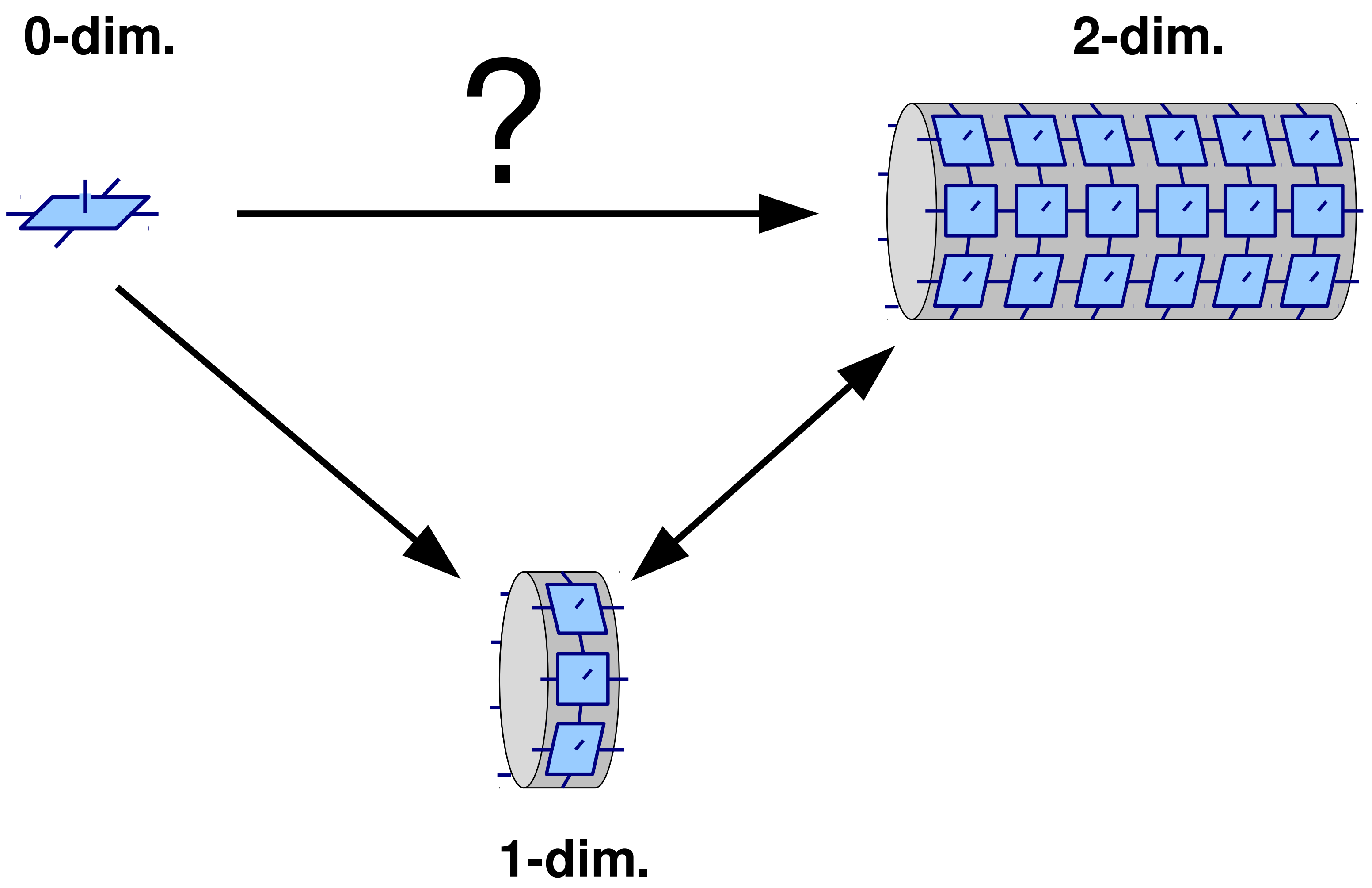
Topological phases in Tensor Networks: A holographic perspective

Norbert Schuch
Max-Planck-Institute of Quantum Optics, Munich

joint work with
K. Duivenvoorden (Aachen), M. Iqbal (Munich) and J. Haegeman & F. Verstraete (Gent)

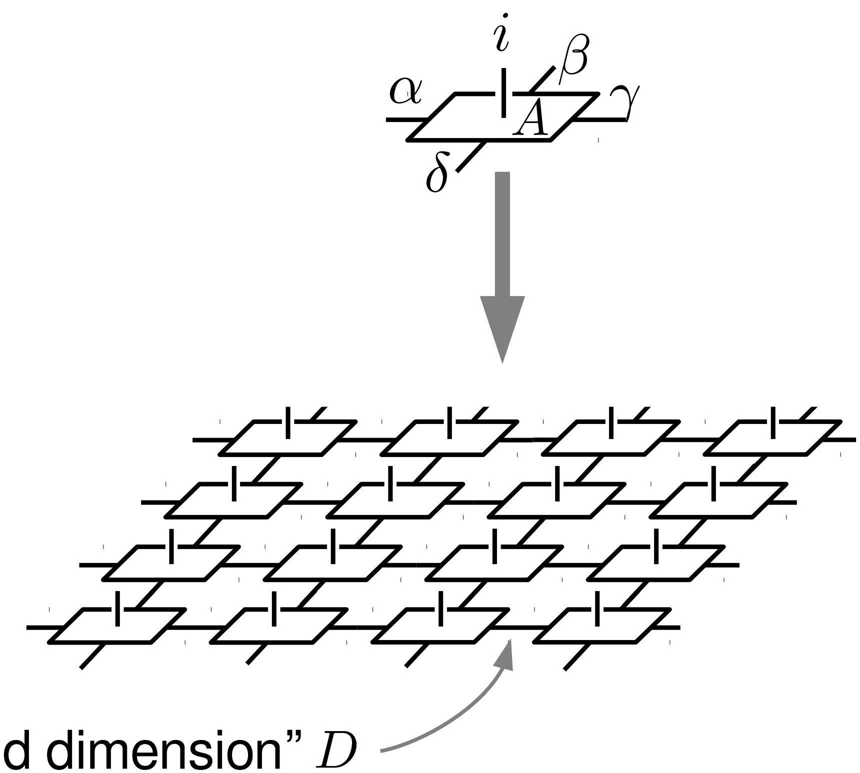
What is this talk about?



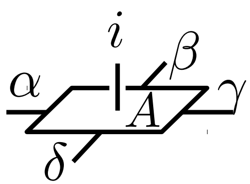
Projected Entangled Pair States

- **Projected Entangled Pair States (PEPS):**

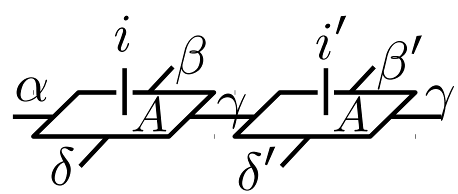
  local description of strongly correlated many-body states



Tensor Network Notation:



$$= A^{i}_{\alpha\beta\gamma\delta}$$



$$= \sum_{\gamma} A^{i}_{\alpha\beta\gamma\delta} A^{i'}_{\gamma\beta'\gamma'\delta'}$$

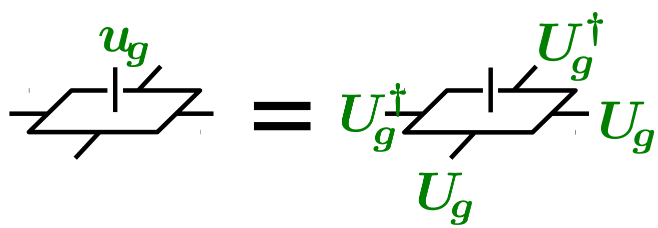
- faithful approximation of low-energy states of local Hamiltonians
  [Hastings PRB '06; Molnar, Schuch, Verstraete, Cirac, PRB '14]

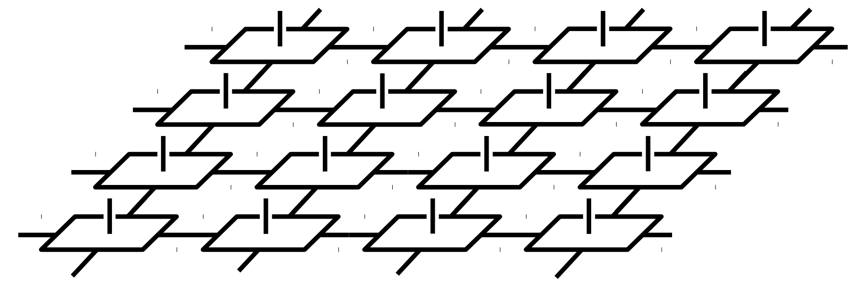
- different boundary conditions (open, periodic, infinite plane, ...)

PEPS: Encoding physics locally

- PEPS allow to **encode physical structure (symmetries) locally**

  E.g.: on-site symmetries:

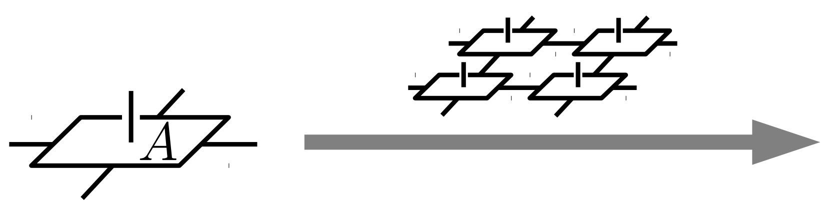




[Perez-Garcia *et al.*, NJP '10]

- local **parent Hamiltonian**: ensure that states looks “locally correct”

  $\Rightarrow$ **inherits all symmetries!**



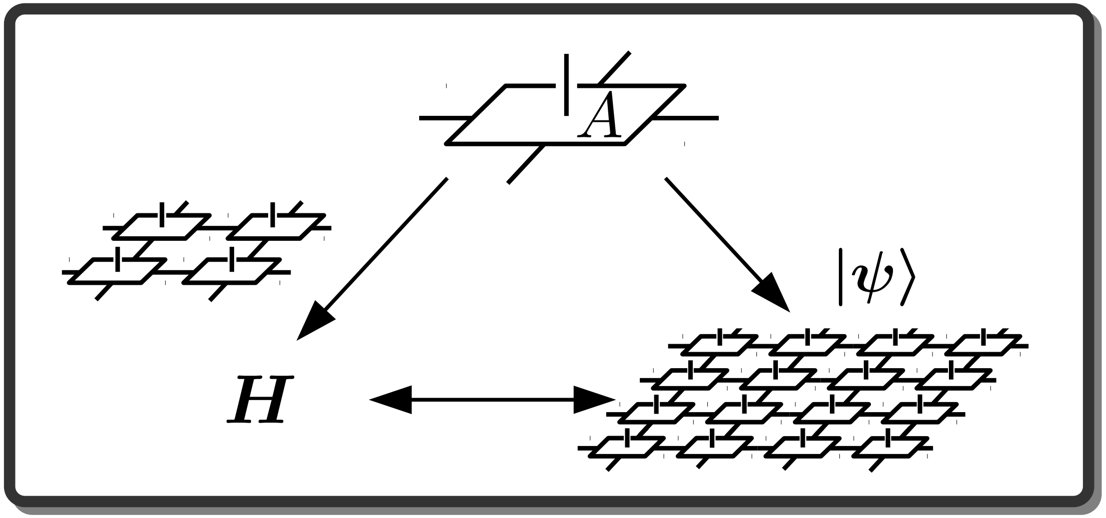
$$H = \sum h_i$$

- conditions (“... - injectivity”) for **controlled ground space structure** exist

Projected Entangled Pair State models

- PEPS: unified description of wavefunction + Hamiltonian from single tensor

  $\rightarrow$ construction of **solvable PEPS models**

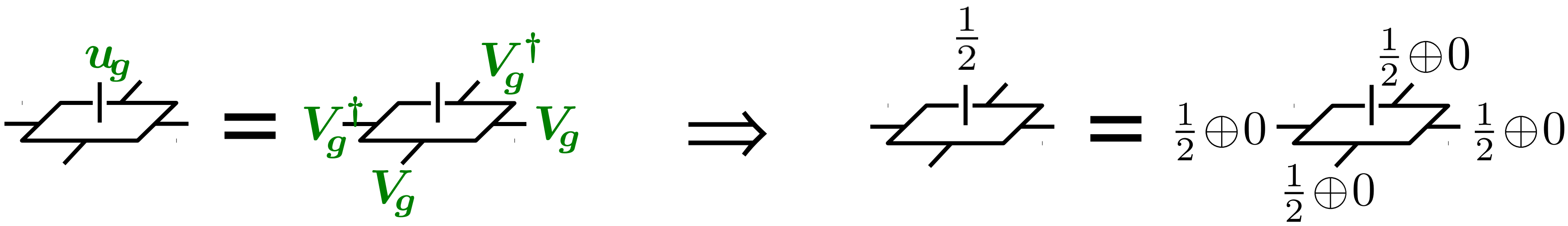


- $\boldsymbol{H}$ **inherits symmetries** of tensor A $\Rightarrow$ model **physics** directly **into tensor**
- framework to **study strongly correlated systems**

How do **local properties** of tensor relate to **globally emerging behavior**?

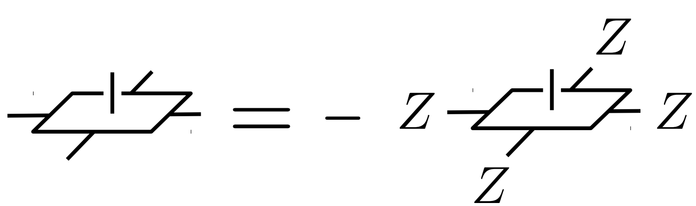
Physical vs. virtual symmetries in PEPS

- Case study: How can we encode **spin - $\frac{1}{2}$** with **SU(2) symmetry**?



$\Rightarrow$ V_g must **combine int. and half-int. representations**, e.g. $V_g \equiv \frac{1}{2}\oplus 0$

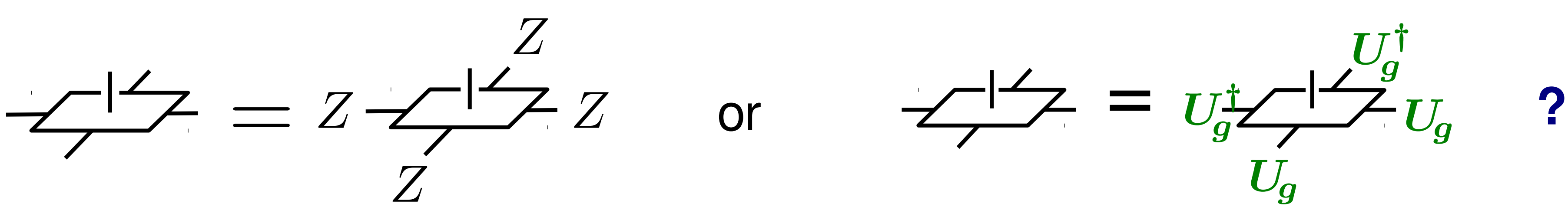
- odd of half-int. representations $\Rightarrow$ emergent **virtual $\mathbb{Z}_2$ symmetry**



$$Z = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

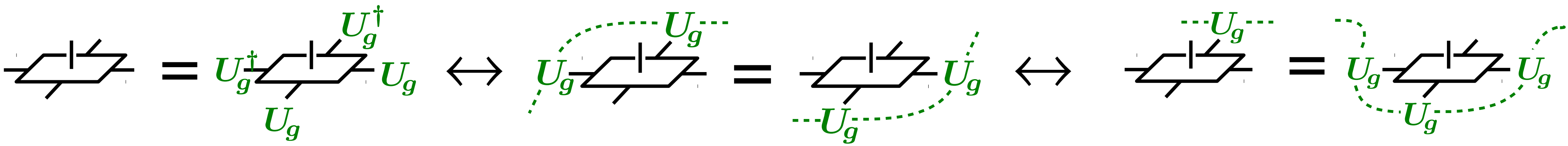
Z counts half-int. spins

What are the implications of a purely virtual symmetry

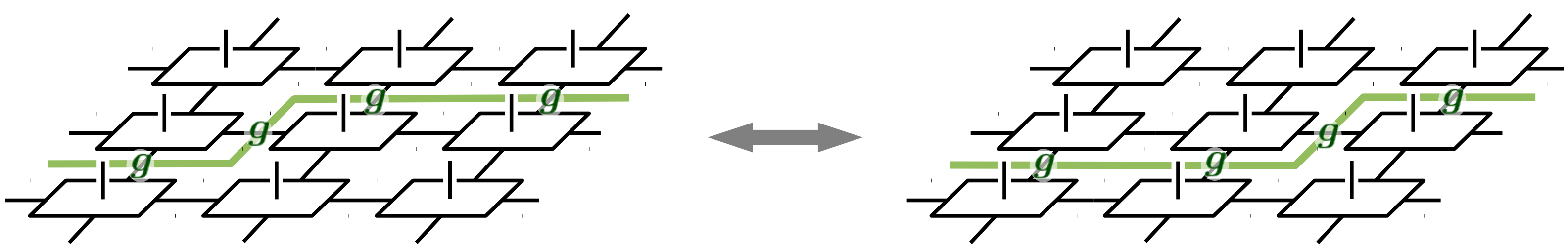


Symmetry and pulling through condition

- Symmetry can be rephrased as **"pulling through condition"**:



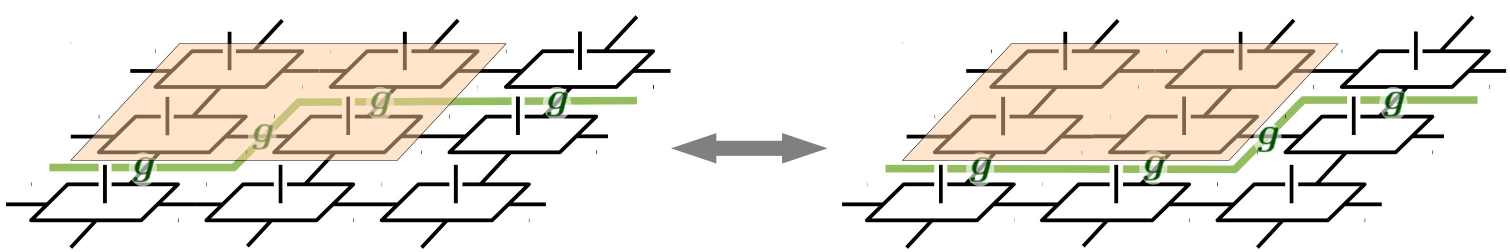
- Consequence of pulling-through: Strings can be **freely moved**!



$(g \equiv U_g)$

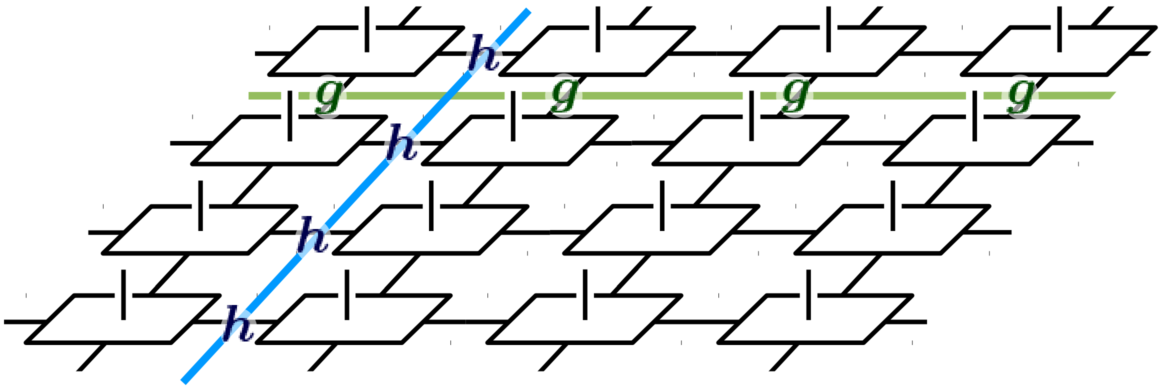
[Schuch, Cirac, Perez-Garcia, Ann. Phys. '10; Sahinoglu *et al.* '14]

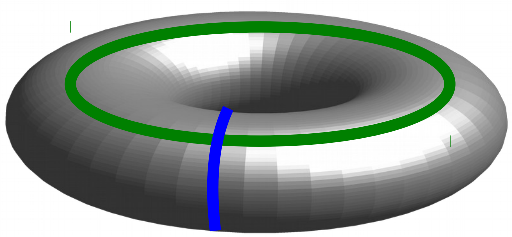
- Strings are **invisible to parent Hamiltonian**



Symmetry strings & ground state manifold

- **Local symmetry** in tensor ⇒ parametrization of **ground space manifold** from a **single tensor**



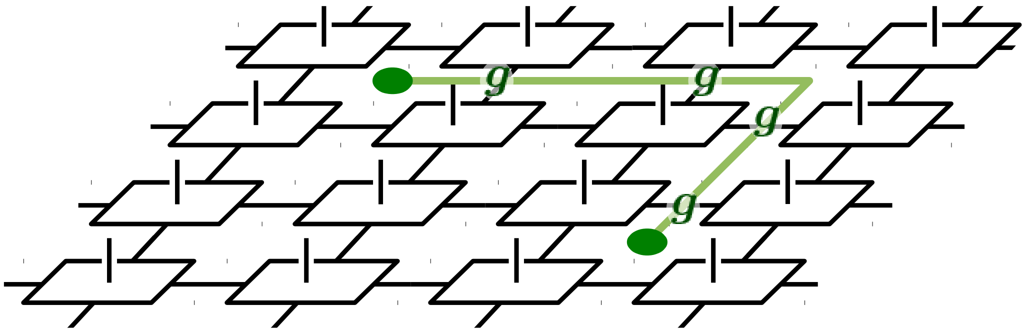


[Schuch, Cirac, Perez-Garcia, Ann. Phys. '10]

- Ground space degeneracy depends on **topology**

Symmetry strings and excitations

- Consider strings w/ open ends:



$\rightarrow$ string invisible to parent Hamiltonian

$\rightarrow$ endpoints (potentially) differ from ground state

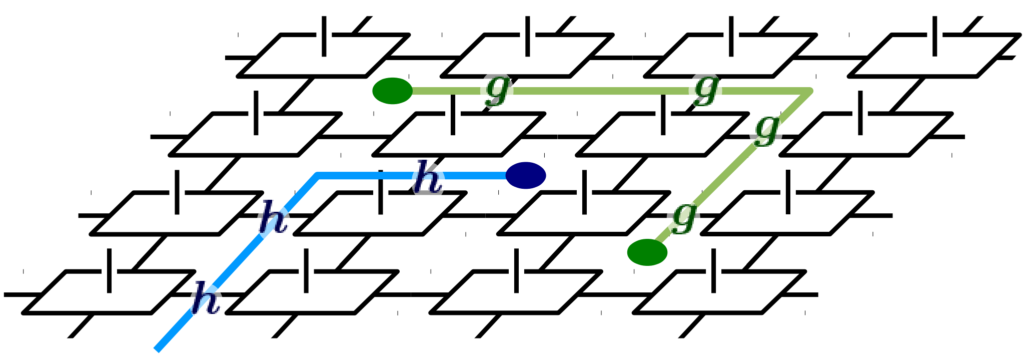
$\Rightarrow$ **localized excitations** which **come in pairs**

$\rightarrow$ labelled by (conjucacy classes of) group elements

$\rightarrow$ **braiding** acts by **conjugation**

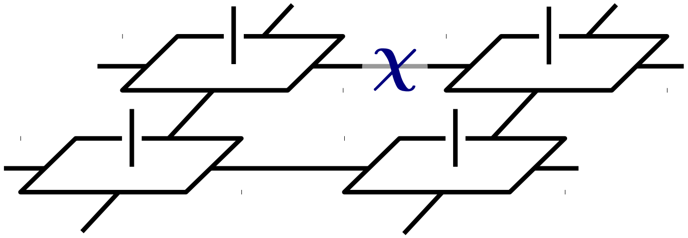
Anyonic excitations!

("magnetic" excitations)



More excitations

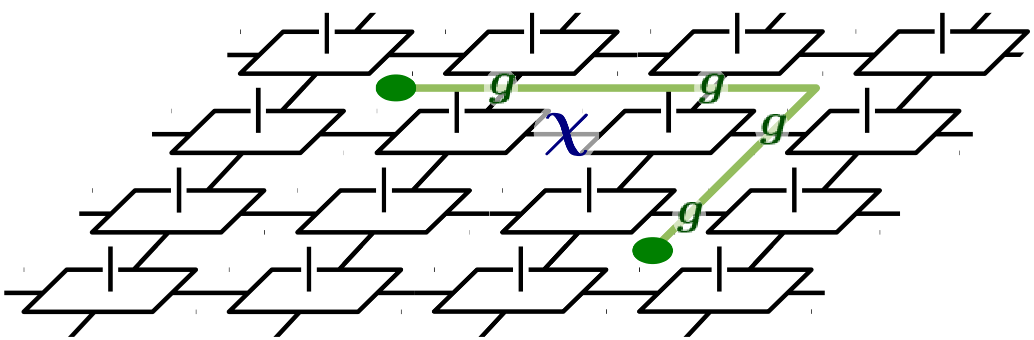
- What **other localized excitations** are there?



→ **local modification** of tensor network, e.g. change one tensor, or place **operator χ on link**

- χ can be **created locally** iff it is a **trivial irrep w.r.t. U_g**

- Topological excitations ↔ **non-trivial irreps χ** (e.g., X for Z strings)



→ non-trivial braiding action $\chi(g)$ by pulling magnetic string through

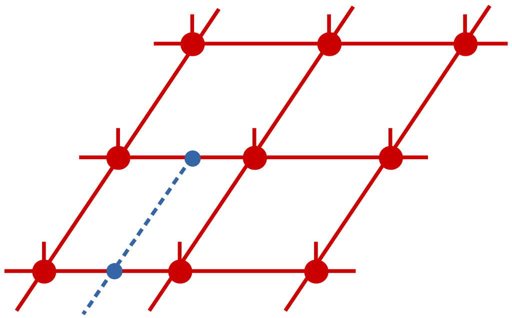
→ come in pairs due to global constr.

→ "electric" excitations

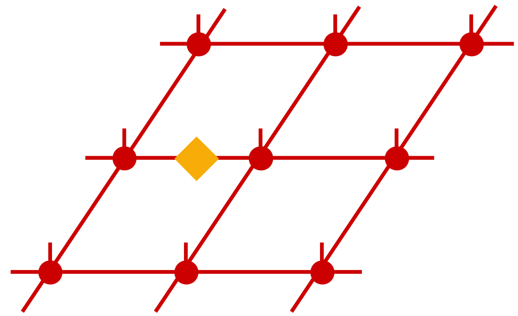
- General (dyonic) excitation: String (group element) + endpoint (irrep)

[Schuch, Cirac, Perez-Garcia, Ann. Phys. '10]

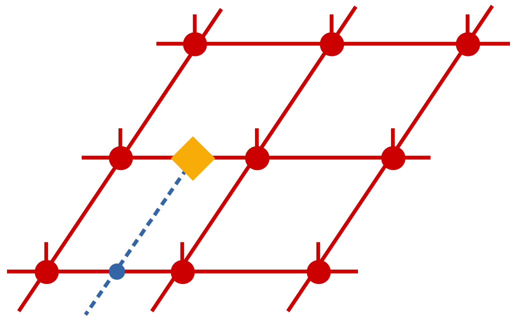
Requirements for physical excitations



string (magnetic) excitations



irrep (electric) excitations



combined (dyonic) excitations

- When do these **virtual objects** describe real **physical excitations**?

  - must be **orthogonal to ground state**:

$$\langle\Omega|\, a_i^\dagger |\Omega\rangle \equiv \boxed{\langle \boldsymbol{a_i^\dagger} \rangle = \mathbf{0}}$$

(if $\langle a_i^\dagger \rangle \neq 0$: **condensed**)

  - **single excitation well-defined**:

$$\boxed{\langle \boldsymbol{a_i a_i^\dagger} \rangle \neq \mathbf{0}}$$

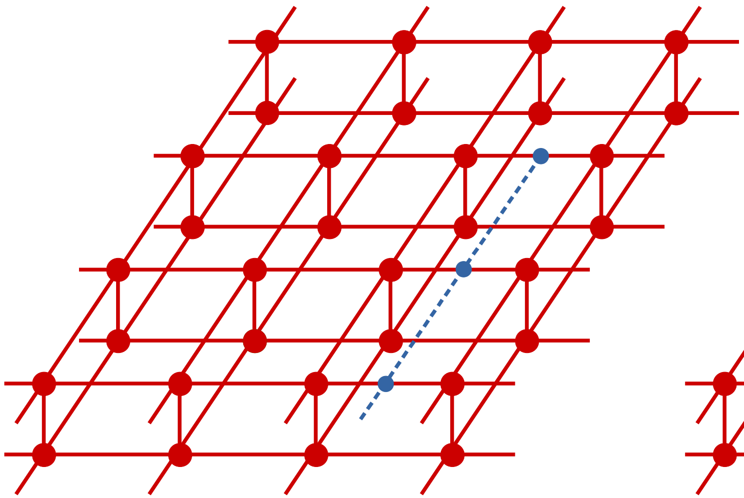
(i.e. massive excitation: $\langle a_i a_j^\dagger \rangle \sim e^{-|i-j|/\xi}$)

(if $\langle a_i a_i^\dagger \rangle = 0$: **confined** – only $\langle a_i a_i^\dagger a_j a_j^\dagger \rangle \sim e^{-|i-j|\xi}$)

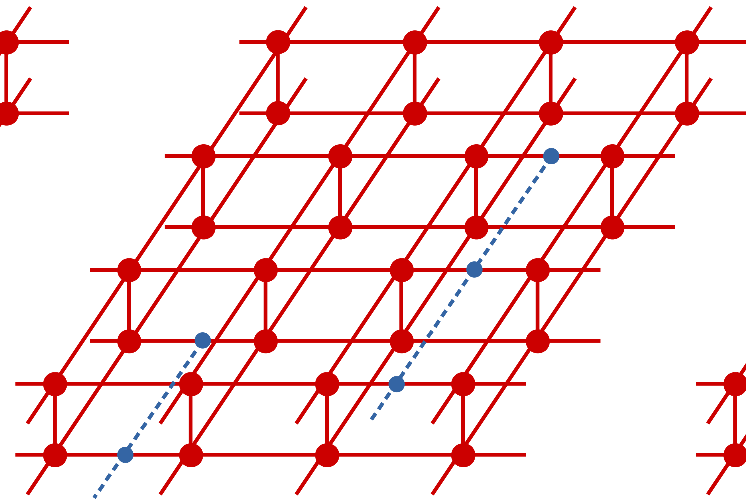
Relevant quantities for topological excitations

- Relevant expectation values for string-like (group action) excitations

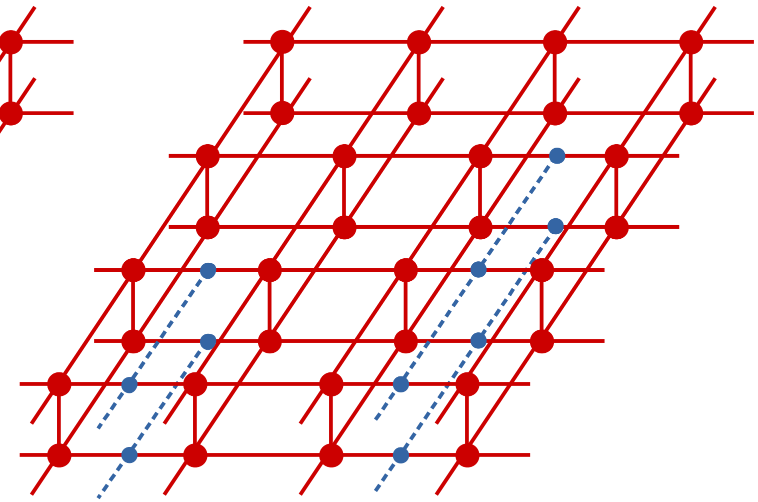
$\langle a_i^\dagger \rangle$



$\langle a_i a_j^\dagger \rangle$

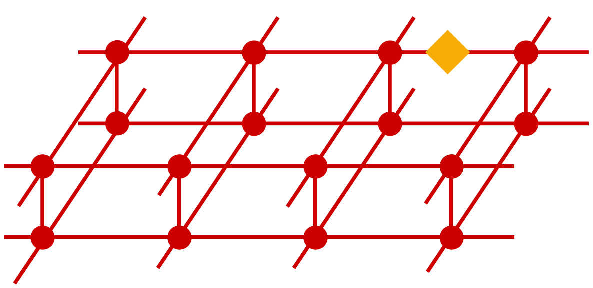


$\langle a_i a_i^\dagger a_j a_j^\dagger \rangle$

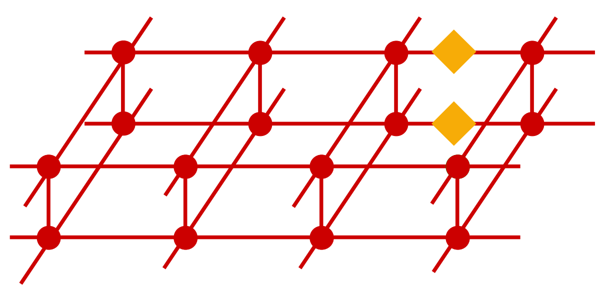


- Relevant expectation values for point-like (irrep action) excitations

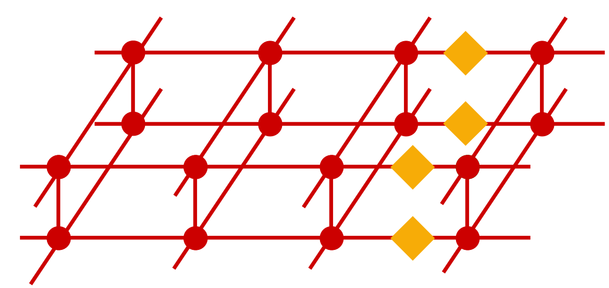
$\langle b_i^\dagger \rangle$



$\langle b_i b_j^\dagger \rangle$

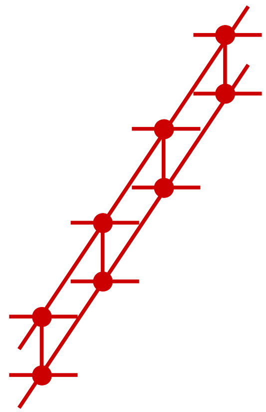


$\langle b_i b_i^\dagger b_j b_j^\dagger \rangle$



The transfer operator

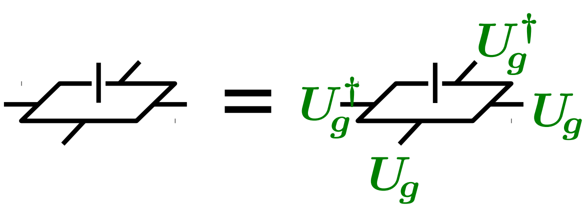
- Central object: **transfer operator**

$\mathbb{T} :=$  $\Rightarrow$ expectation values for strings + irreps evaluated in left & right fixed point of $\mathbb{T}$

- "morally": **transfer operator** $\leftrightarrow$ quasi-local **Hamiltonian** via $\mathbb{T} \sim e^{-H}$

fixed point $\leftrightarrow$ ground state

- **transfer operator inherits symmetries** from tensor:



$$[\mathbb{T}, U_g^{\otimes N} \otimes \mathbb{1}] = 0$$
$$[\mathbb{T}, \mathbb{1} \otimes \bar{U}_g^{\otimes N}] = 0$$

... & similar for physical symmetries

- What are the **possible behaviors** of the fixed point **w.r.t. symmetries**?

Symmetry breaking

- Hamiltonian w/ **symmetry** $[H, Z^{\otimes N}] = 0$ $\rightarrow$ What about **ground states**?

ground state **symmetric**	ground states **break symmetry**
$Z^{\otimes N}$ $\lvert \mathbf{o\ o\ o\ o\ o\ o\ o\ o} \rangle$	$\lvert \mathbf{+\ +\ +\ +\ +\ +\ +\ +} \rangle$ $\lvert \mathbf{-\ -\ -\ -\ -\ -\ -\ -} \rangle$ $\quad Z^{\otimes N}$

semi-infinite string

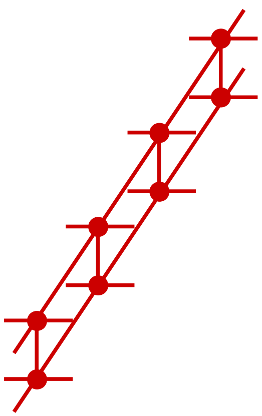
$Z\ Z\ Z\ Z$ $\mathbf{o\ o\ o\ o\ o\ o\ o\ o}$ $\downarrow$ $\mathbf{o\ o\ o\ o\ o\ o\ o\ o}$	$Z\ Z\ Z\ Z$ $\mathbf{+\ +\ +\ +\ +\ +\ +\ +}$ $\downarrow$ $\mathbf{-\ -\ -\ -\ +\ +\ +\ +}$ **infinite domain wall**

order parameter X (w/ $ZX = -XZ$)

$\langle \psi \vert \boldsymbol{X} \vert \psi \rangle = 0$	$\langle \psi \vert \boldsymbol{X} \vert \psi \rangle \neq 0$

Example I: Toric Code

Example: Toric Code



Symmetries of $\mathbb{T}$:

$\mathbb{Z}_2$ in ket, bra, and jointly

$$\left.\begin{array}{l}[\mathbb{T}, Z^{\otimes\infty}\otimes \mathbb{1}] = 0 \\ [\mathbb{T}, \mathbb{1}\otimes Z^{\otimes\infty}] = 0\end{array}\right\} \text{must behave identically}$$

$$[\mathbb{T}, Z^{\otimes\infty}\otimes Z^{\otimes\infty}] = 0$$

Anyons:

group el. \ irreps	1	-1
0	0	e
1	m	f

symmetry of fixed point(s) of $\mathbb{T}$

has all symmetries
$\Rightarrow$ **trivial**
(m condensed)

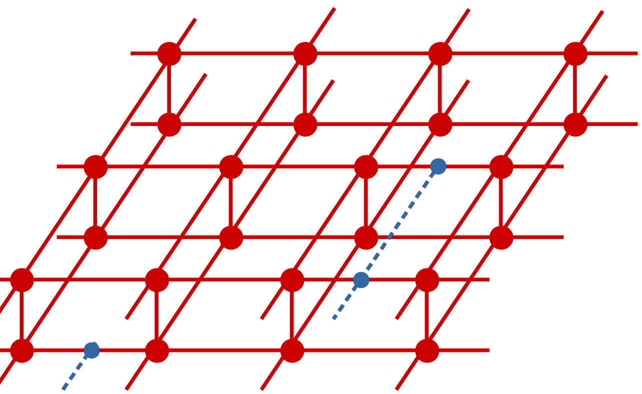
breaks $Z\otimes\mathbb{1}$
conserves $Z\otimes Z$
$\Rightarrow$ **topological**

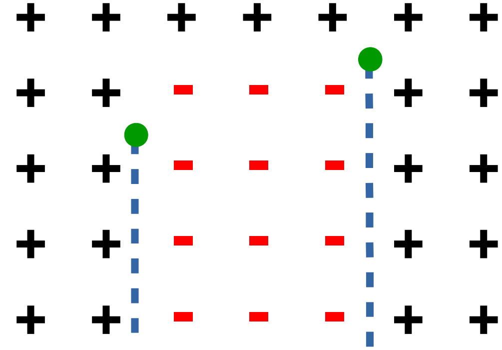
breaks all symmetries
$\Rightarrow$ **trivial**
(e condensed)

Topological phase

- Symmetries: $Z^{\otimes\infty} \otimes \mathbb{1}$ broken, $Z^{\otimes\infty} \otimes Z^{\otimes\infty}$ conserved

$$\langle \boldsymbol{a_i a_j^\dagger} \rangle \sim e^{-|i-j|/\xi}$$



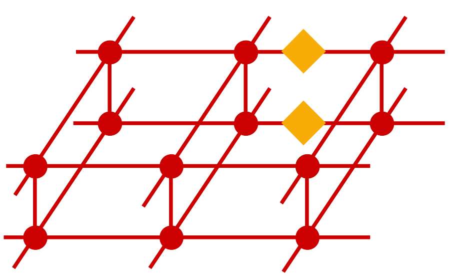


$\Rightarrow$ localized (massive) deconfined magnetic excitations

 $= X \otimes X$

- $\{X \otimes X, Z \otimes \mathbb{1}\} = 0$ and $[X \otimes X, Z \otimes Z] = 0$

  $\Rightarrow$ order parameter for broken sym. $\Rightarrow$ $\langle \boldsymbol{b_i b_i^\dagger} \rangle \neq \boldsymbol{0}$



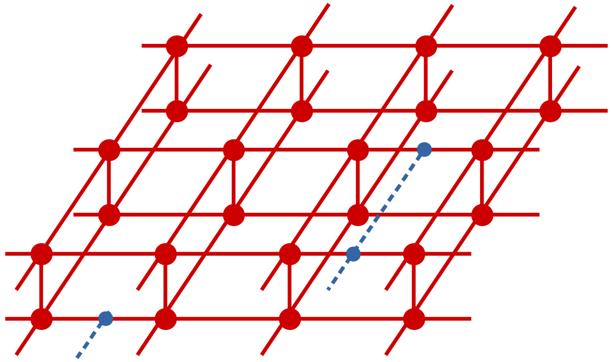
- $\{X \otimes \mathbb{1}, Z \otimes Z\} = 0$, symmetry *not* broken $\Rightarrow$ $\langle \boldsymbol{b_j^\dagger} \rangle = \boldsymbol{0}$

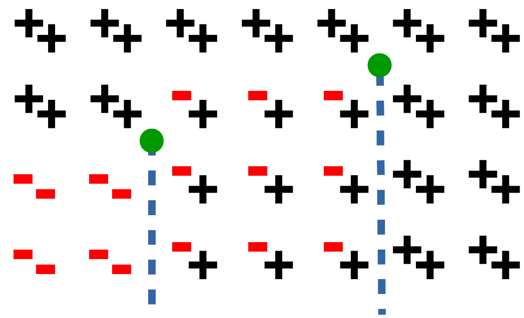
  $\Rightarrow$ $\langle \boldsymbol{b_i b_j^\dagger} \rangle \sim e^{-|i-j|/\xi}$

$\Rightarrow$ localized (massive) deconfined electric excitations

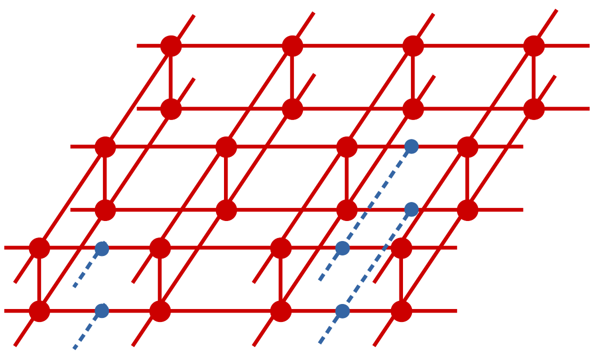
Trivial phase I: Magnon confinement

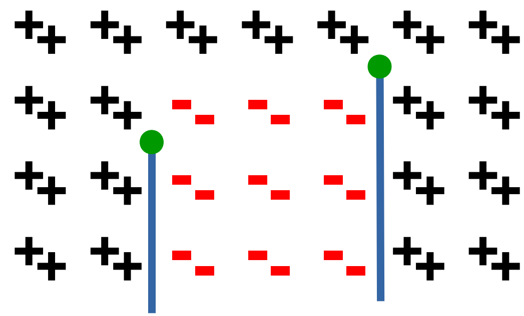
- Symmetries: $Z^{\otimes\infty} \otimes \mathbb{1}$ broken and $Z^{\otimes\infty} \otimes Z^{\otimes\infty}$ broken





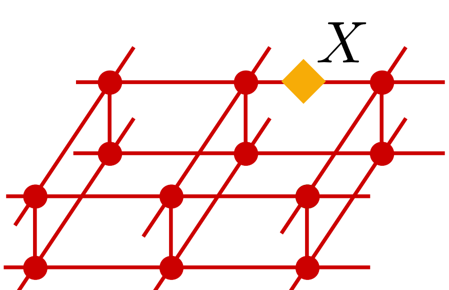
$$\langle \boldsymbol{a_i a_j^\dagger} \rangle = 0$$





$$\langle \boldsymbol{a_i a_i^\dagger a_j a_j^\dagger} \rangle \sim e^{-|i-j|/\xi}$$

$\Rightarrow$ magnetic excitations become confined

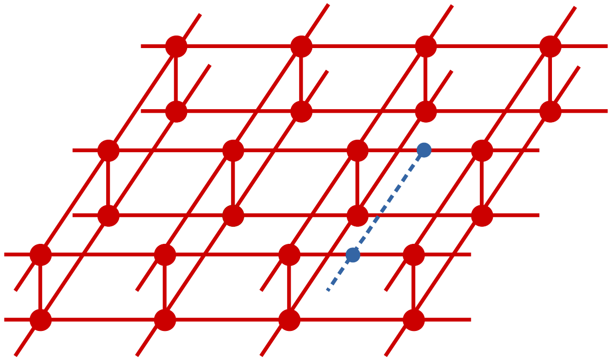


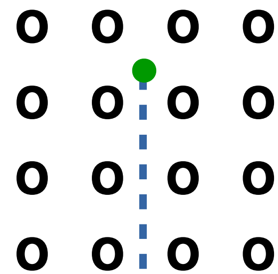
- $\{X \otimes \mathbb{1}, Z \otimes Z\} = 0$, symmetry broken $\Rightarrow$ $\langle \boldsymbol{b_i^\dagger} \rangle \neq 0$

$\Rightarrow$ electric excitations condense into ground state

Trivial phase II: Magnon condensation

- no symmetry broken

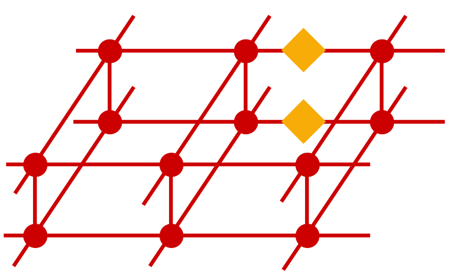




$$\langle a_i^\dagger \rangle \neq 0$$

$\Rightarrow$ magnetic excitations condense into ground state

 $= X \otimes X$



- $\{X \otimes X, Z \otimes \mathbb{1}\} = 0$, symmetry *not* broken $\Rightarrow$ $\langle b_i b_i^\dagger \rangle = 0$
- $\langle b_i b_i^\dagger b_j b_j^\dagger \rangle \sim e^{-|i-j|/\xi}$

$\Rightarrow$ electric excitations become confined

Example II: $\mathbb{Z}_4$ double, Toric Code, Double Semion

- $\mathbb{Z}_4$ double: symmetry group $\mathbb{Z}_4$
- Symmetries of $\mathbb{T}$: $\mathbb{Z}_4$ in ket, bra, and jointly

~~($\mathbb{Z}_4 \cong \{00,13,22,31\}$: forbidden by positivity~~
~~$\rightarrow$ diagonal $\times$ off-diagonal subgroup required)~~

Anyons:

	1	i	-1	$-i$
0	0			
1				
2				
3				

double semion phase:
need to **condense dyon**

$\mathbb{Z}_4$ double model
symmetry: $\mathbb{Z}_4 \cong \{00,11,22,33\}$

Toric Code
$\mathbb{Z}_2 \cong \{00,22\}$

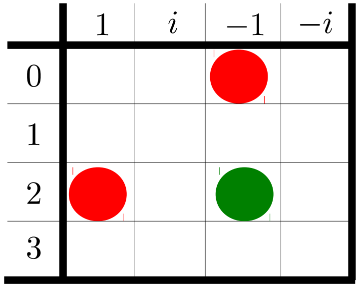
Toric Code or **Double Semion**
$\mathbb{Z}_4 \times \mathbb{Z}_2 \cong \{00,11,22,33,02,13,20,31\}$

trivial
$\mathbb{Z}_1 \cong \{00\}$

trivial
$\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \{00,02,20,22\}$

trivial
$\mathbb{Z}_4 \times \mathbb{Z}_4$

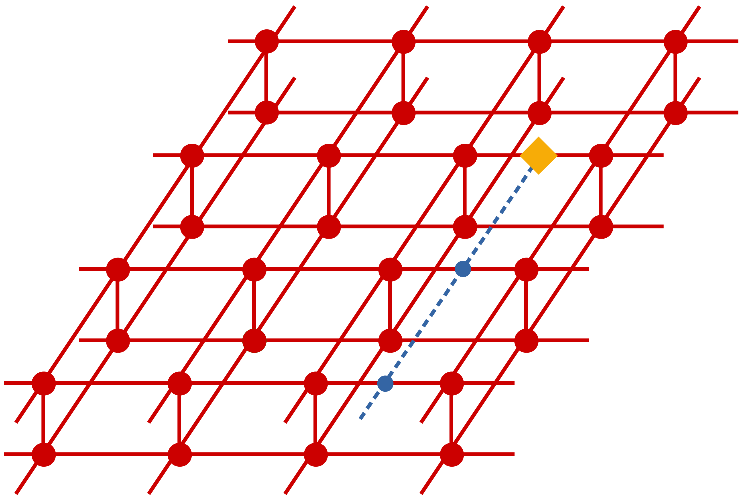
Dyon condensation and SPT phases



- Difference betw. Toric Code & Double Semion?

  $\Rightarrow$ **Identical symmetry,** but **inequivalent fixed points** of $\mathbb{T}$!

- How to **condense a dyon**?



- dyon = **symmetry string + order parameter** at end
- must have non-zero expectation value
- however: both string & order parameter must individually vanish!

All phases under symmetries

ground state **symmetric**

$Z^{\otimes N} |\, o\ o\ o\ o\ o\ o\ o\ o\,\rangle$

$Z\ Z\ Z\ Z$

o o o o o o o o

o o o o o o o o

$\langle Z \cdots Z I \cdots \rangle \neq 0$

$\langle \psi | X | \psi \rangle = 0$

ground states **break symmetry**

$|+\ +\ +\ +\ +\ +\ +\ +\rangle$

$|-\ -\ -\ -\ -\ -\ -\ -\rangle$

$Z^{\otimes N}$

$Z\ Z\ Z\ Z$

+ + + + + + + +

- - - - + + + +

$\langle Z \cdots Z I \cdots \rangle = 0$

$\langle \psi | X | \psi \rangle \neq 0$

Symmetry-protected order

- symmetric ground state
- order parameter: zero exp. value
- semi-infinite string: zero exp. value

- String + order parameter
  $\rightarrow$ **non-zero expectation value!**

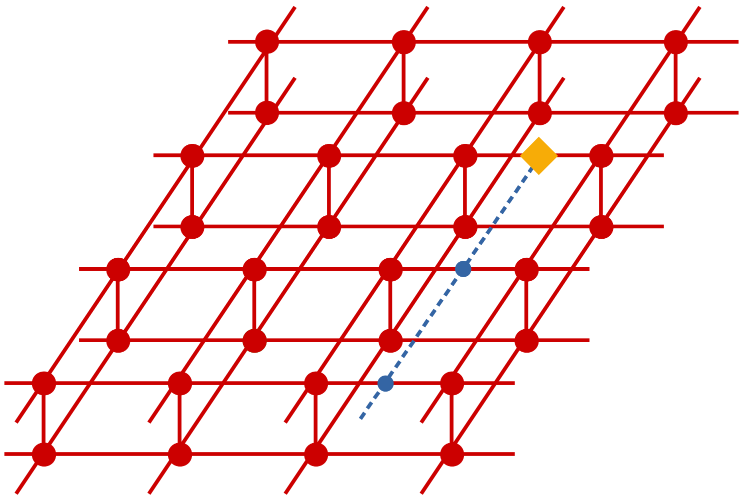
$$\langle Z \cdots Z X I \cdots \rangle \neq 0$$

"String order parameter"

Dyon condensation and SPT phases

MPQ

- How to **condense a dyon**?



- dyon = **symmetry string + order parameter** at end
- must have non-zero expectation value
- however: both string & order parameter must individually vanish!

- **Dyon condensation = symmetry protected order** in fixed point of transfer operator (i.e., at the boundary)

- Classification of **all topological phases** under a given **symmetry** U_g
  $\Rightarrow$ **full "phase diagram"** of $\mathbb{T}$ under **symmetry** $U_g \otimes \bar{U}_g$

- This includes **both symmetry-breaking and symmetry-protected phases** of the unbroken symmetry!

Conclusions

- **PEPS models**: local tensor $\rightarrow$ wavefunction + Hamiltonian
- **topological order** in PEPS $\leftrightarrow$ **symmetry of tensor**
- symmetry of tensor $\Rightarrow$ **symmetry at the boundary** (transfer operator)
- **symmetry breaking & SPT phases** at boundary
  $\leftrightarrow$ **topological phases** in the bulk
- **study topological phases & phase transitions "holographically"**
  through 1D phases at the boundary