THE QUANTUM BERNOULLI FACTORY

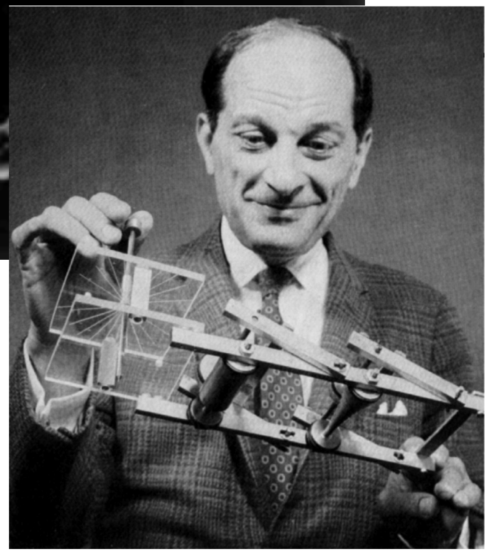
Terry Rudolph

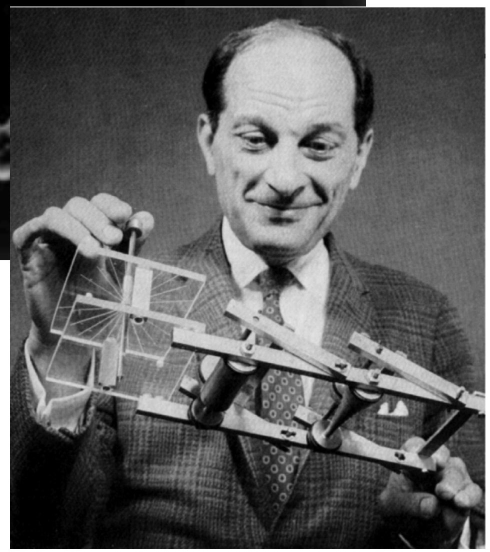
1. arXiv:1509.06183 [pdf, other]

Provable Quantum Advantage in Randomness Processing

Howard Dale, David Jennings, Terry Rudolph

Comments: A much improved version of the paper is available in Nature Communications. (10.1038/ncomms9203)





90's – Invasion of the Math Lords

Invasion I

Provably rapidly mixing
Markov chains

-permanents of +'ve matrices
-integrating log-concave functions
-volumes of convex bodies
-ferromagnetic Ising
(Jerrum, Sinclair, Vigoda, Applegate, Kannan, Dyer, Frieze,...)

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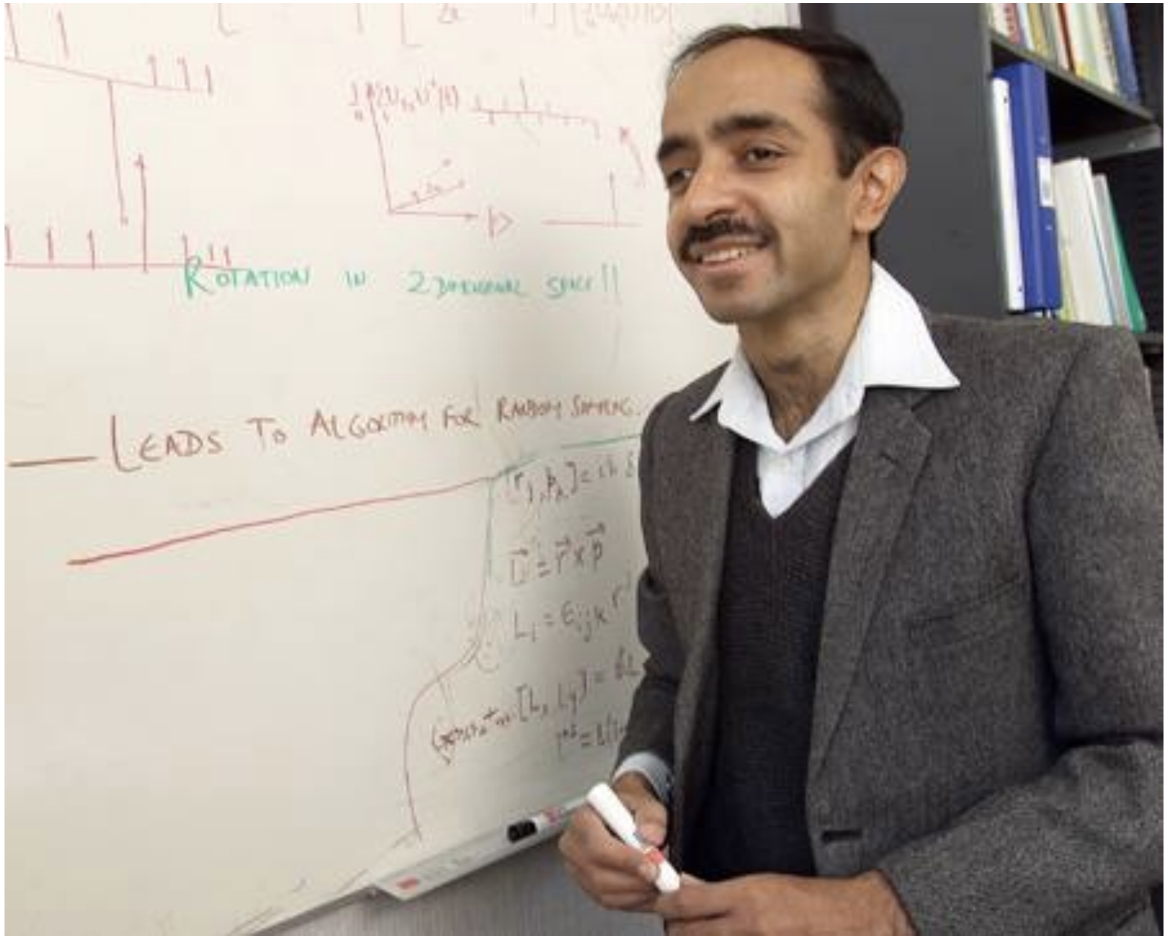

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i'th configuration $\Omega_i = b_i^1 \ldots b_i^n$ has probability $P(b_i^1 \ldots b_i^n)$

Self Reducible: Efficiently calculate $P(b_i^n = 0 | b_i^1 \ldots b_i^{n-1})$

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arXiv.org > quant-ph > arXiv:quant-ph/0208112

Quantum Physics

Creating superpositions that correspond to efficiently integrable probability distributions

Lov Grover, Terry Rudolph

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Q-SAMPLE

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Interesting?

$$\sum_j \sqrt{e^{-\beta E_j}}|j\rangle$$

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$$H^{\otimes n} \sum_j \sqrt{e^{-\beta E_j}}|j\rangle$$

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$$= \sum_i \sqrt{P(b_i^1 \ldots b_i^n)}|b_i^1 \ldots b_i^n\rangle$$

Q-SAMPLE

Interesting?

$$P(k) = \frac{1}{Z2^n}\left|\sum_j (-1)^{j\cdot k} e^{-\beta E_j/2}\right|^2$$

Coogee question 1: How crazy is this?

90’s – Invasion of the Math Lords

Invasion II

Exact sampling

-”coupling from the past” (Propp, Wilson)

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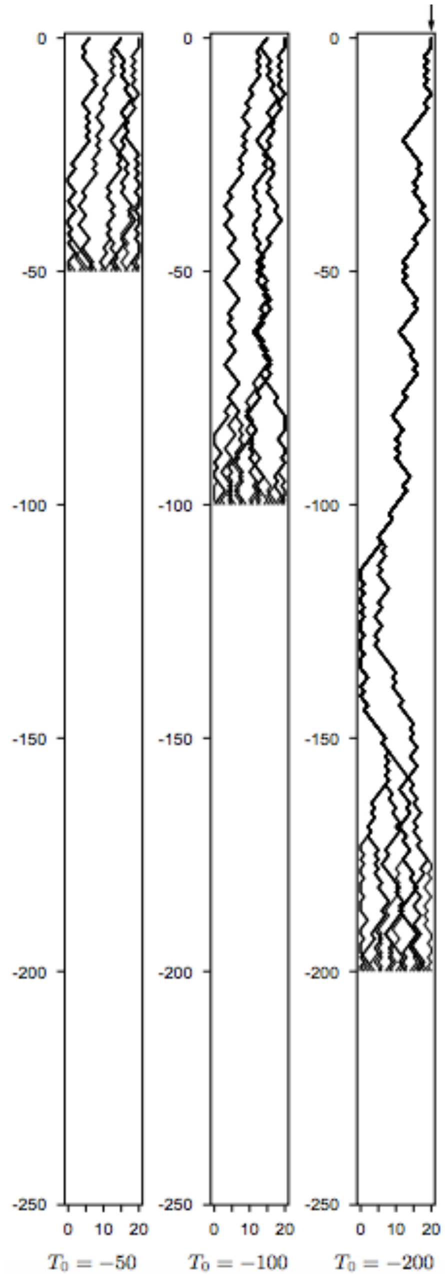

From McKay, Information Theory

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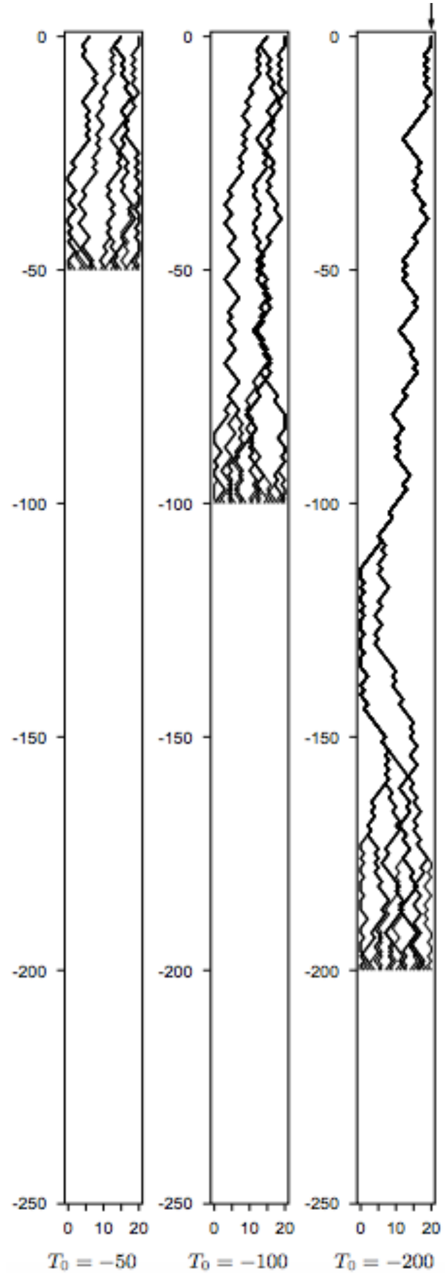
Coogee question 2:

Can we "quantize" CFTP to make exact Q-samples?



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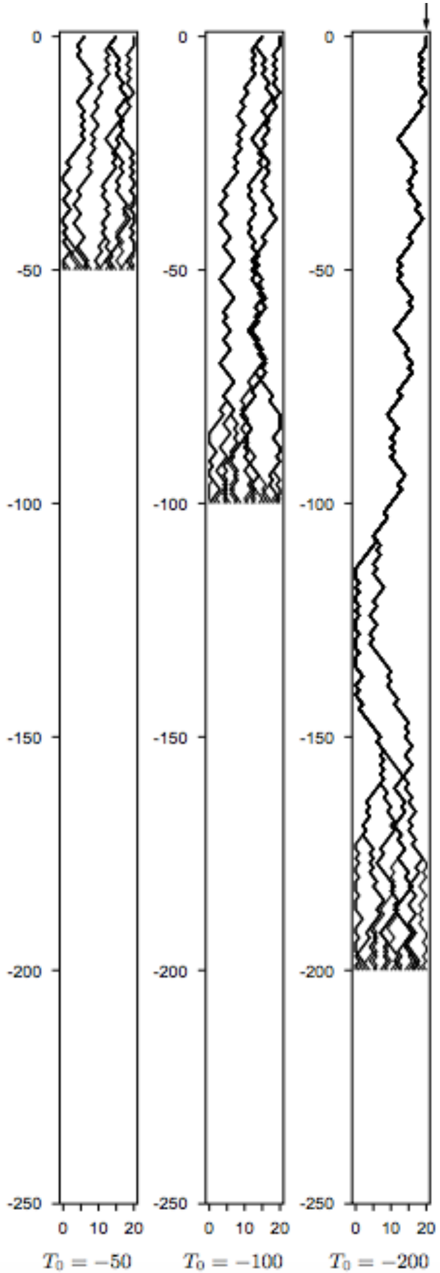

From McKay, Information Theory

Bernoulli Factory Problem:

Given the ability to exactly sample a distribution, what other distributions can also be exactly sampled by suitable processing?

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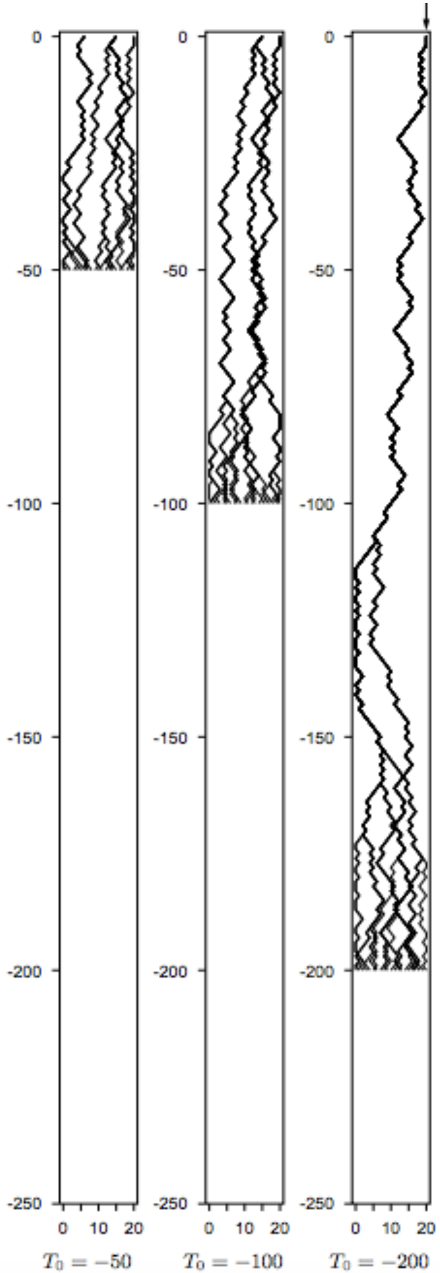
Bernoulli Factory Problem:

Given the ability to exactly sample a distribution, what other distributions can also be exactly sampled by suitable processing?

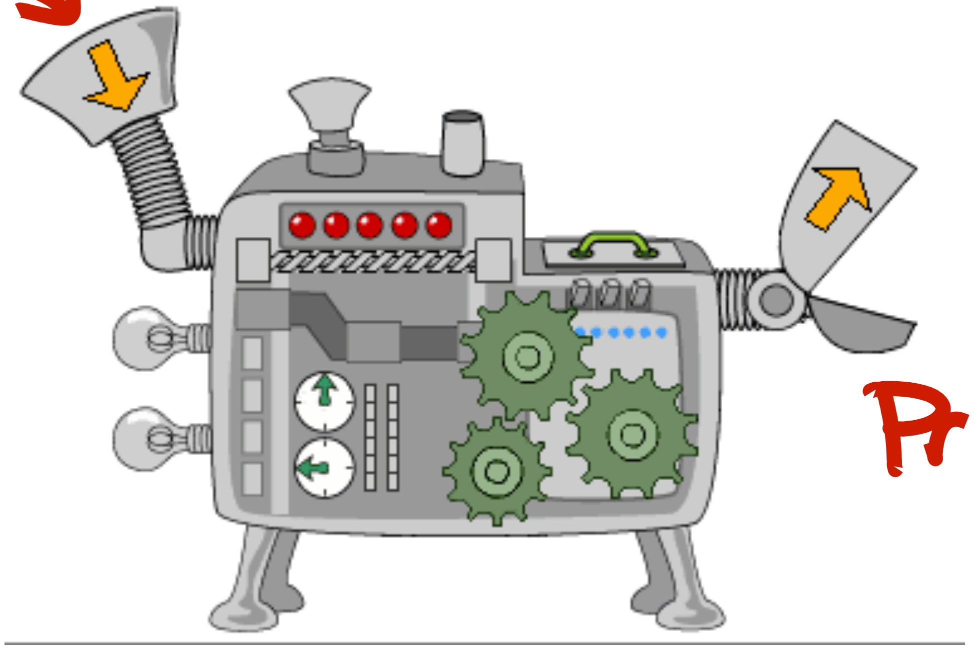
Most attention given to:

Given the ability to repeatedly sample a coin which is heads with probability *p*, for which functions *f* can we output a single new coin with probability of heads *f(p)*?



From McKay, Information Theory

$$\left.\begin{matrix}\vdots & \vdots & \vdots \\ T & H & T \\ H & T & T\end{matrix}\right\} \Pr(H) = p$$



H

$\Pr(H) = f(p)$

NOTE: unbounded input so not equivalent to Turing machine model

$$f(p) \quad = \quad p^2$$

$$
\begin{aligned}
f(p) &= p^2 \\
f(p) &= 3p(1-p)
\end{aligned}
$$

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f(p) &= p^2 \\
f(p) &= 3p(1-p) = 3p(1-p)^2 + 3p^2(1-p)
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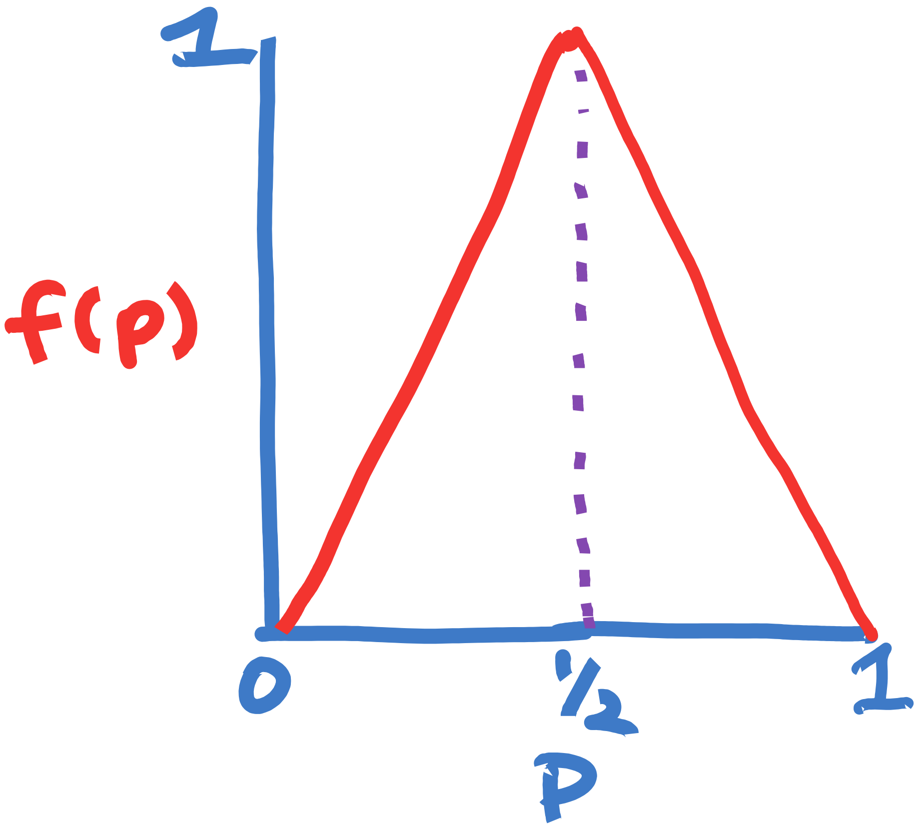
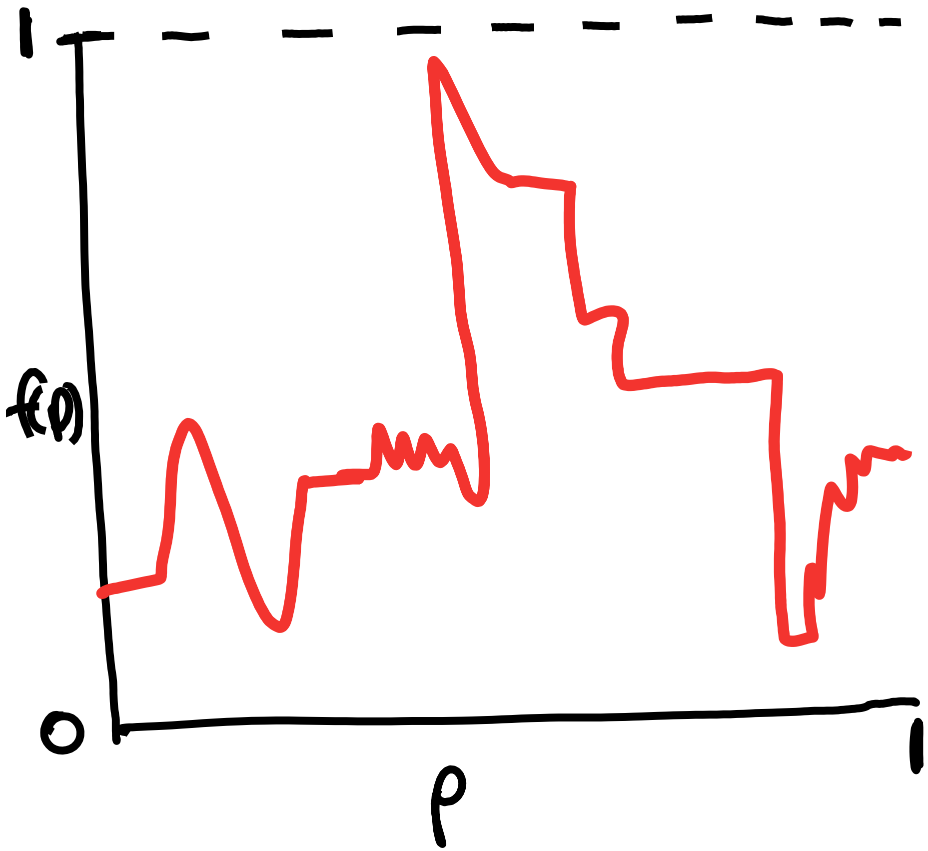
f(p) &= 3p(1-p) = 3p(1-p)^2 + 3p^2(1-p) \\
f(p) &= 1/2 \\
f(p) &= \sqrt{p} = \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{2^{2k+1}(k+1)} (1-p)^{k+1}
\end{aligned}
$$

$$f(p) = \begin{cases} 2p & p \in (0, \tfrac{1}{2}] \\ 2(1-p) & p \in [\tfrac{1}{2}, 1) \end{cases}$$



"Probability Amplification"

GENERAL f(p)?



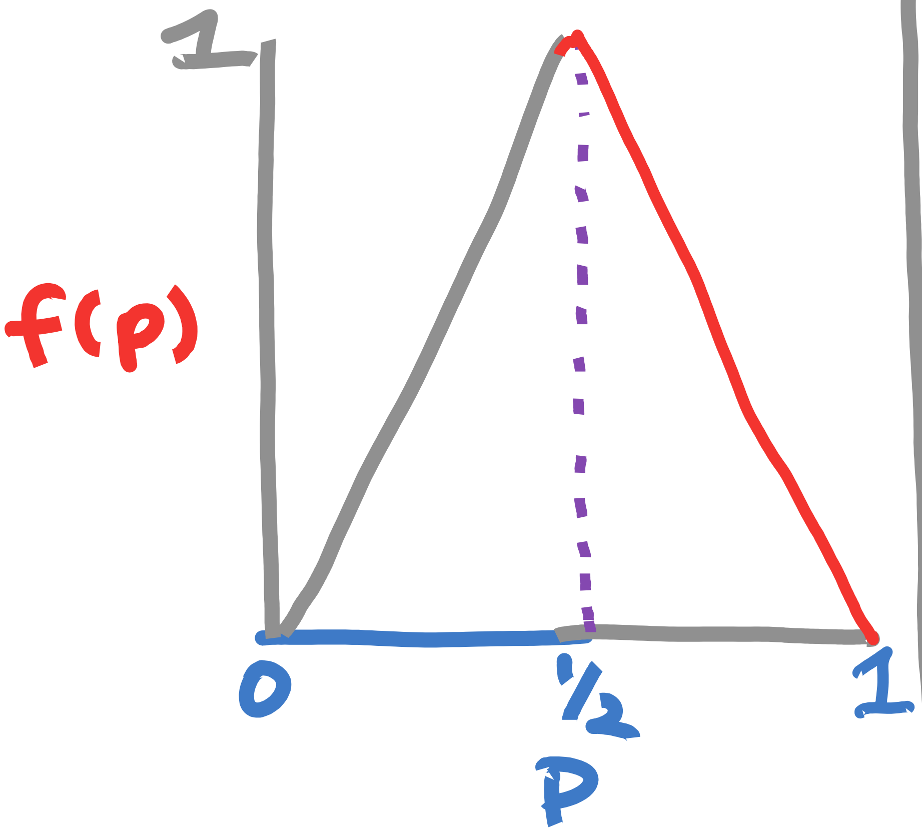
THEOREM

(Keane, O'Brien, '94)

$f(p)$ can be 'manufactured' iff f continuous & either f is constant, OR $\forall p, \exists k$ such that:

$$\min(f(p), 1-f(p)) \geq \min(p^k, (1-p)^k)$$

$$f(p) = 2p$$



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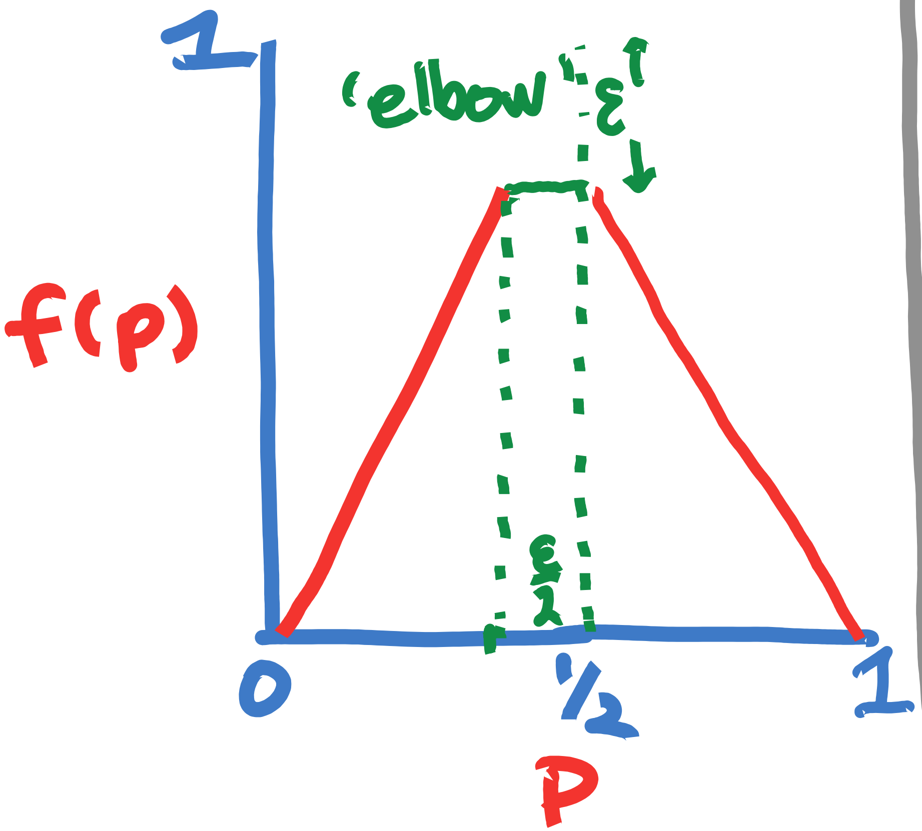

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Simulating Events of Unknown Probabilities via Reverse Time Martingales

Krzysztof Łatuszyński
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envelopes of f. To run the algorithm one has to construct sets of $\{0,1\}$ strings of appropriate cardinality based on coefficients of the polynomial envelopes. Unfortunately its naive implementation requires dealing with sets of exponential size (we encountered e.g. $2^{2^{26}}$) and thus is not very practical. Hence the authors

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Exact sampling for intractable probability distributions via a Bernoulli factory

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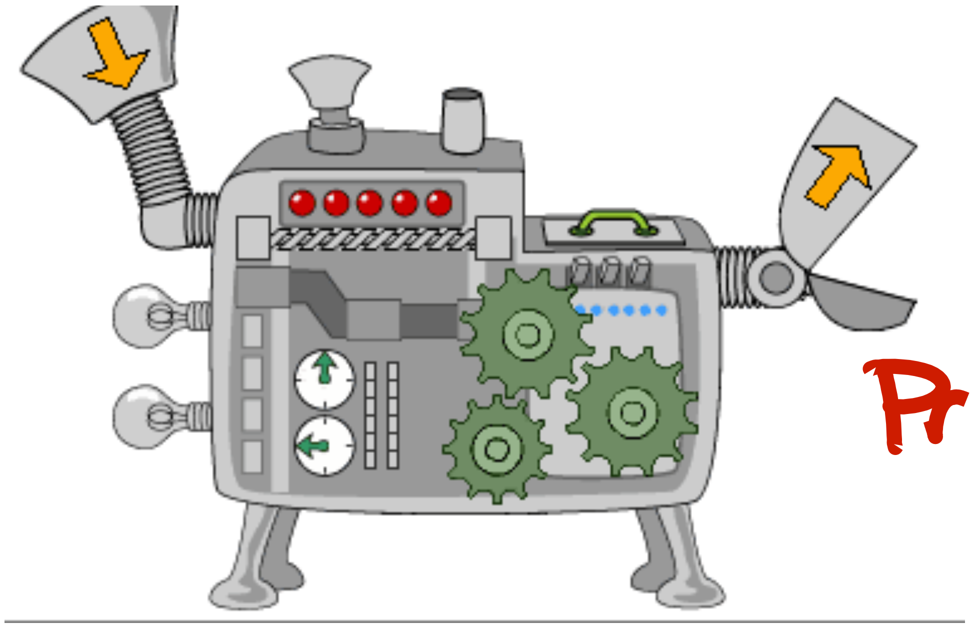
eration settings provided by Roy and Hobert (2007). This example is ill-suited using the proposed algorithm because of computational limitations related to the Bernoulli factory and in obtaining a practical ε. Specifically, we found (in simpler examples) obtaining a single draw from π sometimes required millions of i.i.d. τ variates. Unfortunately, even using non-constant $s(x)$, the probit example requires about 14,000 Markov chain draws per τ (Flegal and Jones, 2010). Hence obtaining a single draw from π would require an obscene number of draws from X. Implementation for more complicated Markov chains, such as this, likely requires further improvements, or a lot of patience.

COIN $p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$

$$\Downarrow$$

QUOIN $\sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle$

$$\sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle$$



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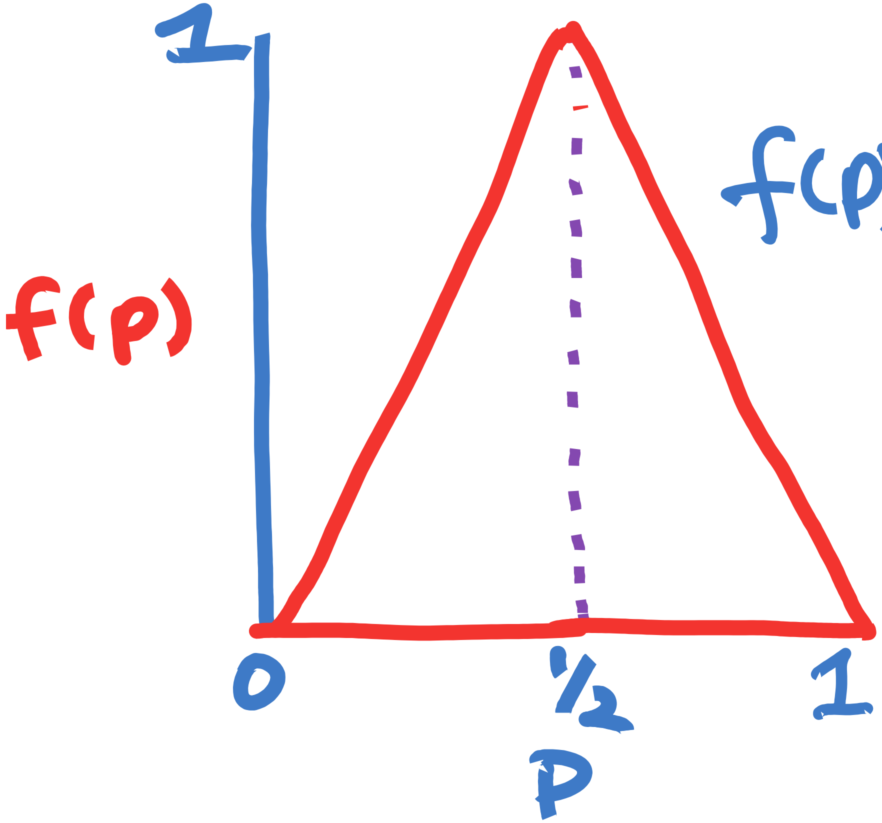
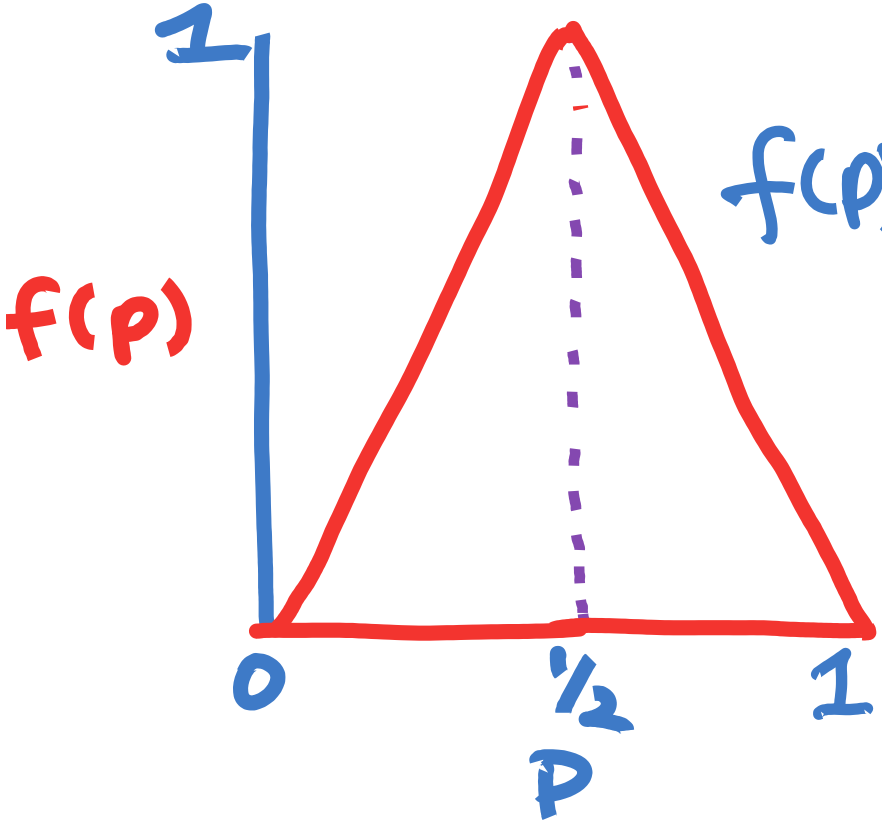


$$f(p) = 1 - \sqrt{1 - 4p(1-p)}$$



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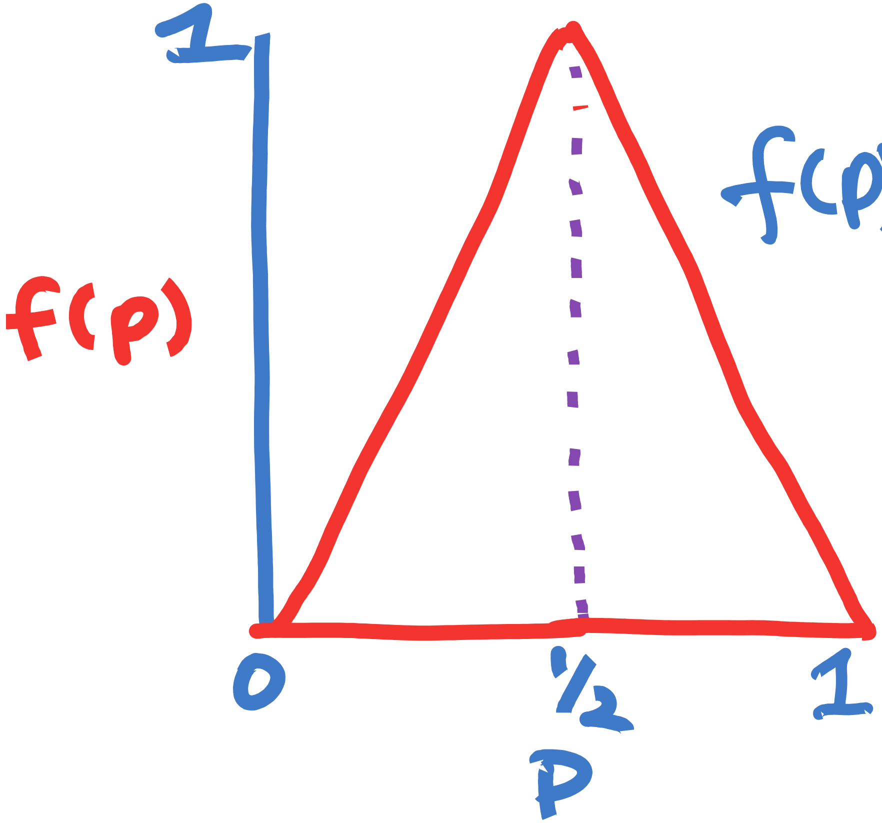

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$$\left(\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle\right)^{\otimes 2} = p|00\rangle + \sqrt{p}\sqrt{1-p}\sqrt{2}\,|\psi^+\rangle + (1-p)|11\rangle$$

$$Pr(|\phi^+\rangle) = \frac{1}{2}(p + (1-p))^2 = \frac{1}{2}$$

$$Pr(|\phi^-\rangle) = \frac{1}{2}(p - (1-p))^2 = \frac{1}{2}(2p-1)^2$$

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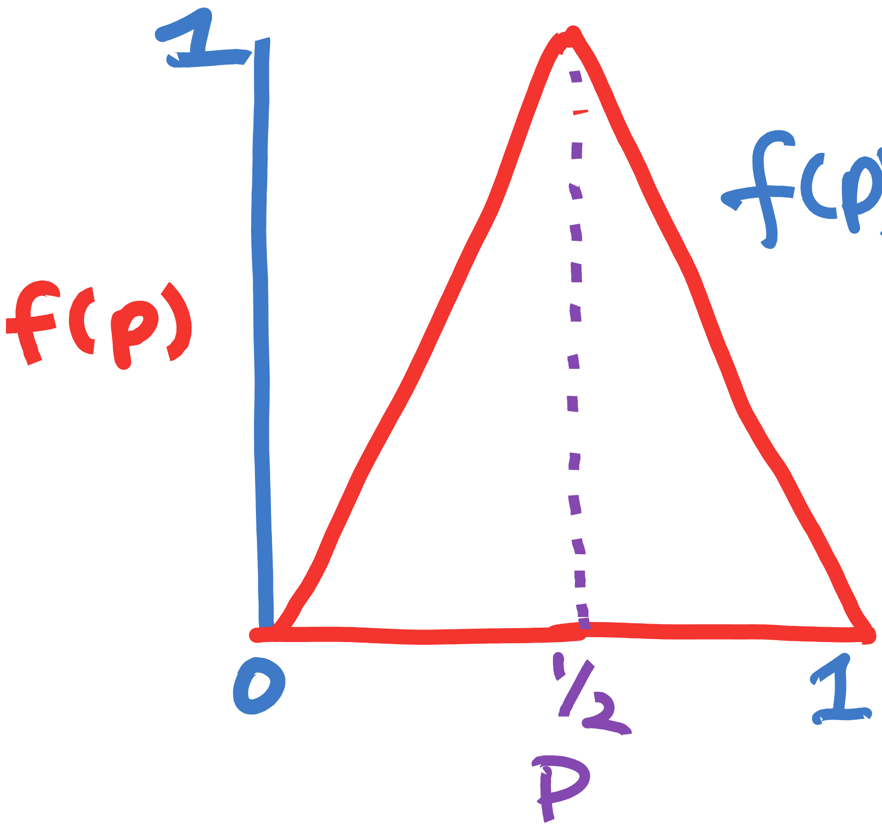

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$$\Pr(\psi^+ | \psi^+ \text{ or } \phi^+) = \frac{2p(1-p)}{1/2} = 4p(1-p)$$

FULL SET OF QUANTUM CONSTRUCTIBLE f(p)?

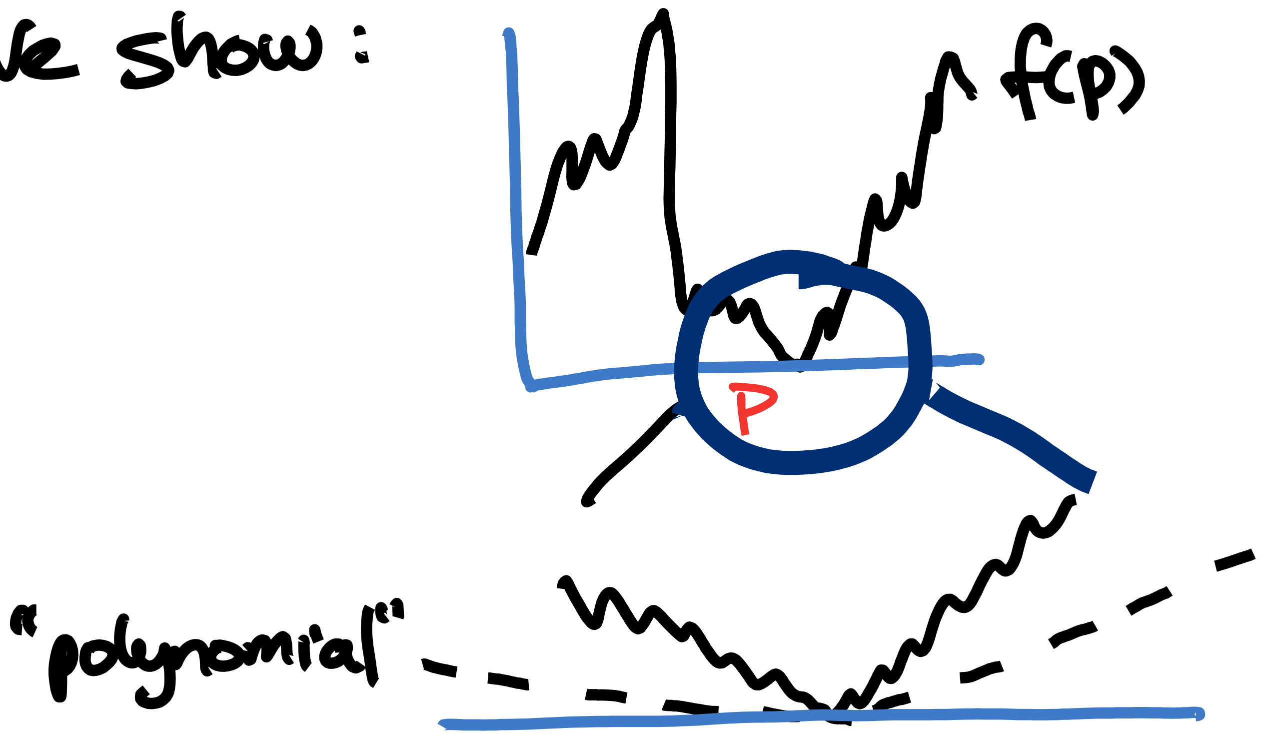
Theorem 1: a function f:[0, 1]→[0, 1] is constructible with quoins and a finite set of single-qubit operations if and only if the following conditions hold:

1. f is continuous.
2. Both $Z=\{z_i : f(z_i)=0\}$ and $W=\{w_i : f(w_i)=1\}$ are finite sets.
3. $\forall z \in Z$ there exists constants c, $\delta>0$ and and integer $k<\infty$ such that $c(p-z)^{2k} \le f(p) \forall p \in [z-\delta, z+\delta]$.
4. $\forall w \in W$ there exist constants c, $\delta>0$ and an integer $k<\infty$ such that $1-c(p-w)^{2k} \ge f(p) \forall p \in [w-\delta, w+\delta]$.

FULL SET OF QUANTUM CONSTRUCTIBLE f(p)?

constructible $\Leftrightarrow$ probability need to flip ∞ number of c/quoins is 0

We show:



Outline of the theorem proof:

Imagine we have constructible bounding functions:

$$L(p) \leq f(p) \leq U(p)$$

$$\max_{p \in [0,1]} (U(p) - L(p)) < \frac{1}{2}.$$

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Define sequence of constructible functions:

$$\begin{aligned}
f_1(p) &= f(p) \\
g_k(p) &= \frac{L_k(p)}{1 - U_k(p) + L_k(p)} \\
f_{k+1}(p) &= \frac{4}{3}\left(f_k(p) - \frac{1}{4} g_k(p)\right)
\end{aligned}$$

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f(p) can be convexly decomposed:

$$f = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^{k-1} \frac{1}{4} g_k(p).$$

First guess for L(p) and U(p), Bernstein polynomial approximants:

$$
\begin{aligned}
A_n(p) &= \sum_{k=0}^{n} \frac{2}{3} f\left(\frac{k}{n}\right) \binom{n}{k} p^k (1-p)^{n-k} \\
B_n(p) &= \sum_{k=0}^{n} \left(\frac{1}{3} + \frac{2}{3} f\left(\frac{k}{n}\right)\right) \binom{n}{k} p^k (1-p)^{n-k}
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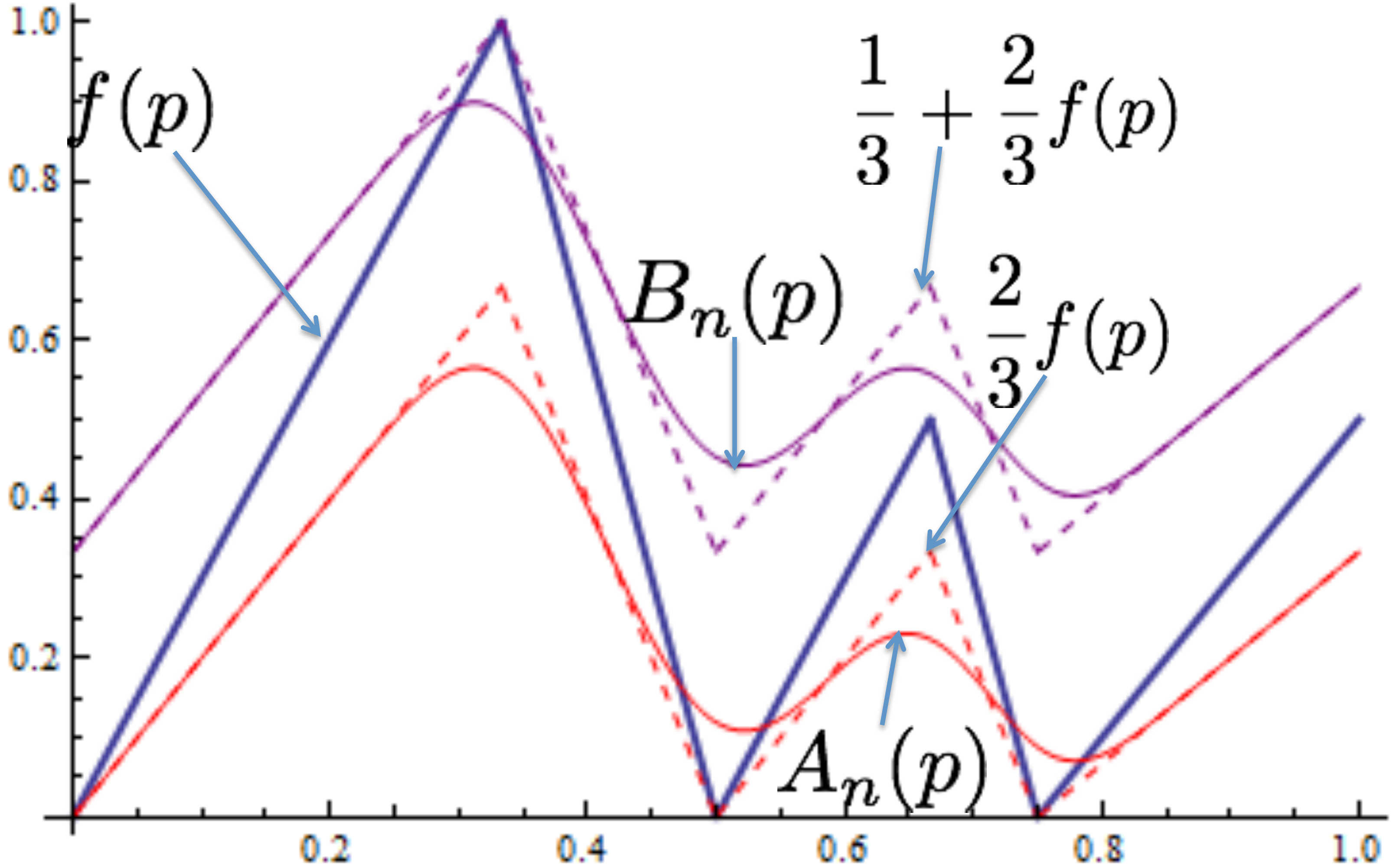
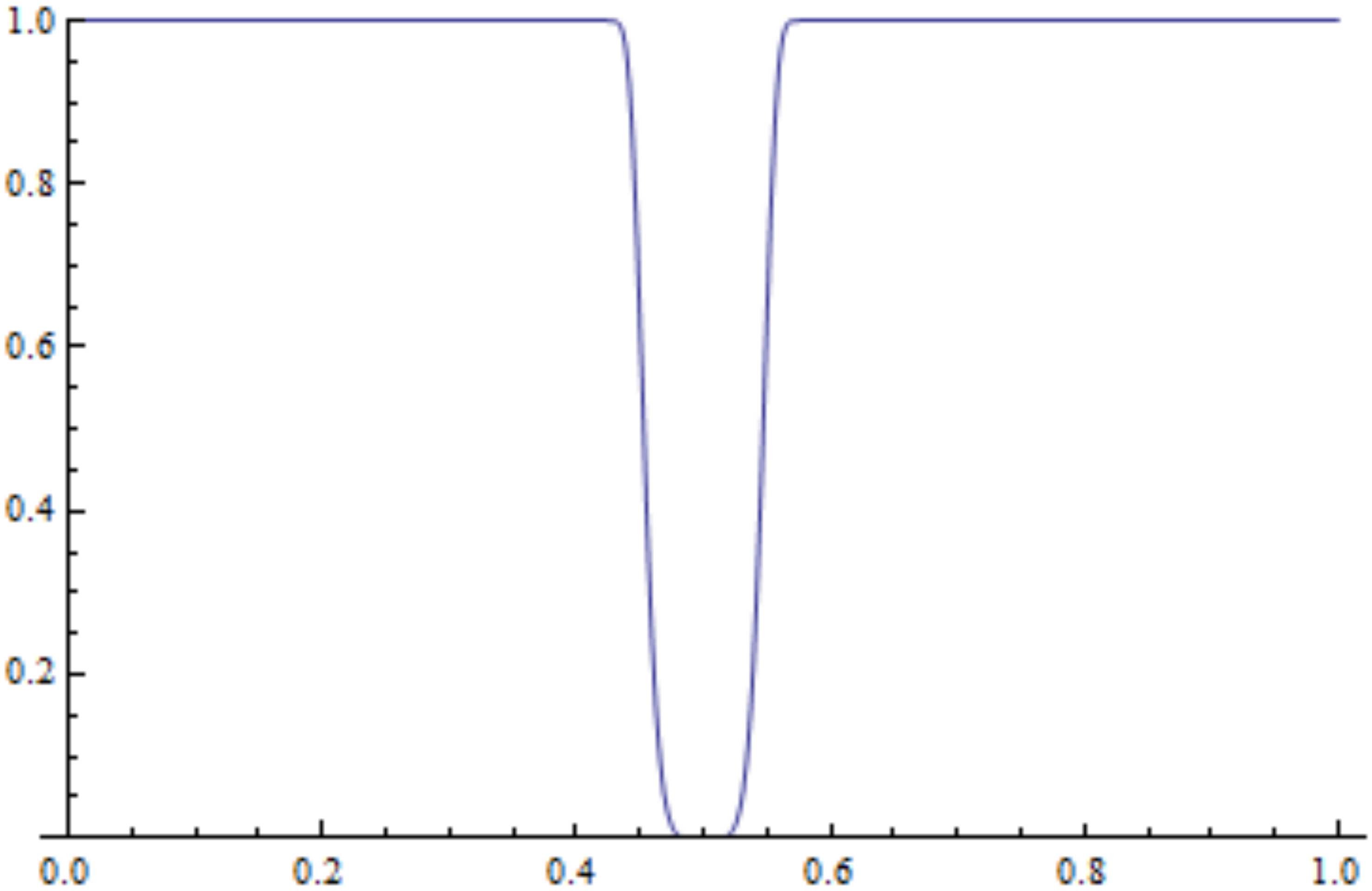


$$T_{m,M,z}(p) = (1 - (1 - h_z(p))^m))^M$$

$$h_z(p) = (\sqrt{1-z}\sqrt{p} - \sqrt{z}\sqrt{1-p})^2$$



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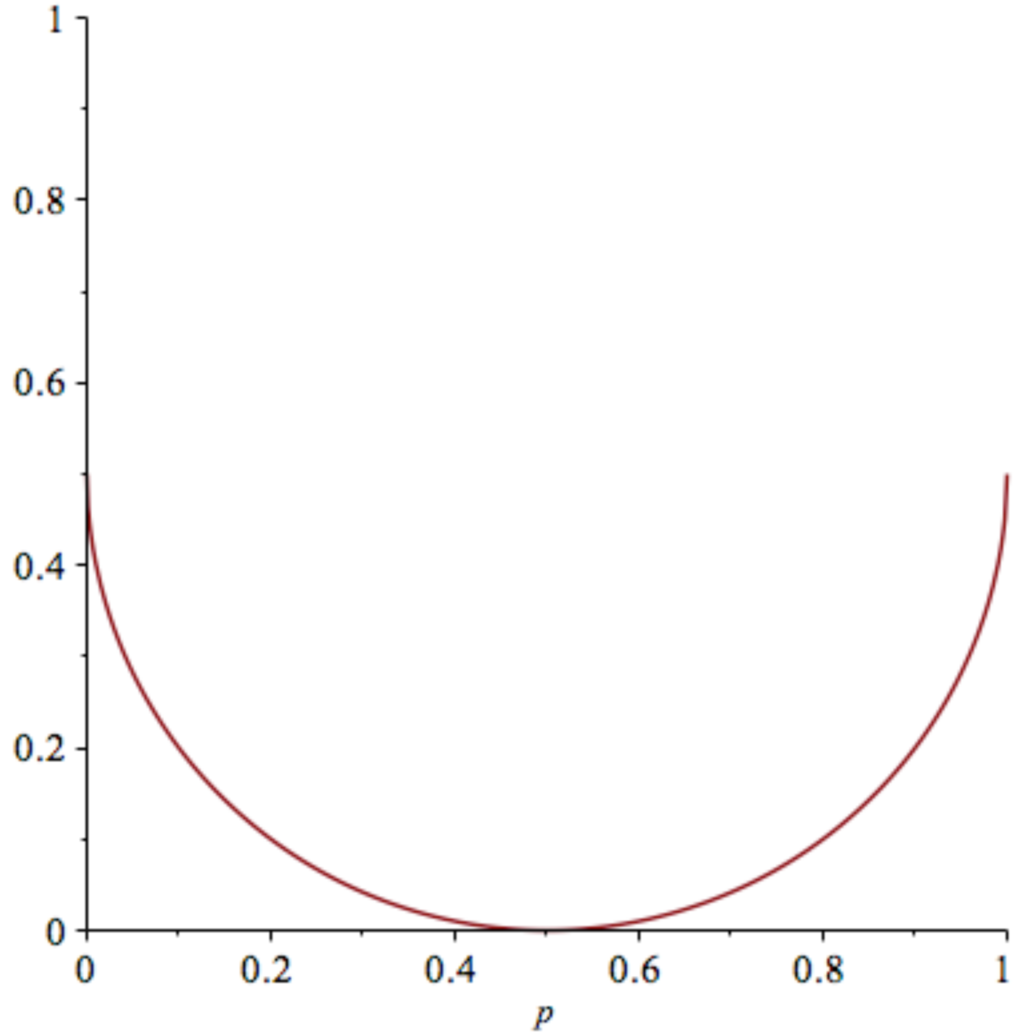

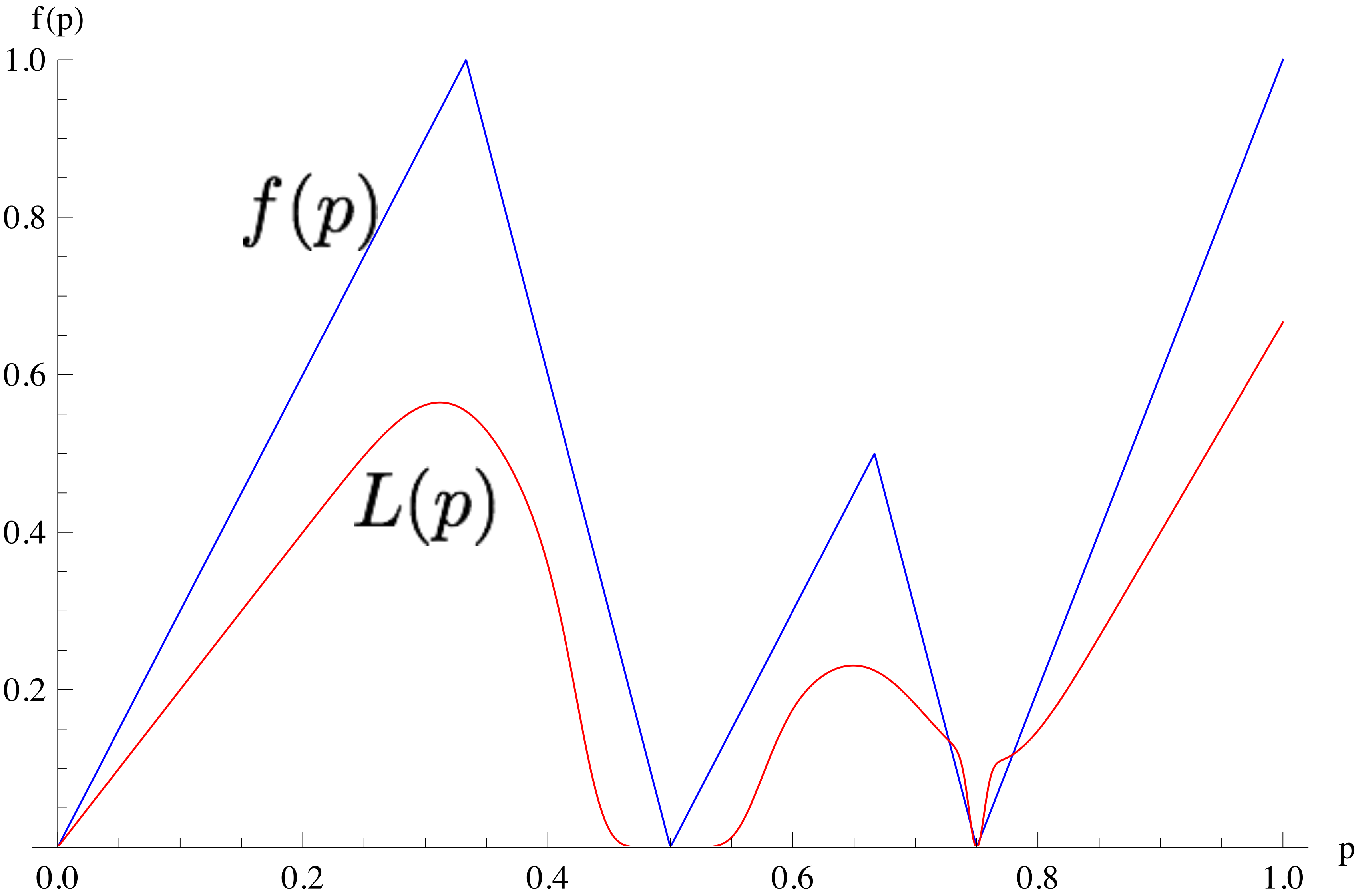

f(p)
1.0
0.8
0.6
0.4
0.2
0.0
0.2
0.4
0.6
0.8
1.0
p
$f(p)$
$L(p)$

CONCLUSIONS

- Bernoulli factory mainly interesting because it leads to *provable* differences between classical info and quantum info

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Coogee questions 4-6:

- Practical advantage?

- What the heck is the class of Q-sampleable distributions?

- For what set of functions *f(p)* can we implement:

$$\left(\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle\right)^{\otimes\infty} \longrightarrow \sqrt{f(p)}|0\rangle + \sqrt{1-f(p)}|1\rangle$$

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