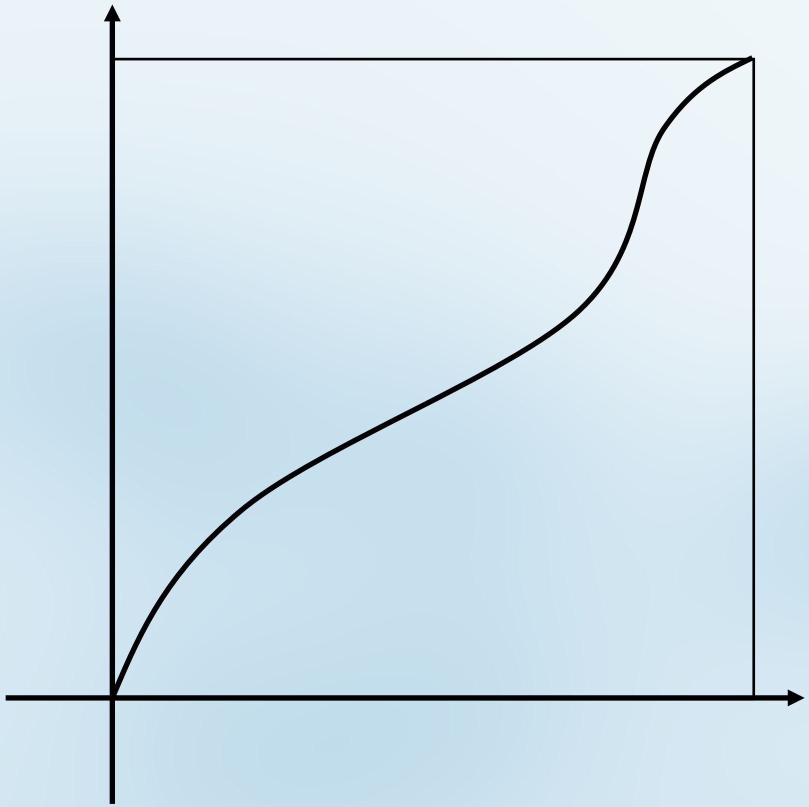
$f(x)$



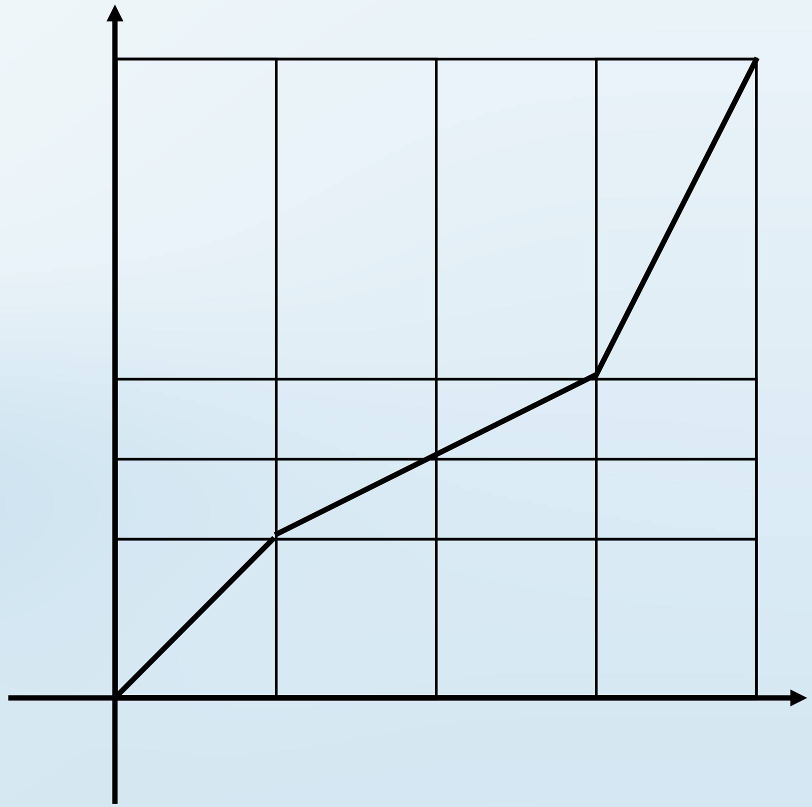
$A_1(x)$

see e.g., A. Akhmedov and M. P. Cohen, arXiv:1508.04604

Proposition ("well known"): let $f \in \mathrm{diff}_+(S^1)$. Then $\exists$ sequence $A_n(x) \in T$ s.t. $\|A_n - f\|_\infty \to 0$.



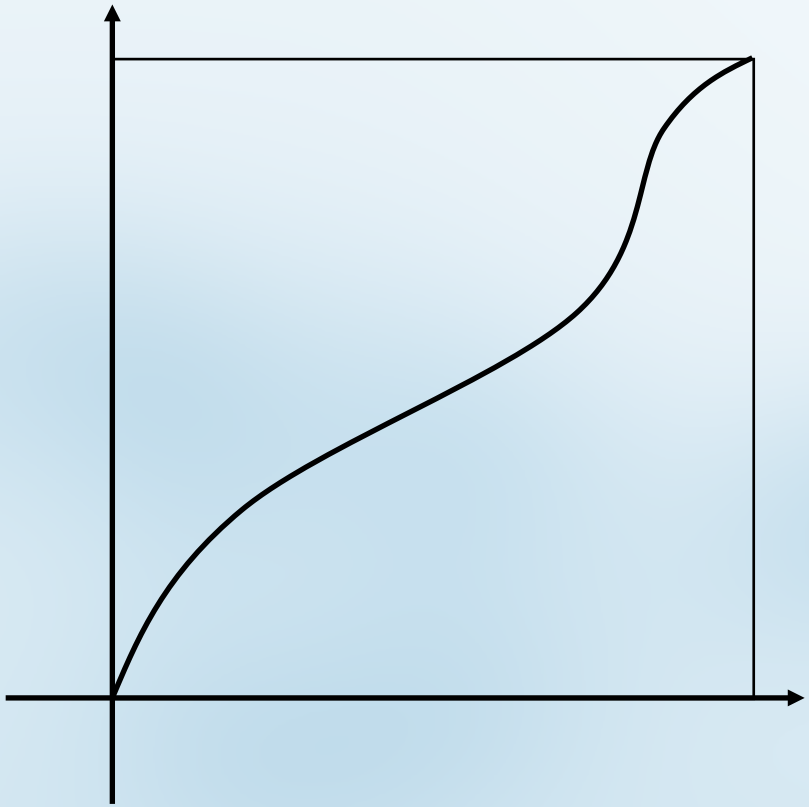
$f(x)$



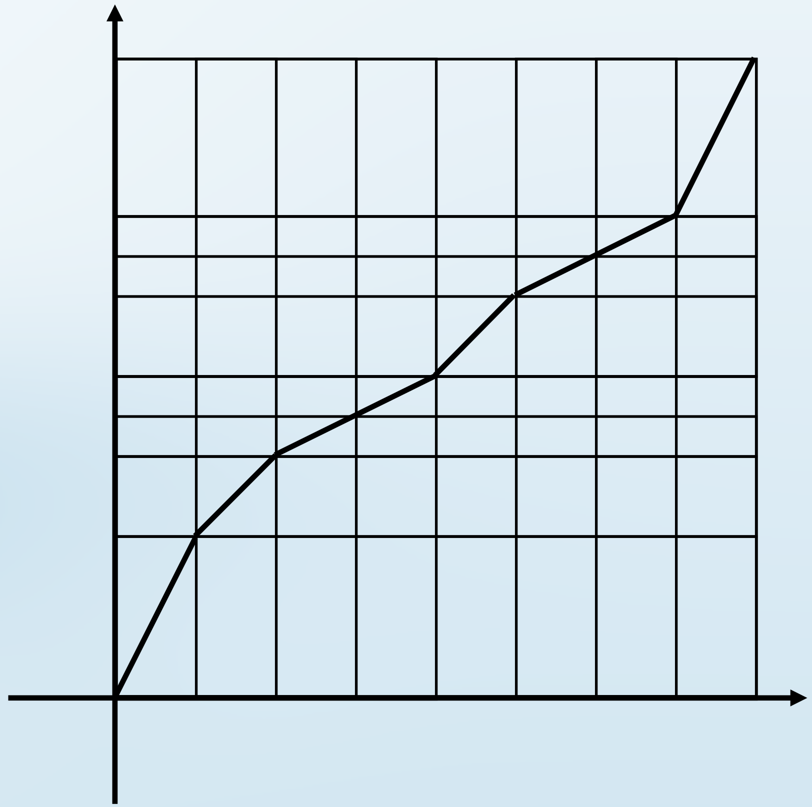
$A_2(x)$

see e.g., A. Akhmedov and M. P. Cohen, arXiv:1508.04604

Proposition ("well known"): let $f \in \mathrm{diff}_+(S^1)$. Then $\exists$ sequence $A_n(x) \in T$ s.t. $\|A_n - f\|_\infty \to 0$.



$f(x)$



$A_3(x)$

see e.g., A. Akhmedov and M. P. Cohen, arXiv:1508.04604

Unitary representations of T on $\widehat{\mathcal{H}}$

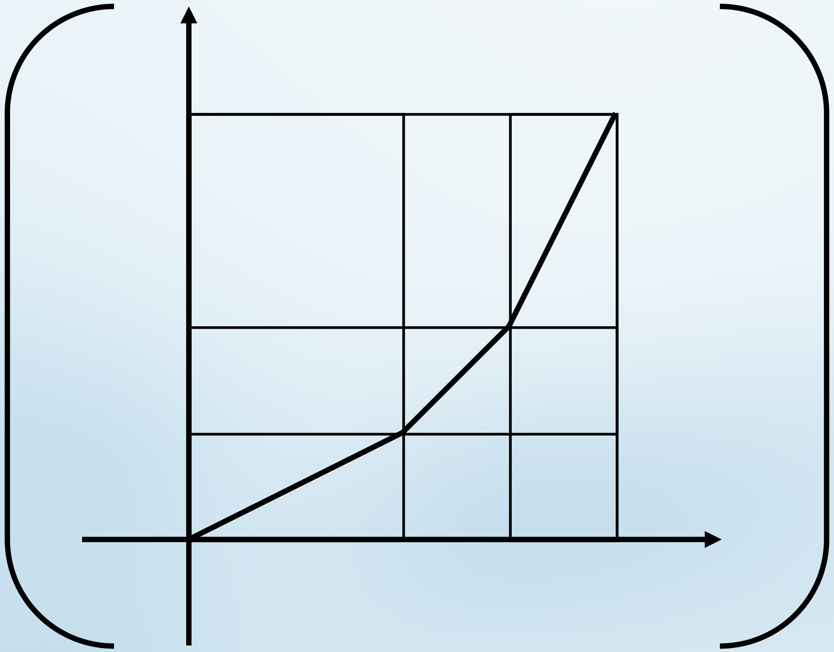
$$\forall f \in T:$$

$$\pi(f)[|\psi\rangle_P] = \left[|\psi\rangle_{f(P)}\right]$$

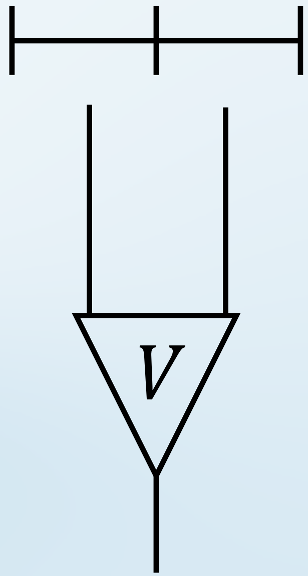
V. F. R. Jones, arXiv:1412.7740 (2014)

Example:

π



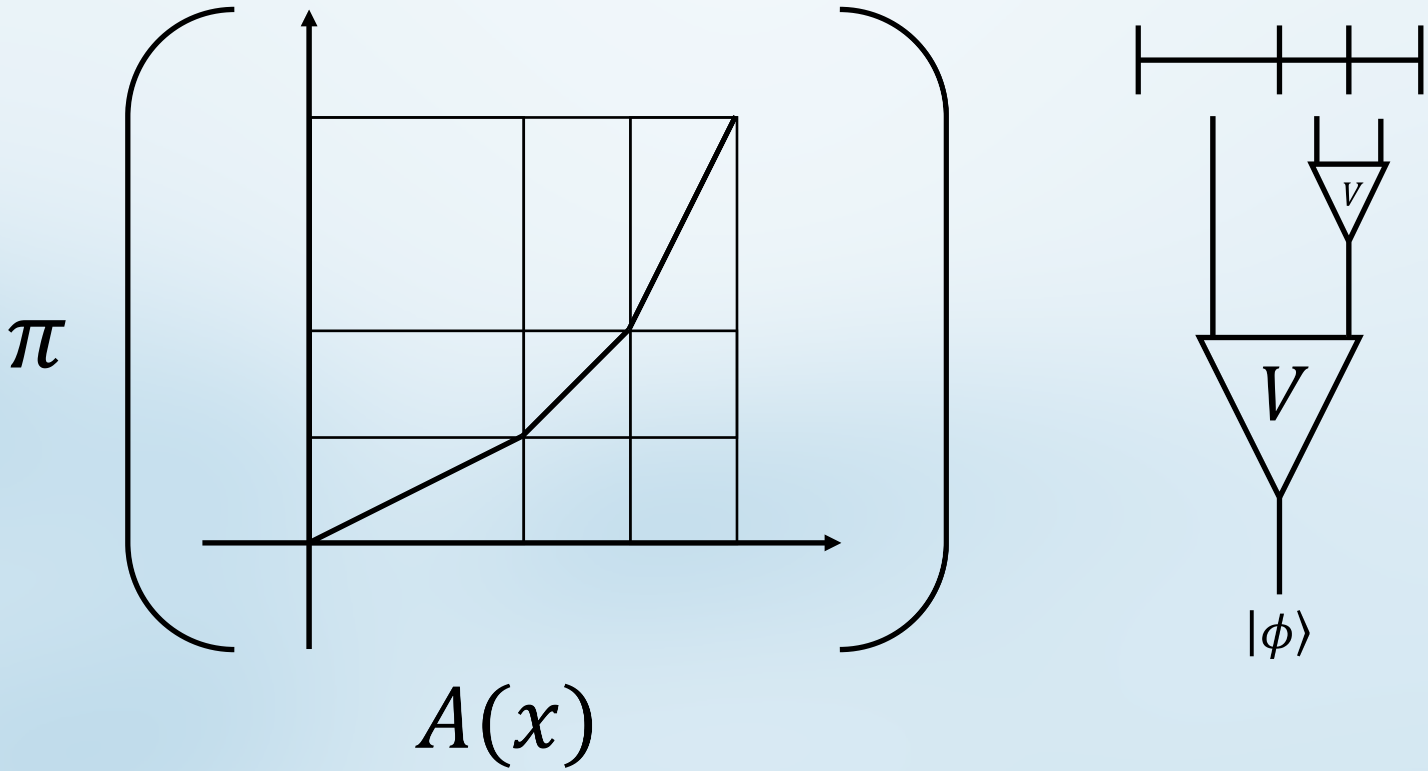
$A(x)$



$|\phi\rangle$

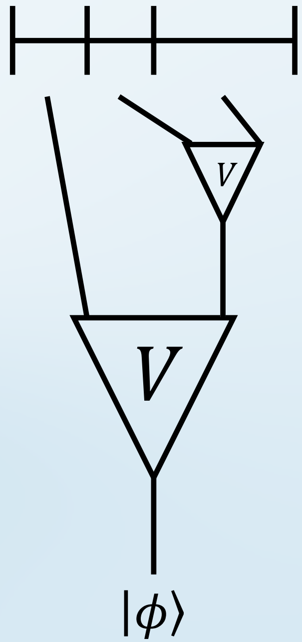
V. F. R. Jones, arXiv:1412.7740 (2014)

Example:



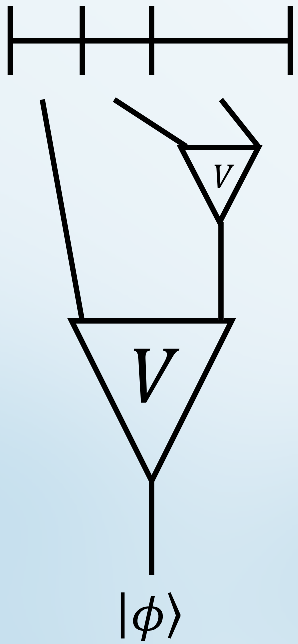
V. F. R. Jones, arXiv:1412.7740 (2014)

Example:



V. F. R. Jones, arXiv:1412.7740 (2014)

Is



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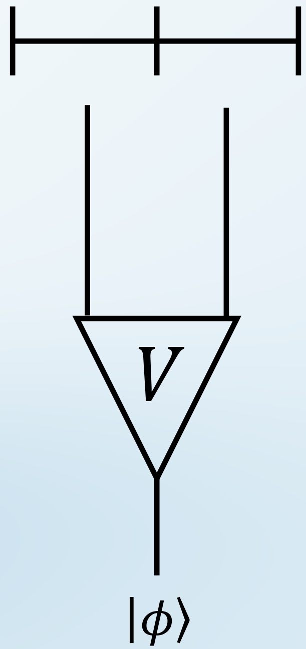

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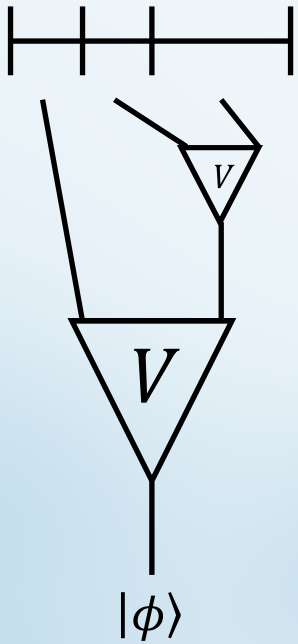
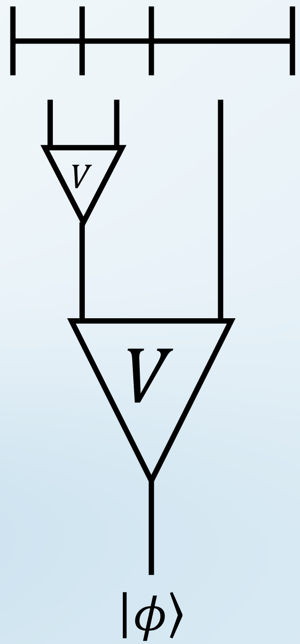
V. F. R. Jones, arXiv:1412.7740 (2014)

Is 

= 

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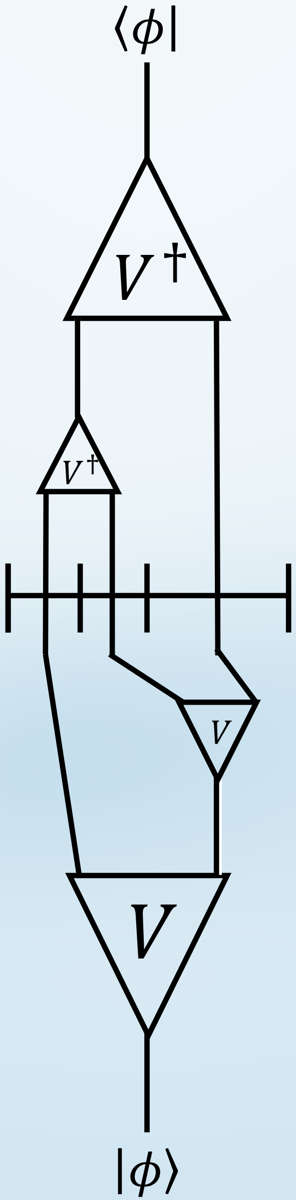
V. F. R. Jones, arXiv:1412.7740 (2014)

Fidelity:

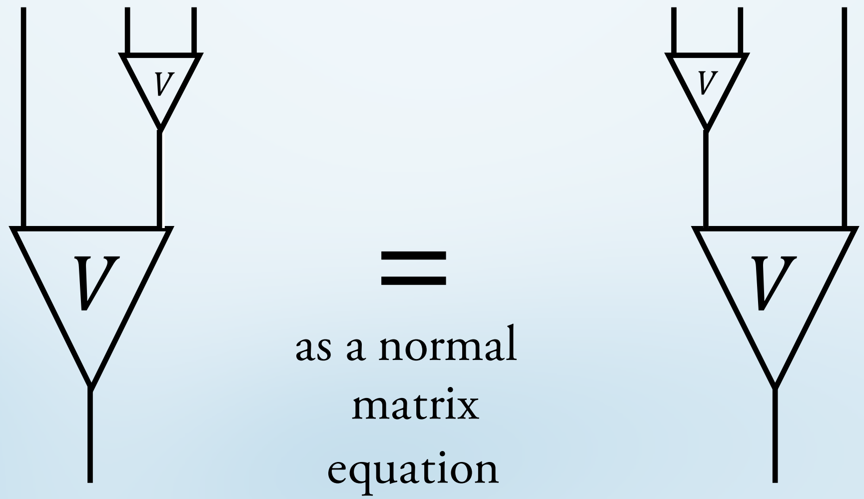


V. F. R. Jones, arXiv:1412.7740 (2014)

Thompsonian invariance:

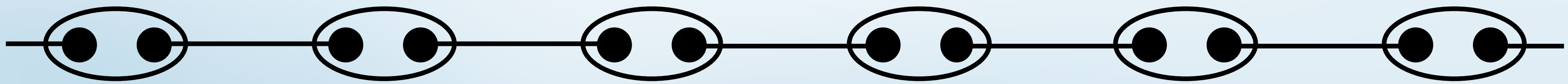
$$\pi(a)[|\Omega\rangle_P] = [|\Omega\rangle_P]?$$

Necessary condition for **exact Thompsonian invariance:** V must satisfy



Physical argument: it is also sufficient

Conjecture: fixed points are **product states**



Unitary action is not continuous!
(With respect to compact-open topology on T and standard topology on $\widehat{\mathcal{H}}$)

Translation: we can always **distinguish** between $\pi(f)|\psi\rangle$ and $\pi(f')|\psi\rangle$ no matter how close f and f'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DREAMS
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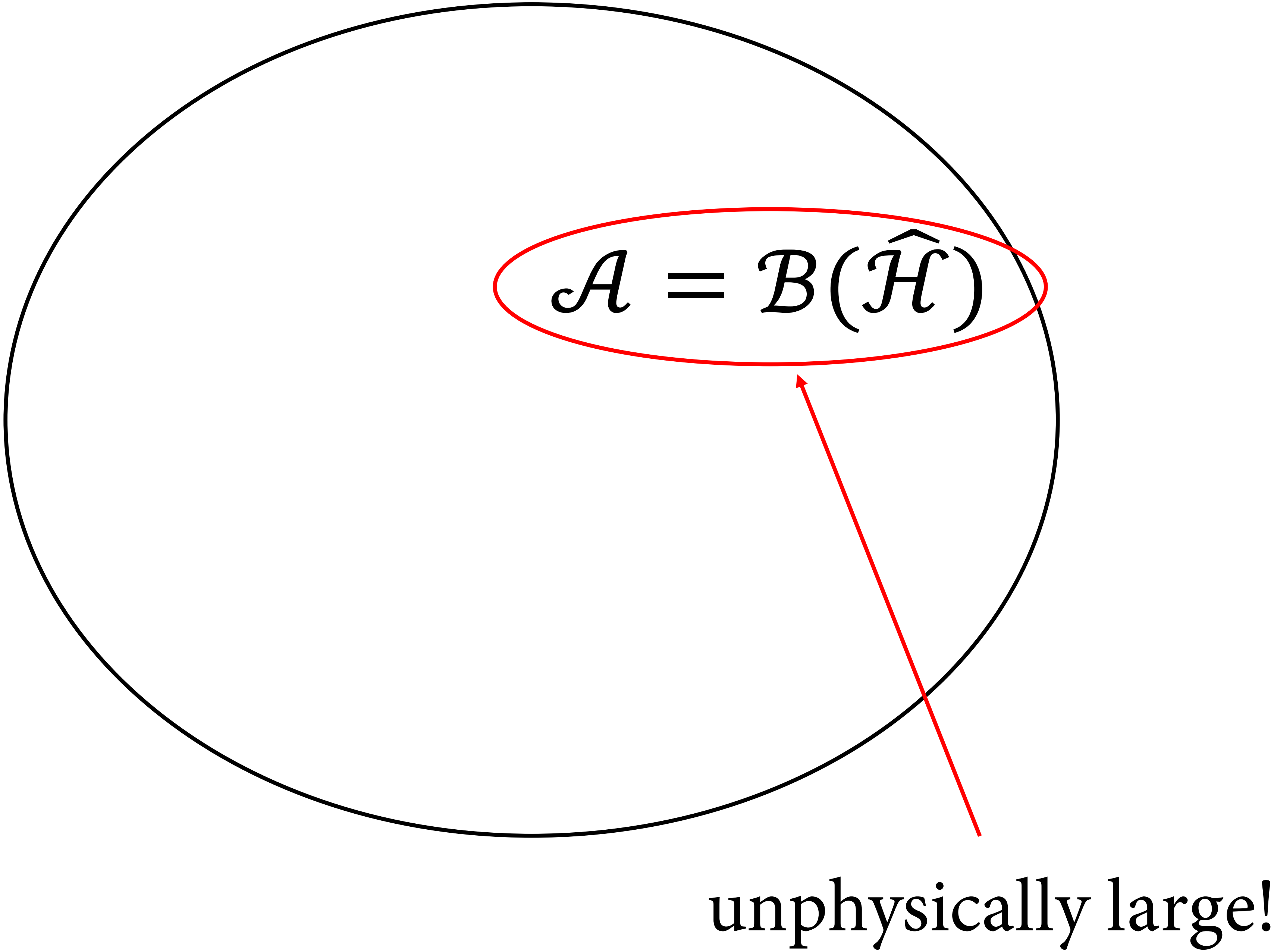
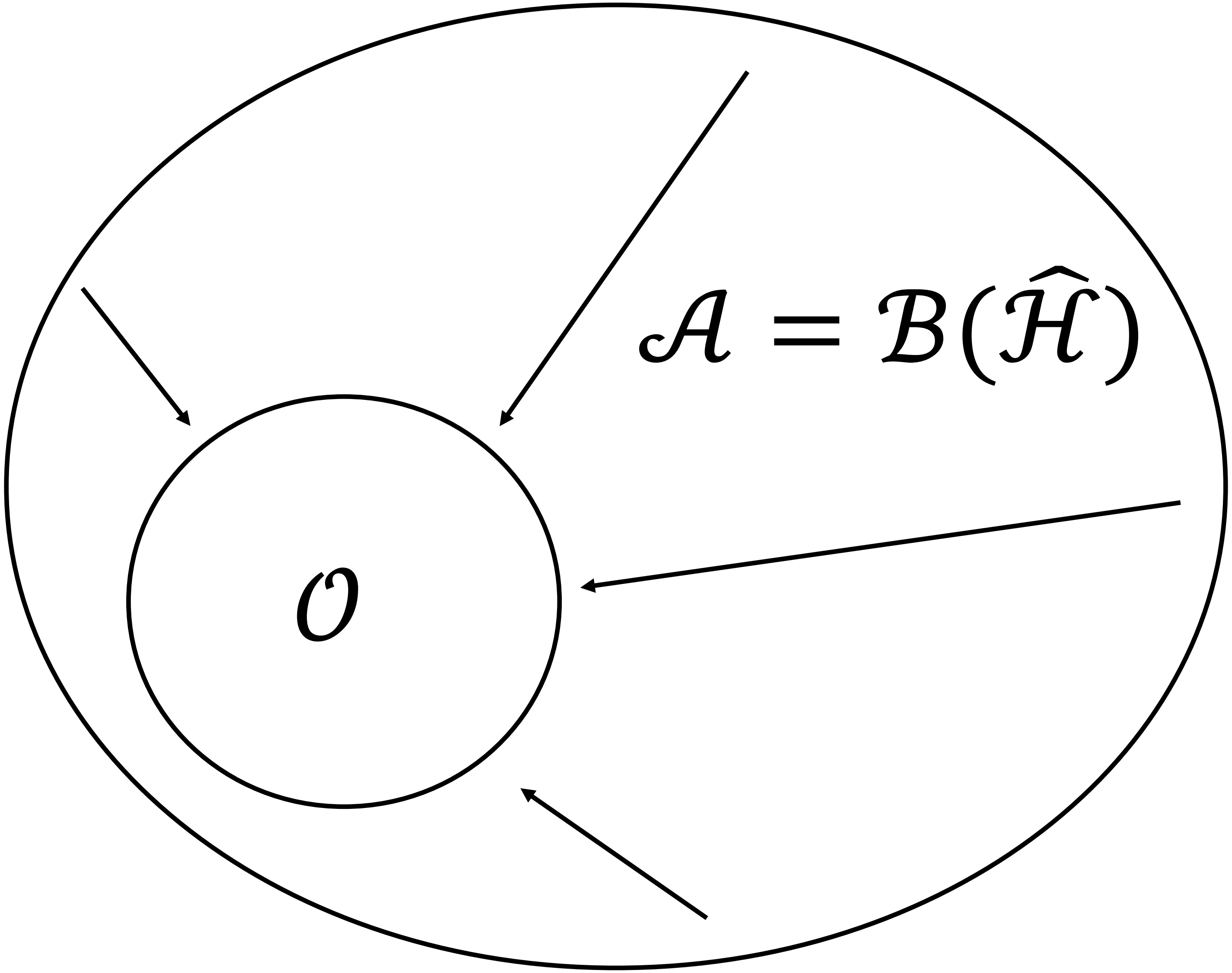
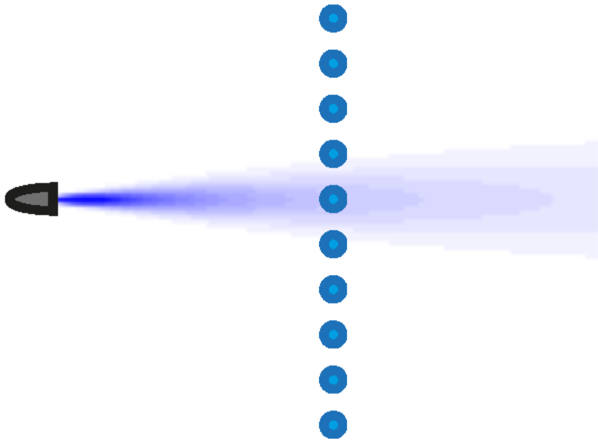
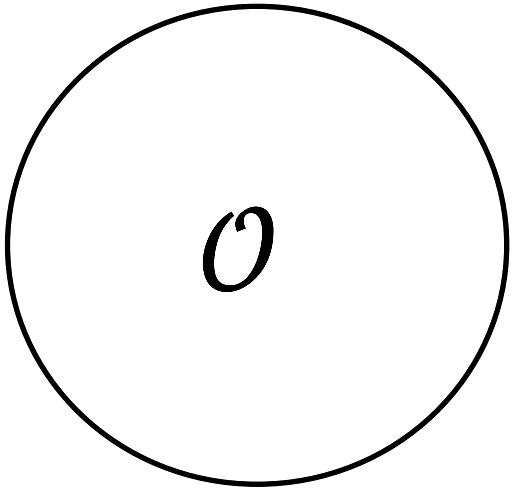

$\mathcal{A} = \mathcal{B}(\widehat{\mathcal{H}})$
unphysically large!


$\mathcal{A} = \mathcal{B}(\widehat{\mathcal{H}})$
$\mathcal{O}$



$a = 1$ $a = 1/2$ $a = 1/4$



https://github.com/tobiasosborne/Continuous-Limits-of-Quantum-Lattice-Systems

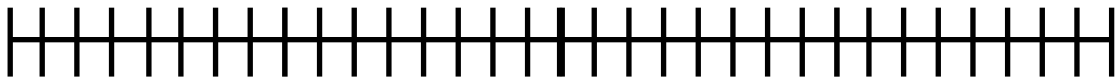
Definition: let $\mathcal{H} = \bigotimes_{j=1}^{2^n} \mathbb{C}^d$. The **local Fluctuation** k-mode observable of $A \in \mathcal{B}(\mathbb{C}^d)$ is

$$F_k^n(A) = \frac{1}{2^{n\left(\frac{1}{2}+\delta_A\right)}} \sum_{j=0}^{2^n-1} e^{2\pi i jk/2^n} \left(A_j - \langle A_j \rangle \mathbb{I}\right)$$

A translated to jth cell

M. Broidioi, B. Momont and A. Verbeure, J. Math. Phys. 36, 6746 (1995)

Denote the regular dyadic partition of 2^n elements as $\mathcal{P}_n$:



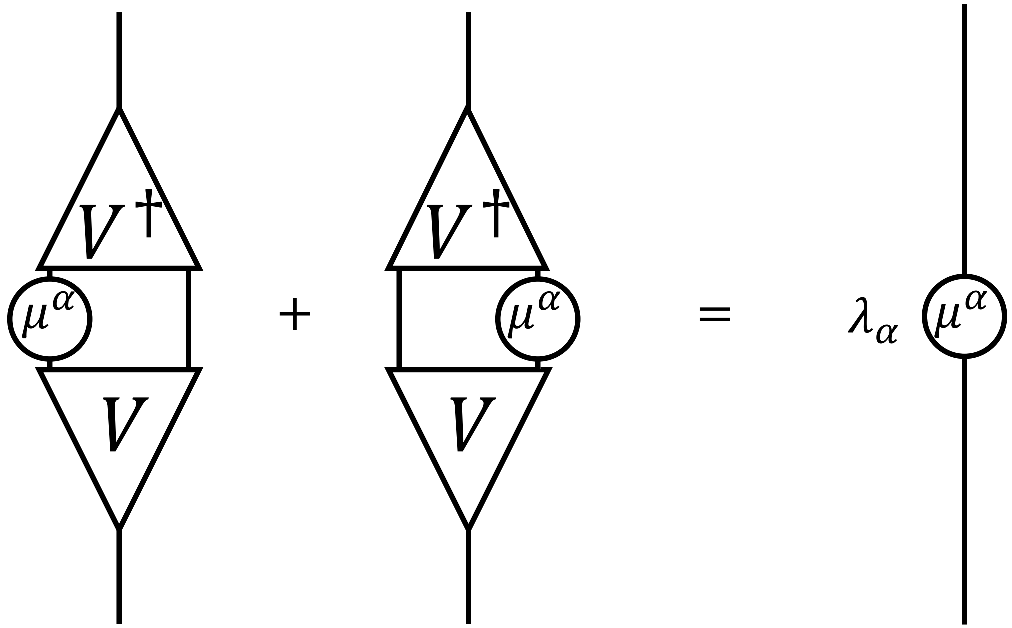
Because $\mathcal{H}_{P_n} \hookrightarrow \widehat{\mathcal{H}}$ any state $|\Omega\rangle$ in $\widehat{\mathcal{H}}$ induces a state ω_n on $\mathcal{P}_n$.

The *anomalous dimensions* δ_A are chosen so that

$$\lim_{n\to\infty} \omega_n(F_k^n(A)^2) < \infty$$

Definition: an *ascending operator* $\mu_\alpha \in \mathcal{B}(\mathcal{H})$ is an eigenvector of the ascending channel:

$$V^\dagger(\mu^\alpha \otimes \mathbb{I} + \mathbb{I} \otimes \mu^\alpha)V = \lambda_\alpha \mu^\alpha$$



Lemma: the anomalous dimensions for μ^{α} are given by

$$\delta_{\alpha} = \log_{2}(\lambda_{\alpha}) - \frac{1}{2}$$

The ascending operators $\mu_\alpha \in \mathcal{B}(\mathcal{H})$ form a lie algebra (equivalent to $\mathfrak{u}(d)$):

$$\left[i\mu^\alpha, i\mu^\beta\right] = i g_\gamma^{\alpha\beta} \mu^\gamma$$

When $\delta_\alpha = \log_2(\lambda_\alpha) - \frac{1}{2}$ the following limit of the *generating function* exists

$$\lim_{n\to\infty} \omega_n \left(e^{i \sum_{\alpha=0}^{d^2-1} \sum_{k\in\mathbb{Z}} z_\alpha(k) F_k^n(\mu^\alpha)} \right) = W\left(z_\alpha(k)\right)$$

(Argument uses Banach fixed point theorem *a la* Hartman and Grobman and a tensor diagrammatic calculus for Taylor series.)

“Reconstruction” theorem: the limiting generating function may be written as

$$W\left(z_\alpha(k)\right) = \widetilde{\omega}\left(e^{i\sum_{\alpha=0}^{d^2-1}\sum_{k\in\mathbb{Z}} z_\alpha(k)T_k^\alpha}\right)$$

where

$$[iT_k^a, iT_l^b] = if_c^{ab}T_{k+l}^c + d_{kl}^{ab}K,$$

$$a, b, c = 1, 2, \ldots, d^2 - 1$$

and

$$\delta_{k,0}K = T_k^0 = \delta_{k,0}\mathbb{I}$$

Structure constants f_c^{ab}:

(i) $f_c^{ab} = 0$ if $\frac{1}{2} + \delta_a + \delta_b - \delta_c > 0$

(ii) $f_c^{ab} = g_c^{ab}$ if $\frac{1}{2} + \delta_a + \delta_b - \delta_c = 0$

(iii) $g_c^{ab} = 0$ if $\frac{1}{2} + \delta_a + \delta_b - \delta_c < 0$

Structure constants f_c^{ab}: give Inönü-Wigner contraction $\mathfrak{g}$ of lie algebra $\mathfrak{u}(d)$

Structure constants d_{kl}^{ab}: is a 2-cocycle on loop algebra $\widehat{\mathfrak{g}}$:

$$d_{kl}^{ab} = i \lim_{n\to\infty} \omega_n\left([\mu^a, \mu^b]\right) \delta_{k,-l}$$

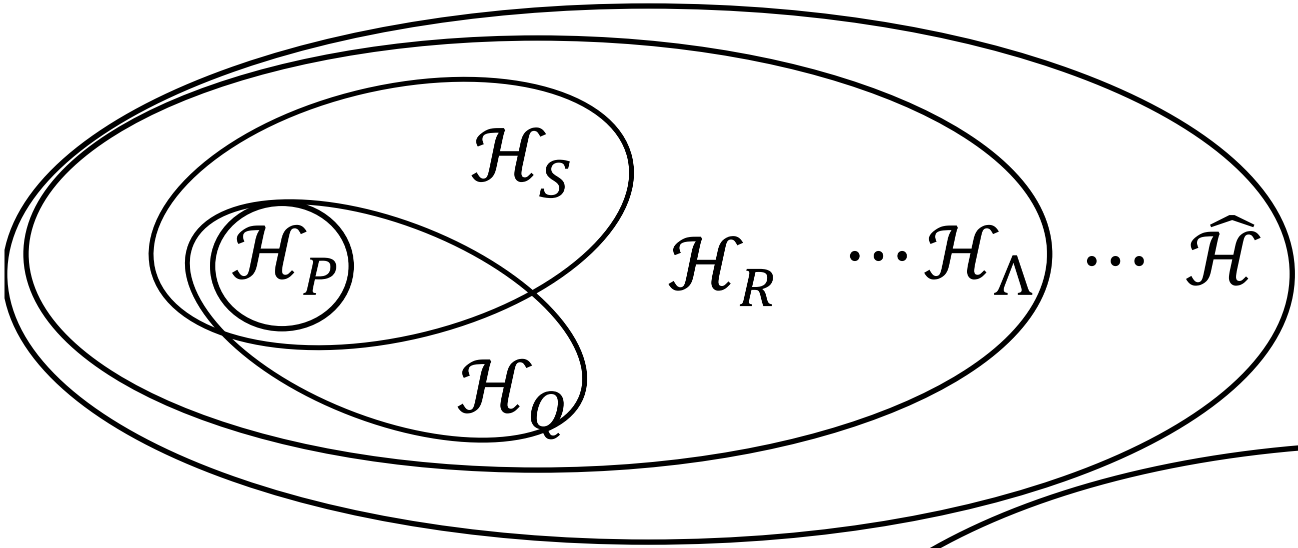
Field operators (currents): let $z \in \mathbb{C}$ define

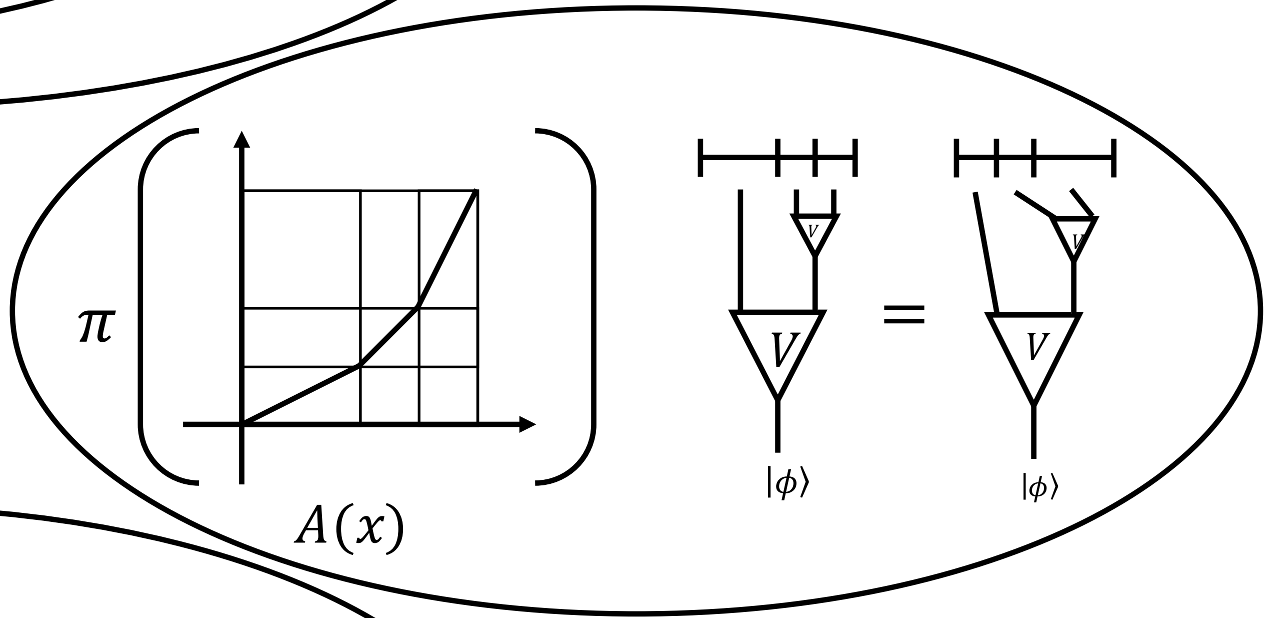
$$\phi_a(z) \equiv \sum_{n \in \mathbb{Z}} z^{-n-\Delta_a} T_n^a$$

Virasoro algebra: obtain via Sugawara construction

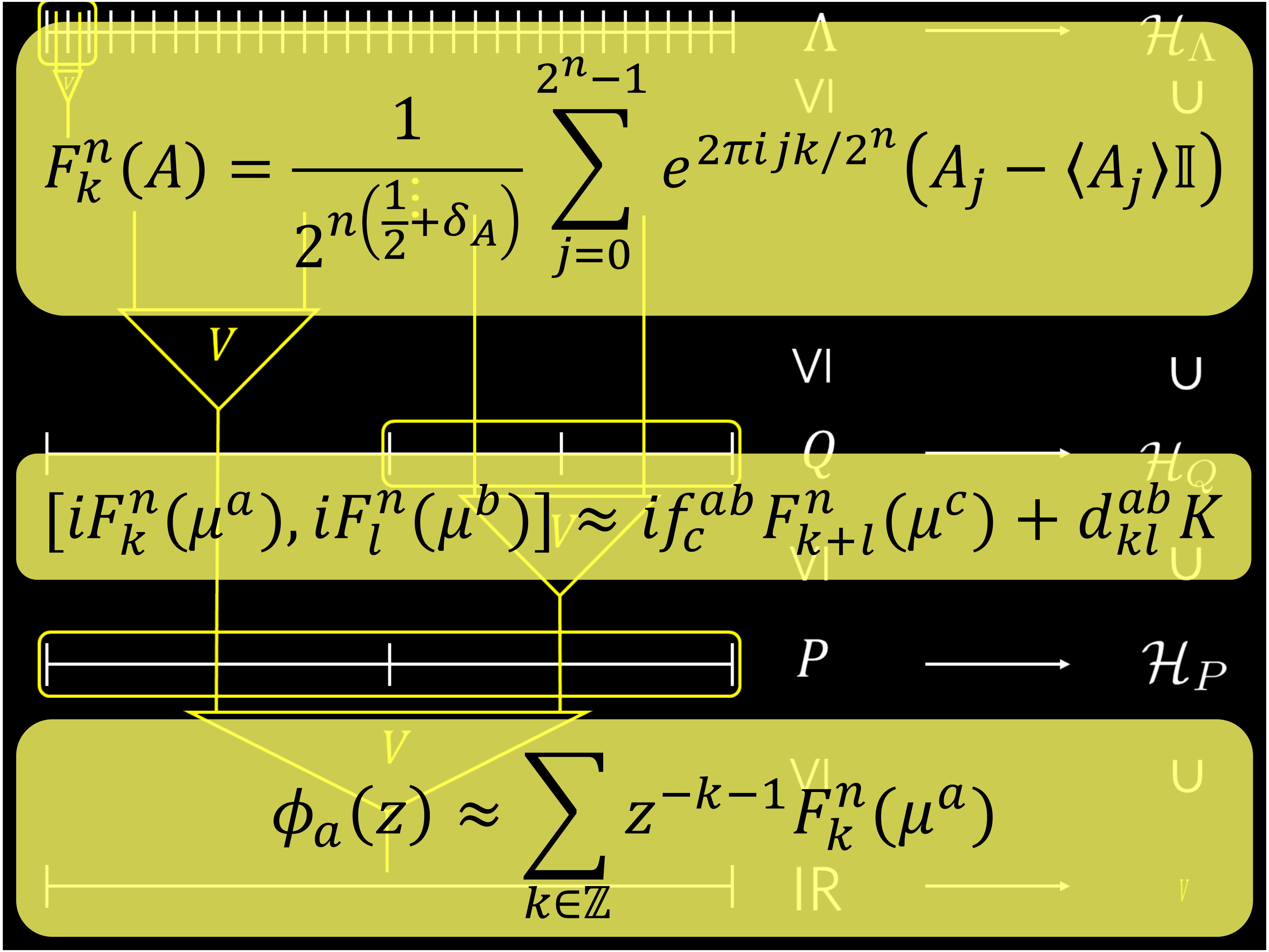
$$L_n \propto \sum_{m\in\mathbb{Z}} \sum_{a=1}^{d^2-1} :T^a_{m+n}\, T^a_{-m}:$$

Conjecture(!): representation of Virasoro algebra via Sugawara is equivalent to that found from unitary representation of $\mathrm{diff}_+(S^1)$ via Thompson.





$$F_k^n(A) = \frac{1}{2^{n\left(\frac{1}{2}+\delta_A\right)}} \sum_{j=0}^{2^n-1} e^{2\pi i jk/2^n}\left(A_j - \langle A_j \rangle \mathbb{I}\right)$$



$$F_k^n(A) = \frac{1}{2^{n\left(\frac{1}{2}+\delta_A\right)}} \sum_{j=0}^{2^n-1} e^{2\pi i jk/2^n}\left(A_j - \langle A_j\rangle \mathbb{I}\right)$$

$$[iF_k^n(\mu^a), iF_l^n(\mu^b)] \approx i f_c^{ab} F_{k+l}^n(\mu^c) + d_{kl}^{ab} K$$

$$\phi_a(z) \approx \sum_{k\in\mathbb{Z}} z^{-k-1} F_k^n(\mu^a)$$