Matrix Product Operators: Algebras and Applications

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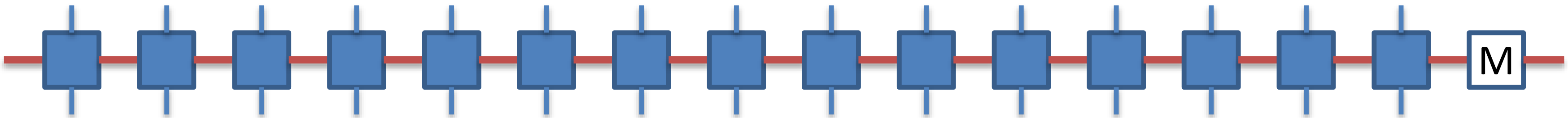
Outline

- Matrix Product Operators: examples of transfer matrices
- Matrix Product Operators: normal forms and diagonalization
- Matrix Product Operator Algebras:
  - Algebraic Bethe Ansatz and Yang Baxter equations
  - Tensor Fusion Categories and Anyons

Matrix Product Operators:

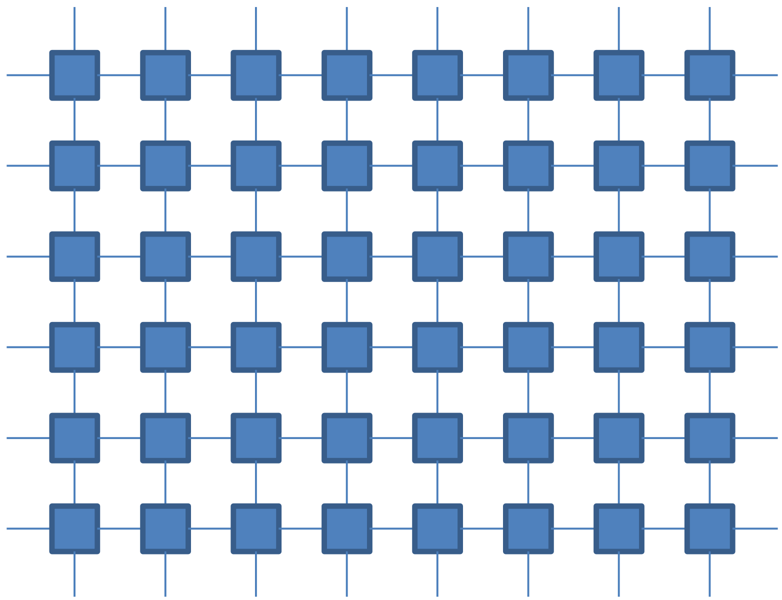


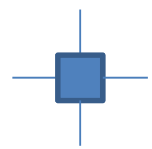
$$\hat{O}(A^{ij}, M) = \sum_{i_1 j_1 i_2 j_2 \cdots} \mathrm{Tr}\left(A^{i_1 j_1} A^{i_2 j_2} \cdots A^{i_N j_N}.M\right) |i_1\rangle\langle j_1| \otimes |i_2\rangle\langle j_2| \otimes \cdots \otimes |i_N\rangle\langle j_N|$$

- The MPO is characterized by the tensor $A^{ij}_{\alpha\beta}$; for given d,D, the dimension of the corresponding manifold is $(d^2-1)D^2$
- MPO's pop up everywhere in many body systems or systems with a tensor product structure: partition functions in statistical physics, counting problems, non-equilibrium steady states, path integrals, …

MPO and statistical physics

- Example: 2-D Ising model: $Z = e^{-\beta F} = \sum_{s_1 s_2 \cdots} e^{-\beta H(s_1, s_2, \cdots)}$
  - Partition function is given by the contraction of the following tensor network:





: $A^{ij}_{\alpha\beta} = \delta_{i\alpha} G_{ij} G_{\alpha\beta}$

$$G = \begin{bmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{bmatrix}$$

  - All the physics is encoded in the eigenstructure of the transfer MPO

- Similarly, the leading eigenvalue of transfer matrix provides the scaling of the number of configurations on the lattice in counting problems:
  - Counting dimer configurations on the square lattice:

$$A^{ij}_{\alpha\beta} = (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)_{\alpha\beta ij}$$

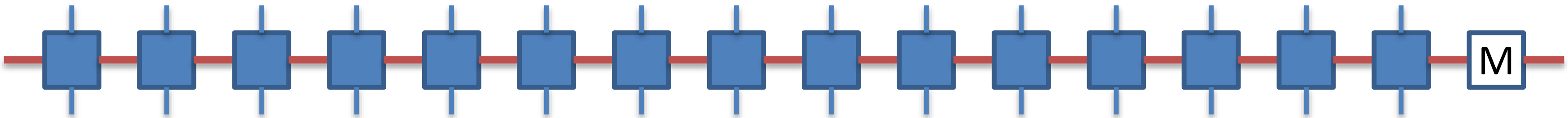
  - Entropy of spin-ice on the square lattice:

$$A^{ij}_{\alpha\beta} = (|0011\rangle + |0110\rangle + |1100\rangle + |1001\rangle + |0101\rangle + |1010\rangle)_{\alpha\beta ij}$$

  - Number of Configurations of “hard disks”:

$$A^{ij}_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}_{i\alpha} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}_{ij} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}_{\alpha\beta}$$

MPO and non-equilibrium physics



- Probabilistic cellular automata, mapping probability functions of n bits to probability distributions:

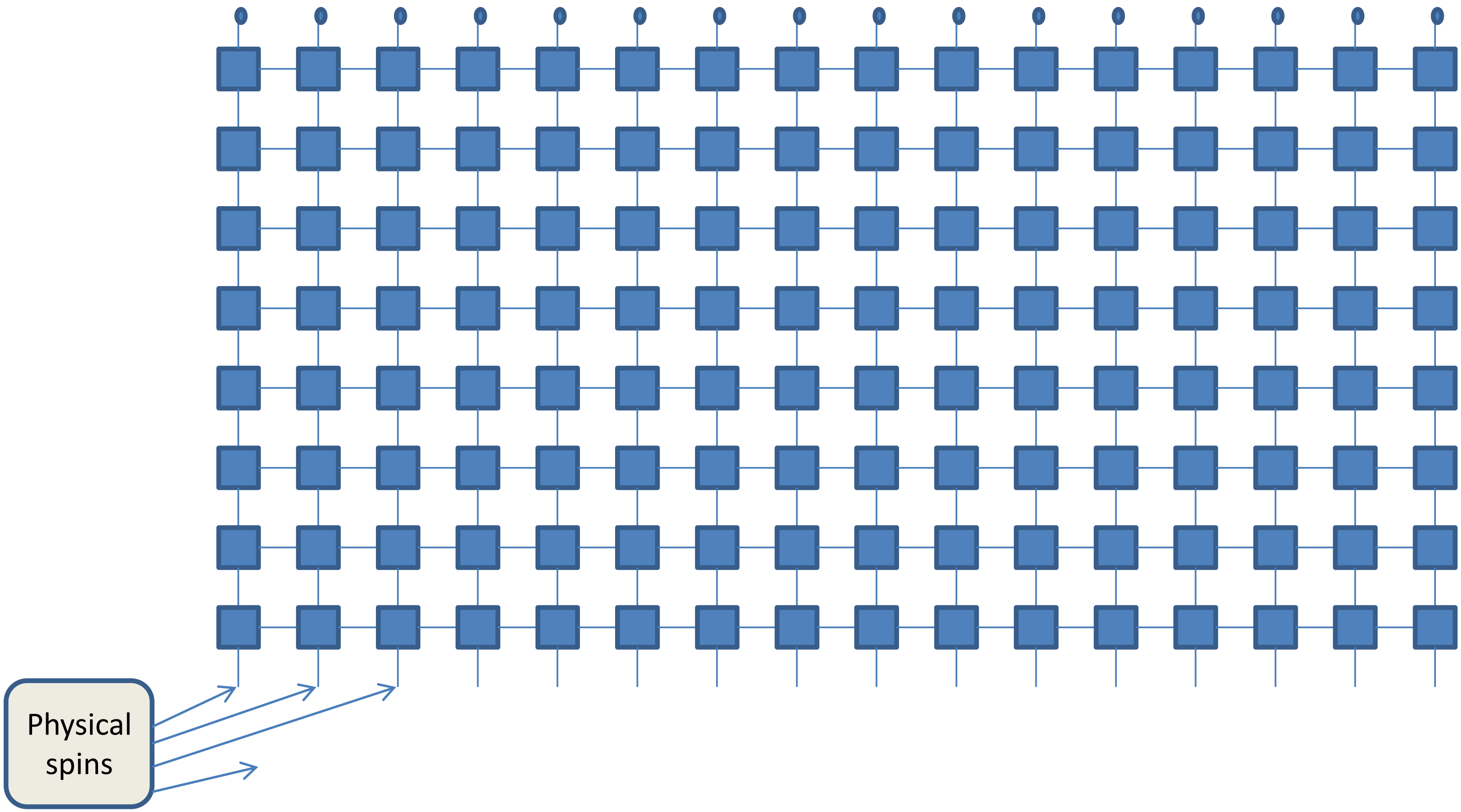
$$p\left(i_1, i_2, \cdots i_N\right)$$

- Typical Examples:
  - Percolation
  - Asymmetric exclusion processes and traffic

- Main purpose: find fixed point (i.e. leading eigenvector) of corresponding MPO
  - Non-equilibrium phase transitions happen when the MPO becomes gapless (with corresponding critical exponents, …); e.g. directed percolation universality class, traffic jams, …

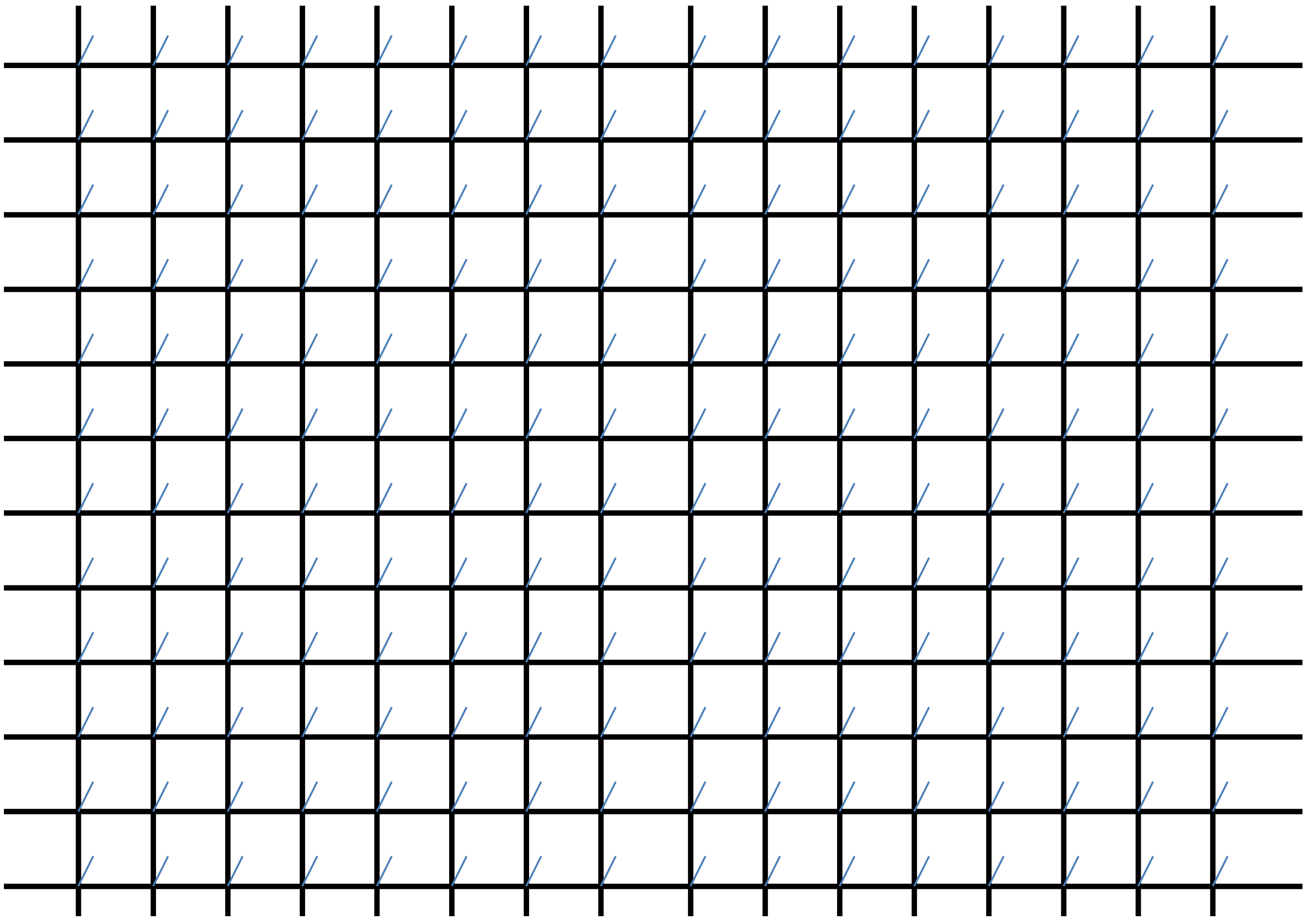
MPO's and path Integral representation of ground states

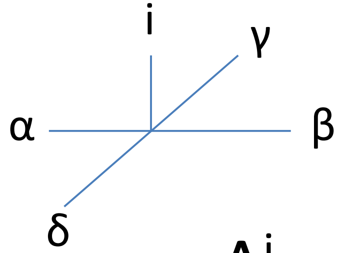
- Let us consider an arbitrarily Hamiltonian of a quantum spin system, and a path integral $\exp\left(-\beta\mathcal{H}\right)|\psi_0\rangle$ representing the ground state for $\beta \to \infty$



MPO's and entanglement degrees of freedom:

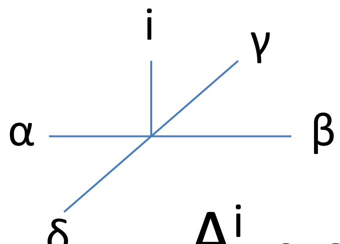
- Consider the following tensor network (PEPS):



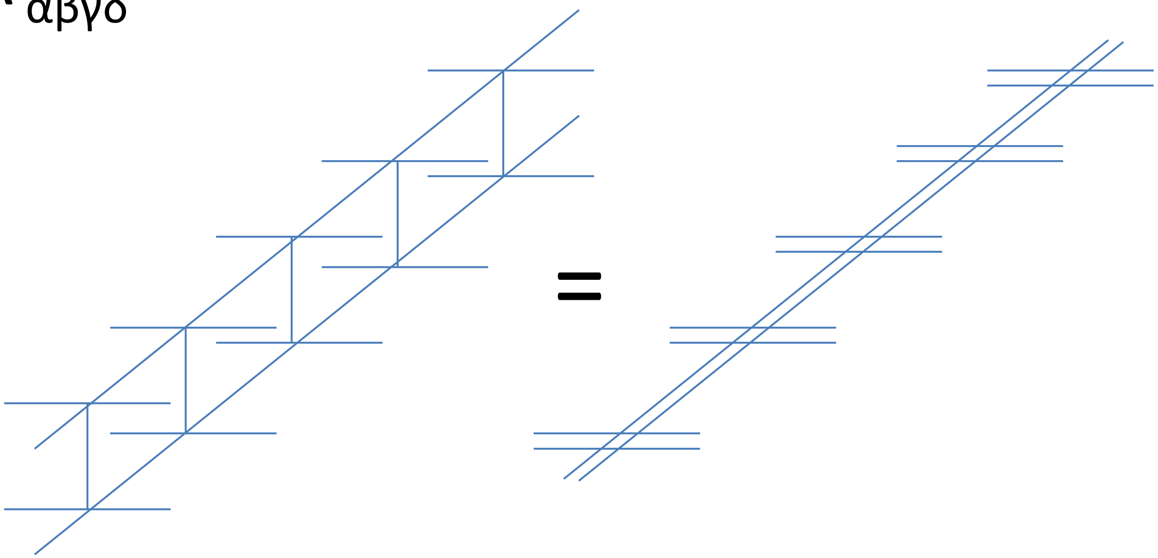


$= A^{i}_{\alpha\beta\gamma\delta}$

- The main features of PEPS are encoded in the eigenvalues and eigenvectors of the corresponding transfer matrices (similar to classical statistical physics, although here we have a "double layer" structure and we deal with a state as opposed to a Hamiltonian)



$A^{i}_{\alpha\beta\gamma\delta}$



- Different phases of matter will be characterized by symmetries and symmetry breaking on the entanglement degrees of freedom ("virtual" level)

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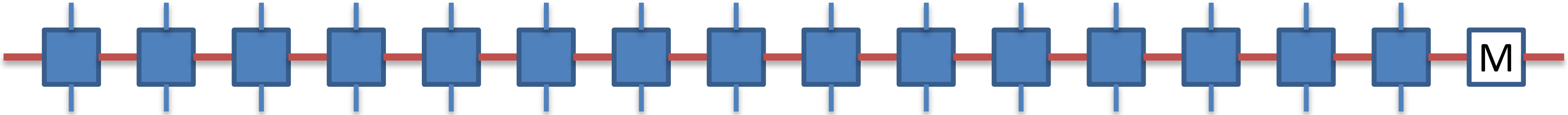
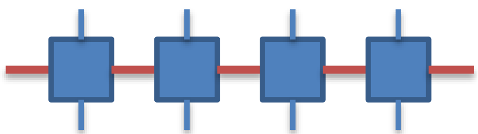
Matrix Product Operators



$$\hat{O}(A^{ij}, M) = \sum_{i_1 j_1 i_2 j_2 \cdots} \mathrm{Tr}\left(A^{i_1 j_1} A^{i_2 j_2} \cdots A^{i_N j_N} . M\right) |i_1\rangle\langle j_1| \otimes |i_2\rangle\langle j_2| \otimes \cdots \otimes |i_N\rangle\langle j_N|$$

- Injectivity: there exists a finite n such that the map from the blue to the red indices is full rank



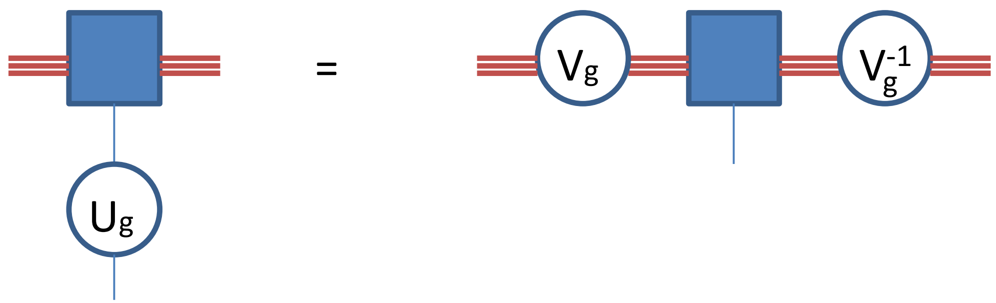
- Fundamental Theorem of Injective Matrix Product States:

$$\hat{O}(A^{ij}) = \hat{O}(B^{ij}) \ \textit{for} \ N \to \infty \ \textit{iff}$$

there exists a gauge transform which transforms A^{ij} *into* B^{ij}*:*

$$A^{ij} X = X B^{ij}$$

- Application: consider an MPS with a global symmetry U_g ; then



$$V_g . V_h = e^{i\omega(g,h)} V_{g.h}$$

- Classification of projective representations leads to classification of 1-D symmetry protected phases (Pollmann, Turner, Berg, Oshikawa ‘10; Chen, Gu, Wen ‘11; Schuch, Perez-Garcia, Cirac ’12)

- We will later need the fundamental theorem in case of non-injective MPS:
  - If MPO is non-injective, there exists a basis in which the MPS is upper block diagonal:

$$U A^{ij} U^\dagger = \left[\begin{array}{cc} A_{11}^{ij} & A_{12}^{ij} \\ 0 & A_{22}^{ij} \end{array} \right]$$

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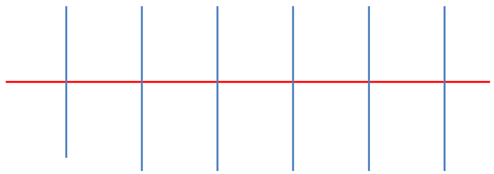
$$U A^{ij} U^{\dagger} = \begin{bmatrix} A_{11}^{ij} & A_{12}^{ij} \\ 0 & A_{22}^{ij} \end{bmatrix}$$

  - The upper triangular blocks do not contribute to the MPO on the physical level, so they can be set to zero, leaving us with a direct sum of 2 MPO's which can again be injective or not. Repeat this until all invariant subspaces are injective.

  - Fundamental theorem of MPS: 2 MPS are equal for all N iff the injective MPS's in the invariant subspaces are equal to each other up to a gauge transform

Example: CZX MPO

Chen, Gu, Wen '12



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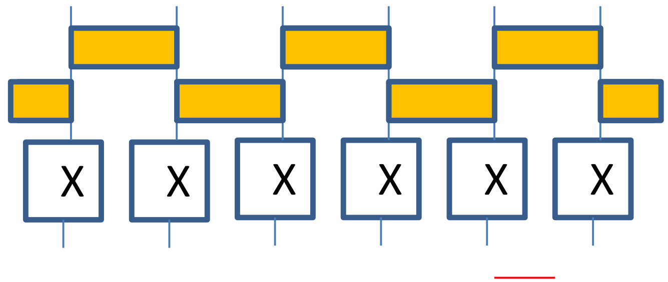
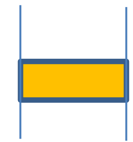


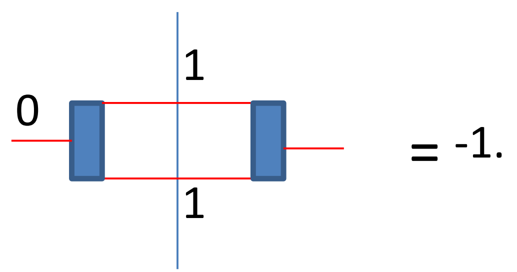


$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$$A^{ij}_{\alpha\beta} = \delta_{\beta i}(-1)^{\alpha . i} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{ij}$$

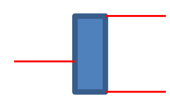
Let us take the square of this operator, giving rise to a non-injective MPO with bond dimension 4: CZX.CZX = $(-1)^N$ .I

$$B^{00} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



$$B^{11} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

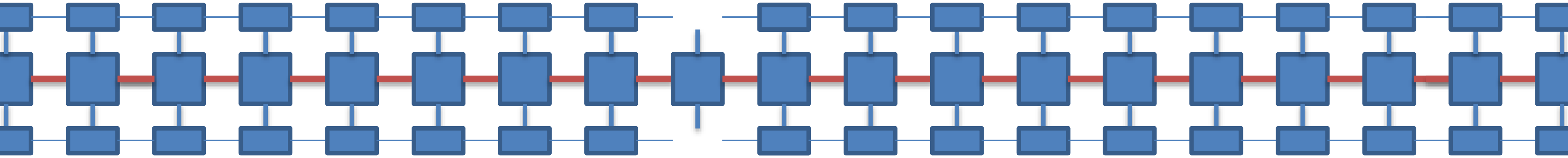
with



$= |0\rangle \left(\langle 0|\langle 1| - \langle 1|\langle 0|\right)$

Diagonalization of MPO's

- For many interesting cases, exact solutions for the eigenvalues of the MPO's have been found using mappings to free fermions (Ising, dimer), Bethe ansatz (spin ice), or algebraic matrix product state methods (ASEP)
- More generally, leading eigenvectors can be approximated very efficiently using variational matrix product state algorithms:



  - We get an effective Hamiltonian for the "entanglement" degrees of freedom
  - Many other alternatives: corner transfer matrices (Nishino), iTEBD (Vidal), iDMRG (McCulloch), ...: converge very fast

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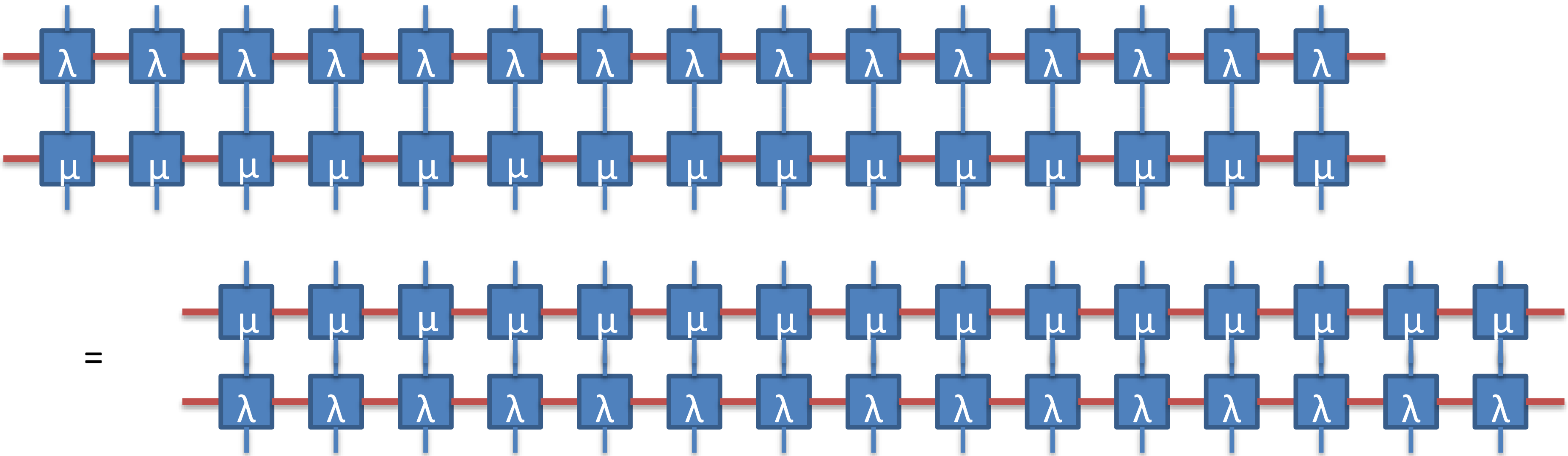
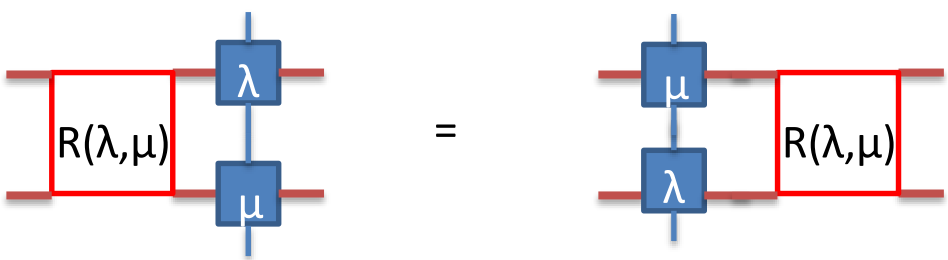
Algebraic Bethe Ansatz

- Central concept in integrable models of statistical mechanics and quantum spn chains: find a 1-parameter set of commuting MPO's:

$$T(\lambda).T(\mu) = T(\mu).T(\lambda)$$

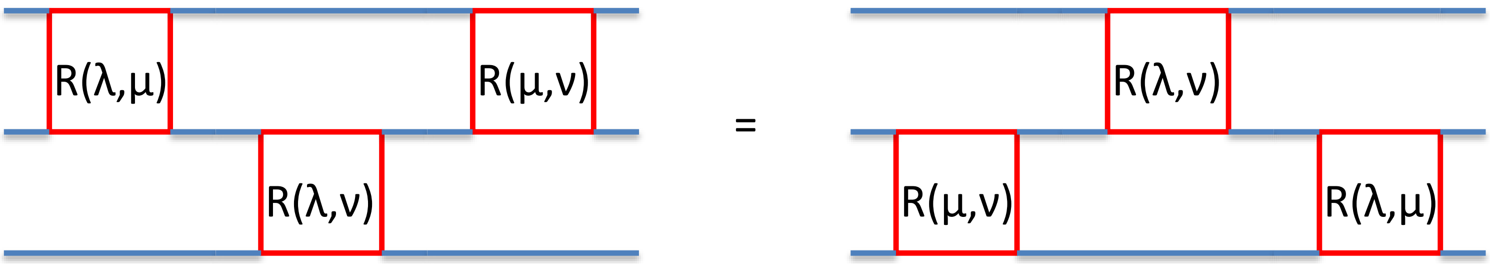


- Fundamental theorem of MPO's: there exists a R(λ,μ) such that

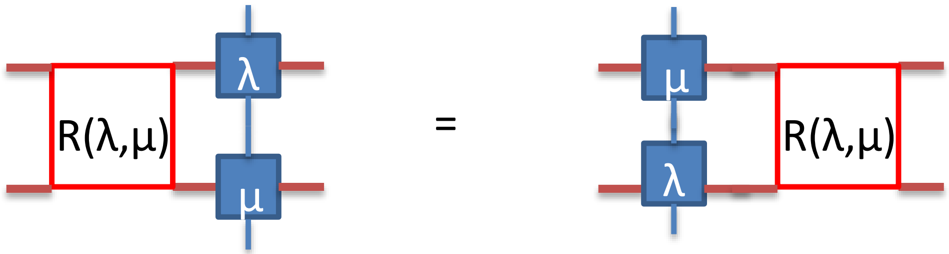


Yang Baxter

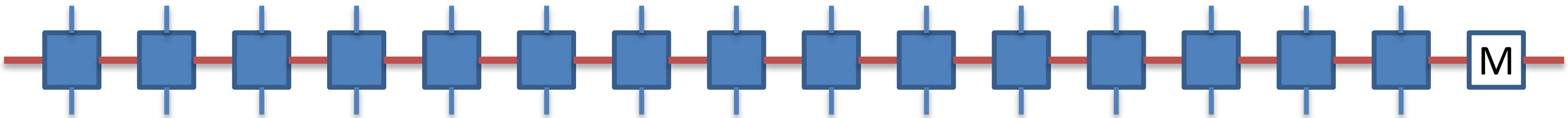
- The R-matrices have to satisfy an associativity condition, which is encoded in the Yang Baxter equation (condition on 3-particle scattering):



- Logic of Bethe ansatz construction: find solution of Yang Baxter, and then use those R's to construct MPO's satisfying the pulling through equation



  – Of course, many other solutions are possible



- In the case of D=2, we construct 4 MPO's with different boundary conditions:
  - A(λ), B(λ), C(λ), D(λ)

$$M_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$M_C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad M_D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- The Yang Baxter solutions for D=2 are of the form

$$\mathrm{R}(\lambda, \mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix}$$

  - Yang Baxter dictates the following commutation relations for the MPO's:

$$[B(\lambda),\ B(\mu)] = 0; \qquad [C(\lambda),\ C(\mu)] = 0;$$
$$A(\mu)B(\lambda) = f(\mu,\lambda)B(\lambda)A(\mu) + g(\lambda,\mu)B(\mu)A(\lambda);$$
$$D(\mu)B(\lambda) = f(\lambda,\mu)B(\lambda)D(\mu) + g(\mu,\lambda)B(\mu)D(\lambda);$$
$$C(\lambda)A(\mu) = f(\mu,\lambda)A(\mu)C(\lambda) + g(\lambda,\mu)A(\lambda)C(\mu);$$
$$C(\lambda)D(\mu) = f(\lambda,\mu)D(\mu)C(\lambda) + g(\mu,\lambda)D(\lambda)C(\mu);$$
$$[C(\lambda),\ B(\mu)] = g(\lambda,\mu)\left\{A(\lambda)D(\mu) - A(\mu)D(\lambda)\right\};$$
$$[A(\lambda),\ A(\mu)] = 0; \qquad [D(\lambda),\ D(\mu)] = 0;$$
$$B(\mu)A(\lambda) = f(\mu,\lambda)A(\lambda)B(\mu) + g(\lambda,\mu)A(\mu)B(\lambda);$$
$$D(\mu)C(\lambda) = f(\mu,\lambda)C(\lambda)D(\mu) + g(\lambda,\mu)C(\mu)D(\lambda);$$
$$A(\lambda)C(\mu) = f(\mu,\lambda)C(\mu)A(\lambda) + g(\lambda,\mu)C(\lambda)A(\mu);$$
$$B(\lambda)D(\mu) = f(\mu,\lambda)D(\mu)B(\lambda) + g(\lambda,\mu)D(\lambda)B(\mu);$$
$$[D(\lambda),\ A(\mu)] = g(\lambda,\mu)\left\{B(\lambda)C(\mu) - B(\mu)C(\lambda)\right\};$$
$$[A(\lambda),\ D(\mu)] = g(\lambda,\mu)\left\{C(\lambda)B(\mu) - C(\mu)B(\lambda)\right\};$$
$$[B(\lambda),\ C(\mu)] = g(\lambda,\mu)\left\{D(\lambda)A(\mu) - D(\mu)A(\lambda)\right\}.$$

Korepin: Quantum Inverse Scattering

$$[B(\lambda),\ B(\mu)] = 0; \qquad [C(\lambda),\ C(\mu)] = 0;$$
$$A(\mu)B(\lambda) = f(\mu,\lambda)B(\lambda)A(\mu) + g(\lambda,\mu)B(\mu)A(\lambda);$$

Using those relations, one can now readily construct eigenstates of the transfer matrix A(λ)+D(λ) of the form

$$\prod_{j=1}^{N} B(\lambda_j)|0\rangle$$

where the λ_j have to satisfy a consistency equation dictated by the g and f's (the so-called Bethe equations)

Example: spin ice

$$T(\lambda)^{ij}_{\alpha\beta} = \left(\lambda.\mathcal{I} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)_{i\alpha,j\beta}$$

Heisenberg antiferromagnet: $H = \sum_i \vec{S}_i \vec{S}_{i+1} = \frac{d}{d\lambda} \log \hat{O}(\lambda)\Big|_{\lambda=0}$

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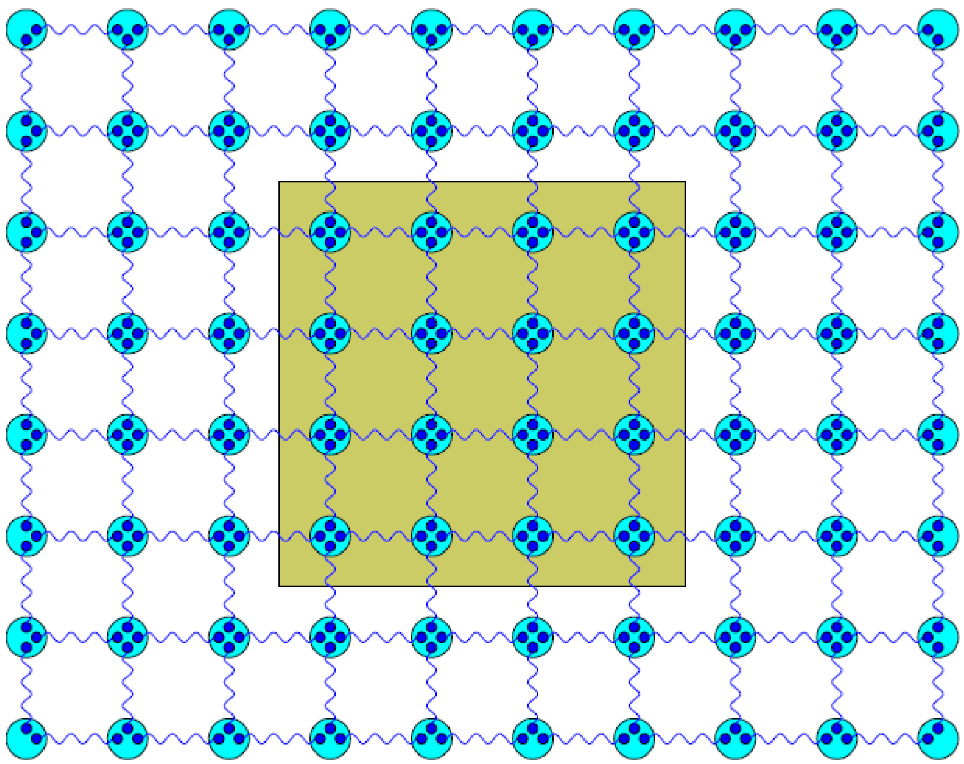
Topological Order

- Yang Baxter relations also play a central role in studies of the braid group, albeit where we deal with discrete labels as opposed to continuous ones
  - Natural question: can we find representations for topological phases using Matrix Product Operators?
  - We will develop well known tensor fusion category theory from the MPO point of view

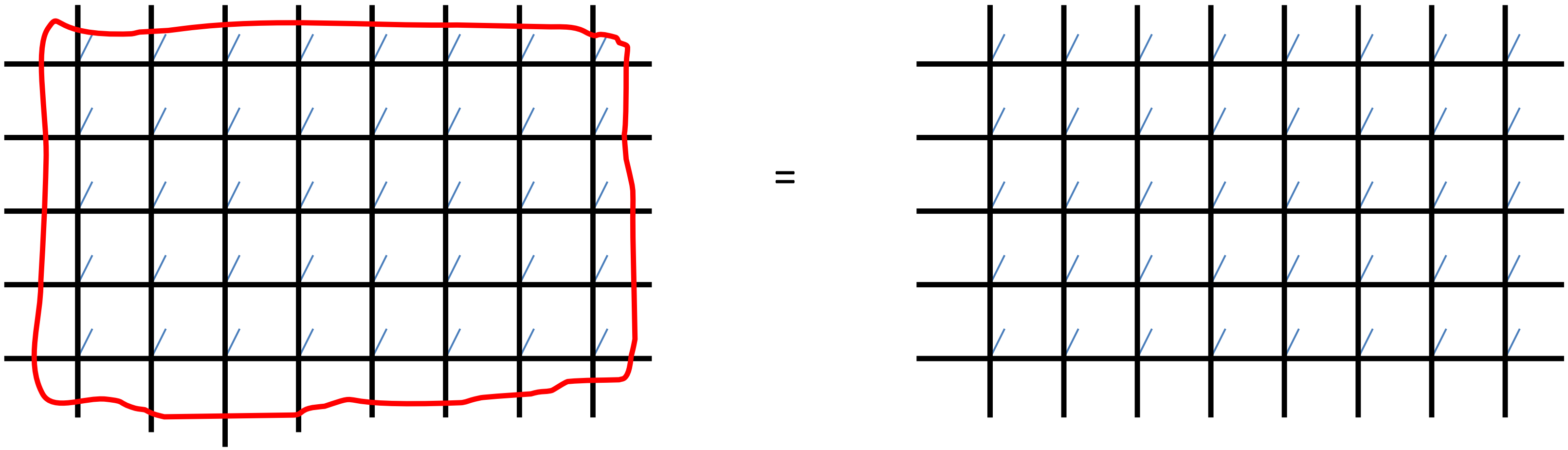
PEPS and topological order

- A large class of 2-D systems with topological order (so-called string nets and quantum doubles) have a simple description in terms of PEPS.
- The defining property of those PEPS is a symmetry on the entanglement (virtual) degrees of freedom which forces the PEPS to be non-injective and giving rise to the topological entanglement entropy

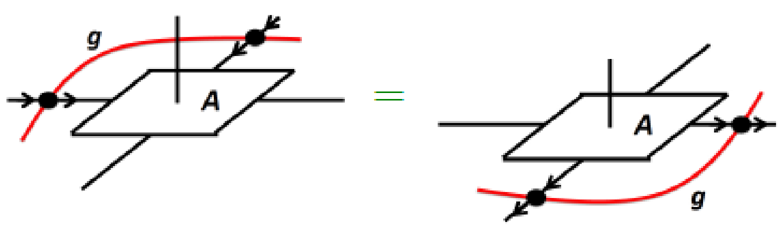


$$S(\rho) = c.L - \sqrt{\sum_i d_i^2}$$

- This symmetry can be completely characterized by a MPO projector:



- Locally, this symmetry is manifested by a “pulling through” equation:

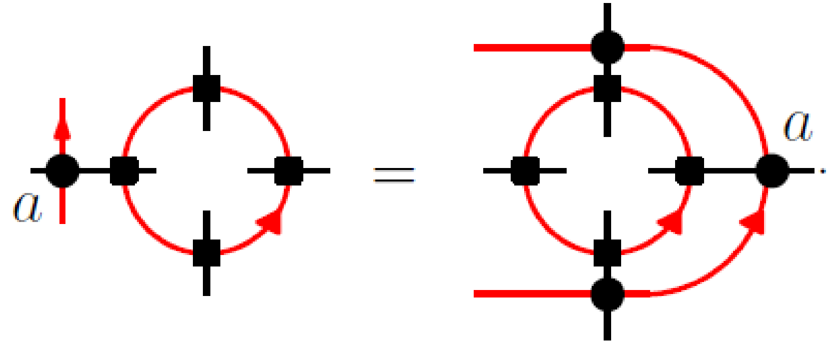


- The fundamental theorem of MPO’s can now be used to characterize the consistency condtions for such an MPO (cfr. Yang Baxter)

Example: Toric Code

- Simplest MPO projector: $$\frac{1}{2}\left(\mathcal{I} + Z^{\otimes N}\right)$$
  - Note that the MPO is NOT injective and is the sum of 2 MPO's with bond dimension 1

- Just like in the case of Bethe ansatz, we construct a PEPS tensor satisfying the pulling through by condition: MPO itself



- Consistency conditions for general MPO's:

Bultinck et al. arXiv 1511.08090

1. MPO is a projector so $P^2=P$
   If we bring P into normal form, $P_L = \sum_{a=1}^{\mathcal{N}} w_a O_a^L$ we get

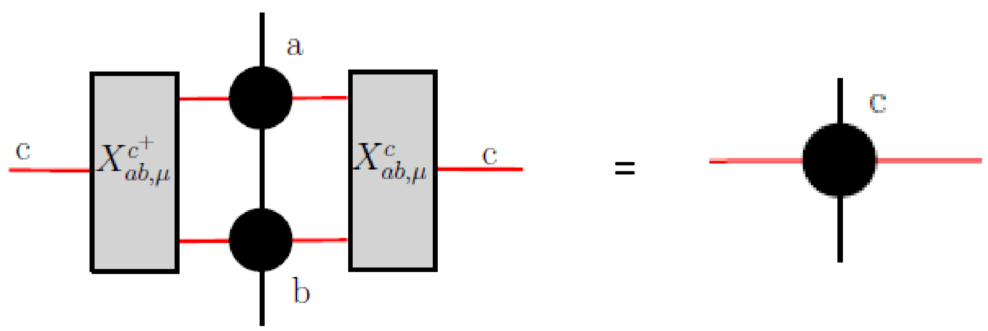
$$P_L^2 = \sum_{a,b=1}^{\mathcal{N}} w_a w_b O_a^L O_b^L = \sum_{a=1}^{\mathcal{N}} w_a O_a^L = P_L$$

and

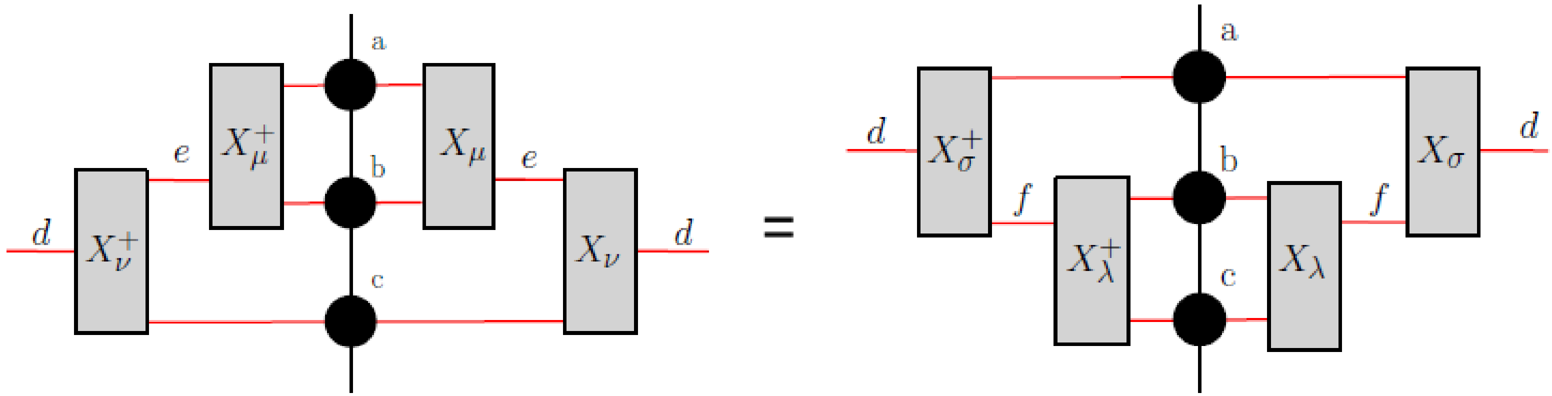
$$O_a^L O_b^L = \sum_{c=1}^{\mathcal{N}} N_{ab}^c O_c^L$$

$$\sum_{a,b=1}^{\mathcal{N}} N_{ab}^c w_a w_b = w_c$$

2. The fundamental theorem of MPS now tells us that P^2 must have the same blocks as P, hence there must exist a gauge transform X s.t.



3. Associativity + injectivity leads to F-symbols:



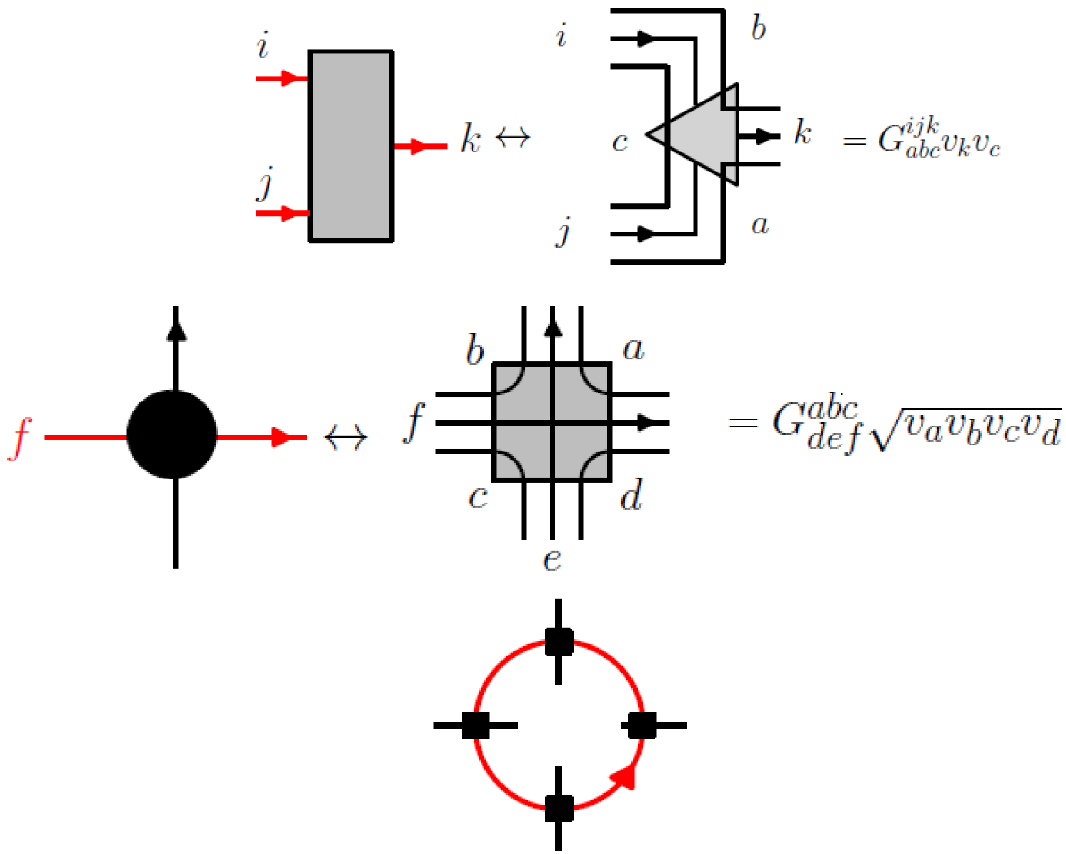
$$\left(X^e_{ab,\mu} \otimes \mathbb{1}_{\chi_c}\right) X^d_{ec,\nu} = \sum_{f=1}^{\mathcal{N}} \sum_{\lambda=1}^{N^f_{bc}} \sum_{\sigma=1}^{N^d_{af}} (F^{abc}_d)^{f\lambda\sigma}_{e\mu\nu} \left(\mathbb{1}_{\chi_a} \otimes X^f_{bc,\lambda}\right) X^d_{af,\sigma}$$

4. Associativity once more leads to conditions on F-symbols: pentagon equation

$$\sum_{h,\sigma\lambda\omega} (F^{abc}_g)^{f\mu\nu}_{h\sigma\lambda} (F^{ahd}_e)^{g\lambda\rho}_{i\omega\kappa} (F^{bcd}_i)^{h\sigma\omega}_{j\gamma\delta} = \sum_{\sigma} (F^{fcd}_e)^{g\nu\rho}_{j\gamma\sigma} (F^{abj}_e)^{f\mu\sigma}_{i\delta\kappa} .$$

This equation has only a discrete number of solutions for a given fusion rule

5. Just as in the case of the Yang Baxter equation in Bethe ansatz, solutions of the pentagon equation allows us to construct fusion tensors and MPO’s + PEPS satisfying all conditions that we wanted: we can find solutions both of associativity of V’s and pulling through of MPO’s by defining all tensors in term of F-symbols

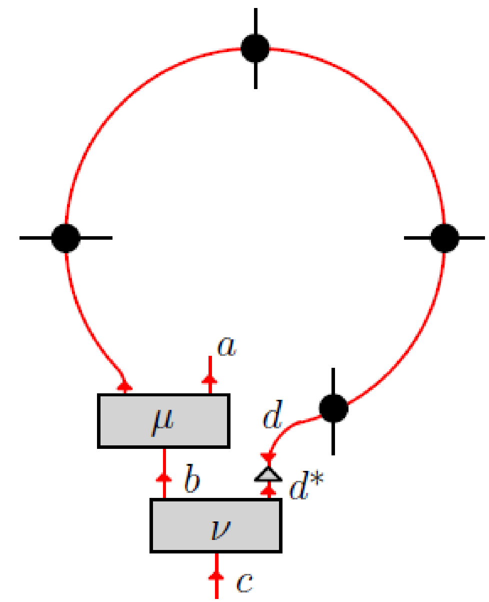


This construction allows us to define string nets (Levin and Wen) and quantum doubles on arbitrary lattices and also to construct examples with string tension; given any input tensor category, this construction leads to a modular tensor category

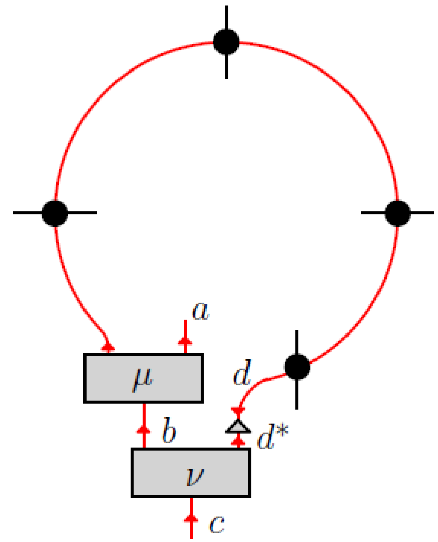
Anyons and Topological Sectors

- Modular tensor category defines a consistent theory of anyons; what is the MPO description of them?

$A_{abcd,\mu\nu} =$

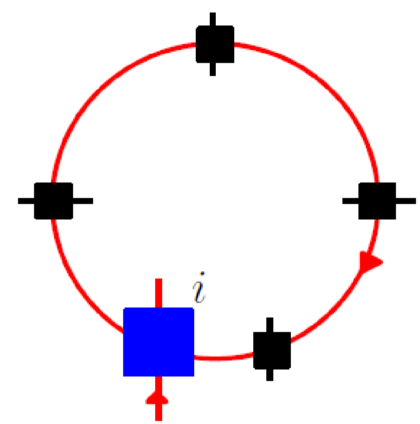


$$A_{abcd,\mu\nu} =$$

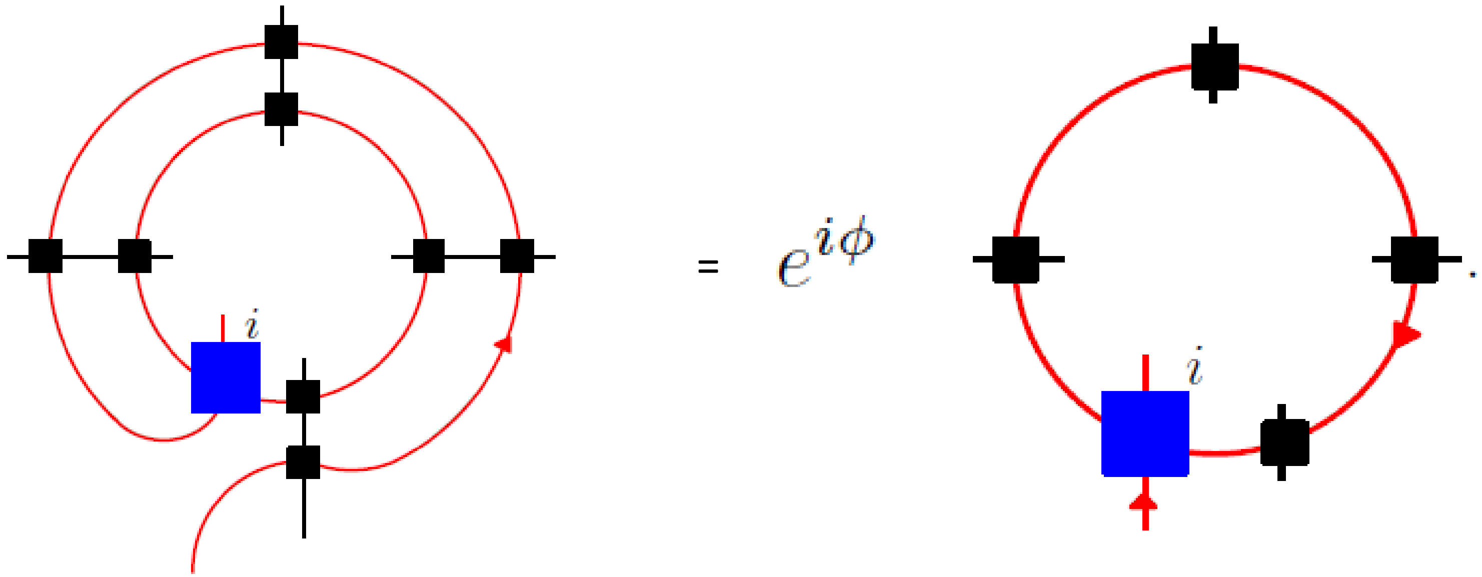


- This "anyon" tensor has 5 indices in and 5 out:
  - defines a C* algebra
  - Elementary excitations / anyons should be locally distinguishable by their charge. So they should be defined as idempotents of this algebra
  - It turns out that the elementary anyons correspond to the central idempotents of this algebra

$$\mathcal{P}_i = \sum_{abd,\mu\nu} c^i_{abd,\mu\nu} A_{abad,\mu\nu}$$

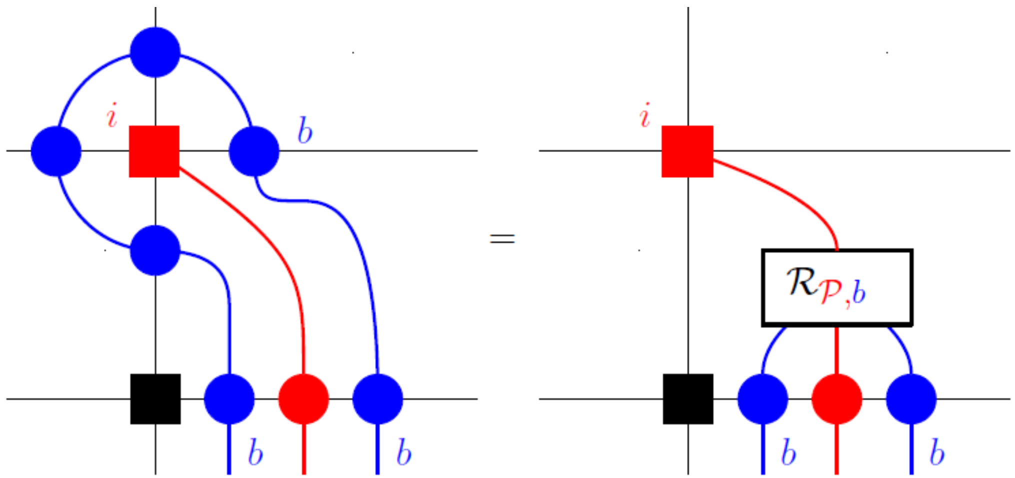


Topological spin:

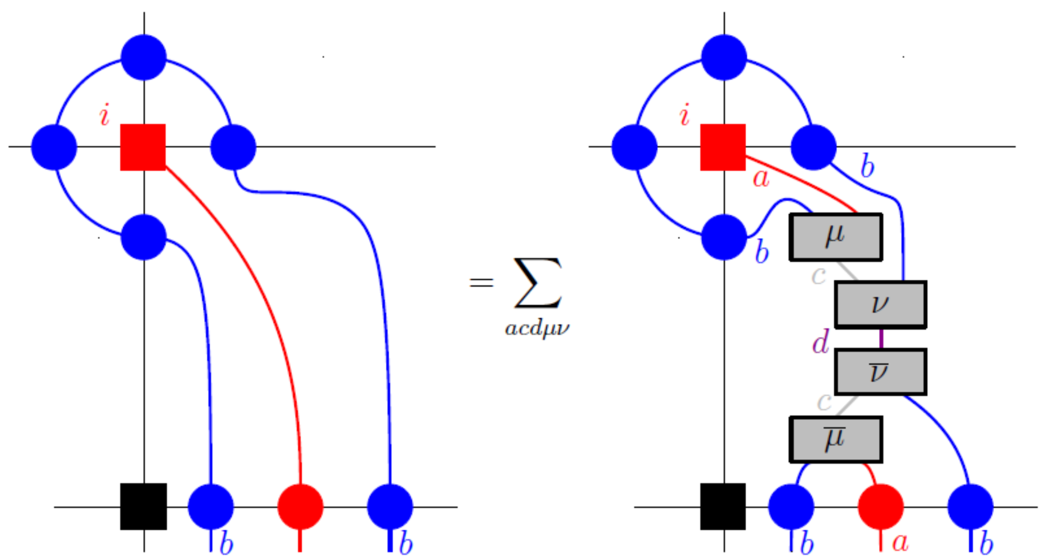


- Example: the 4 central idempotents for the toric code are given by $I \pm Z^{\otimes 4}$ with or without a string of Z's (1,e,m,em) and hence the topological spins are (0,0,0,π)

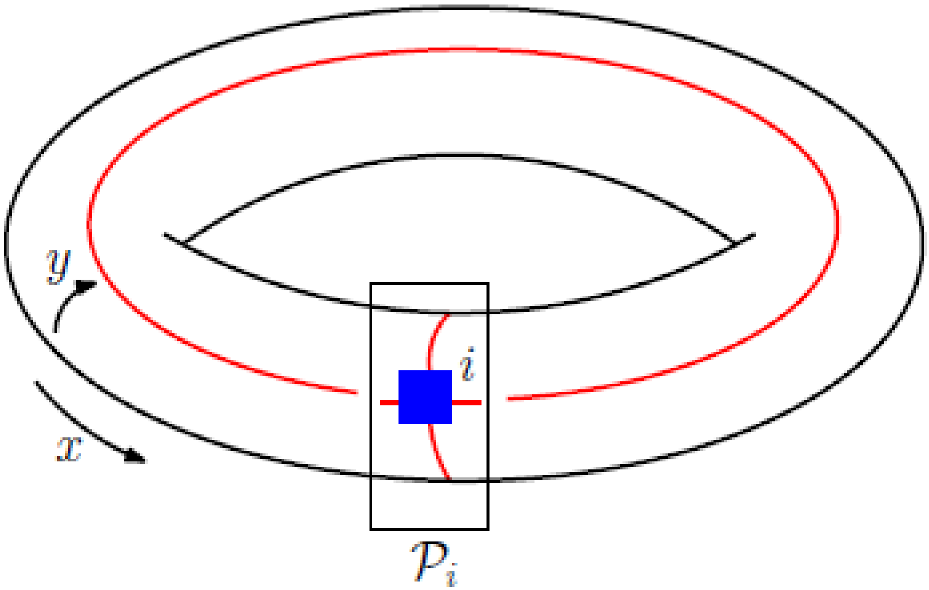
Braiding



- It can readily be seen that the braiding matrix R is itself determined by the central idempotent ("teleportation")



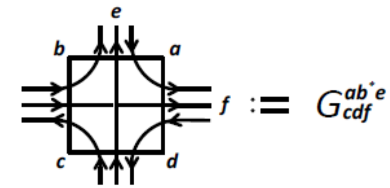
Topological Sectors in the Ground State

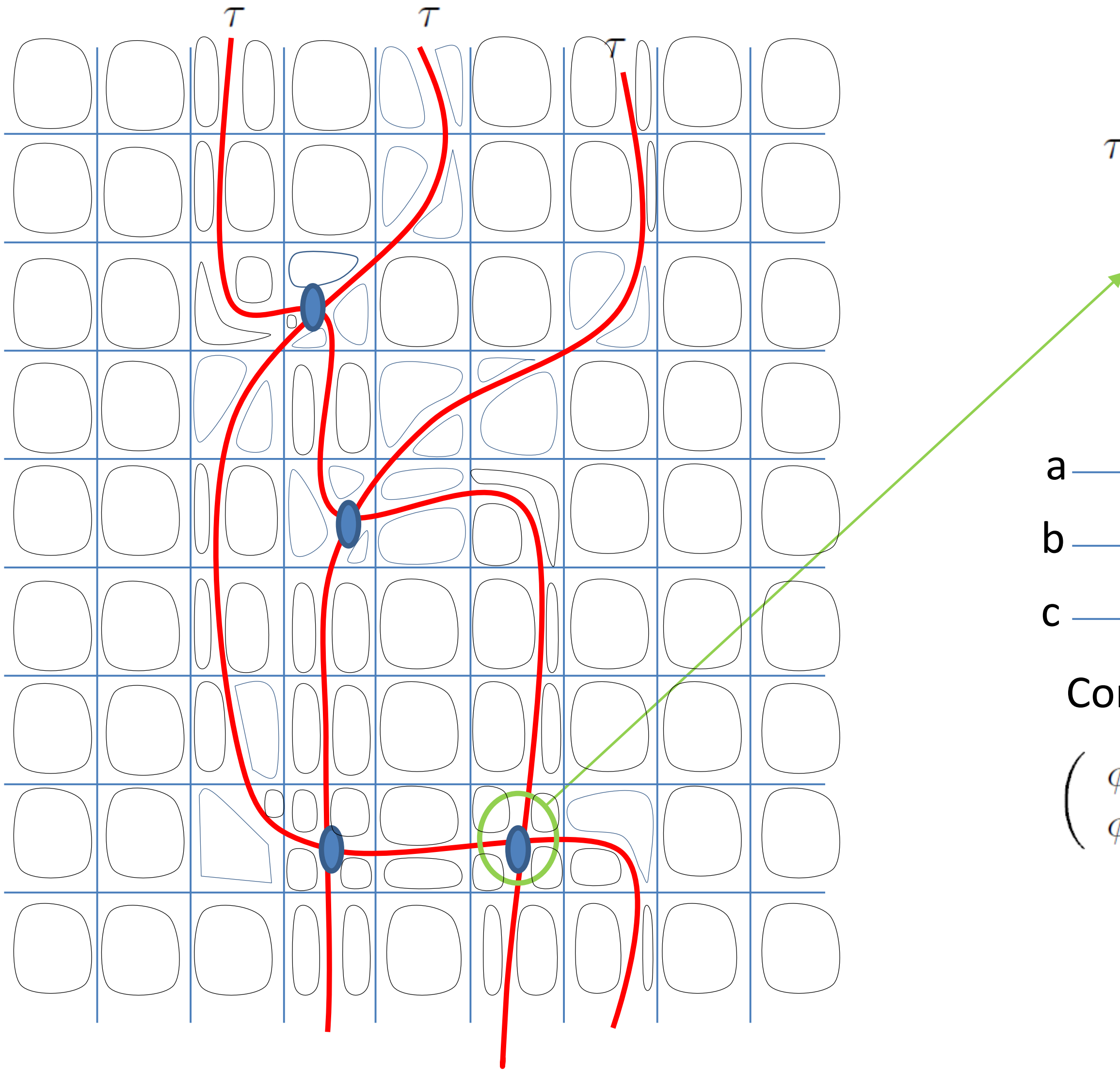


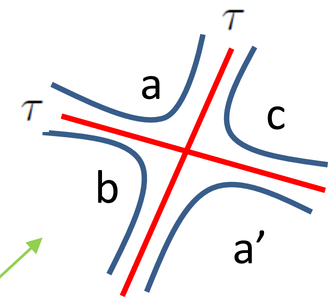
- Gives direct access to the S and T matrices: e.g. Dehn twist is with respect to the MPO and not with respect to the lattice!

Topological Quantum Computation

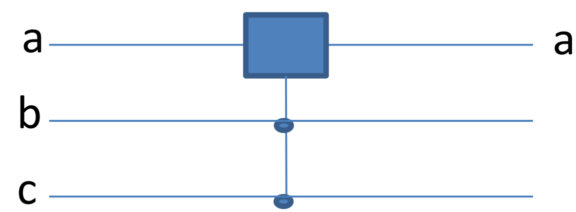
We can identify a tensor product structure of logical qubits with the entanglement (virtual) degrees of freedom; e.g. Fibonacci string net







Braiding tensor is F-symbol



Controlled-Controlled-U gate

$$\begin{pmatrix} \phi^{-1}e^{4\pi i/5} & \phi^{-1/2}e^{-3\pi i/5} \\ \phi^{-1/2}e^{-3\pi i/5} & -\phi^{-1} \end{pmatrix}$$

Freedman, Kitaev, Wang, …

Conclusion

- Matrix Product Operators pop up everywhere when studying many body physics
- Matrix Product Operator Algebras give an explicit representation of tensor fusion categories
- Next step is to study topological quantum phase transitions (e.g. condensation of anyons)