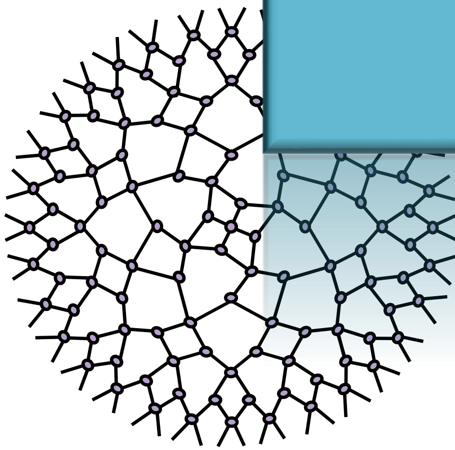
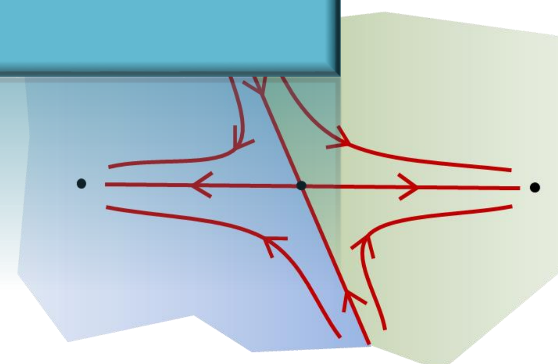
Coogee'16

Sydney Quantum Information Theory Workshop

Feb 2nd - 5th, 2016

Scale invariance on the lattice





Guifre Vidal

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

JOHN TEMPLETON FOUNDATION

compute canada | calcul canada

TENSOR NETWORKS INITIATIVE

SIMONS FOUNDATION

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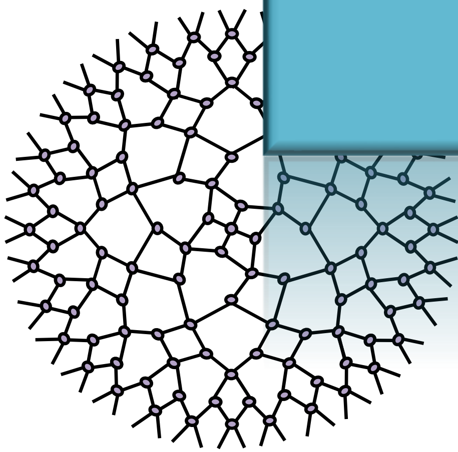

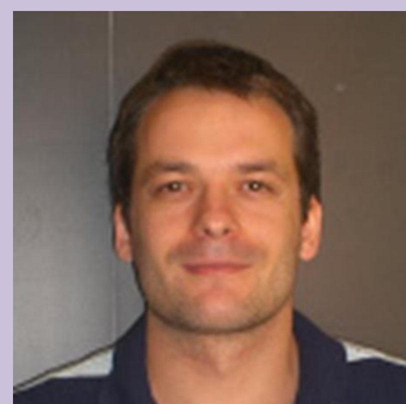

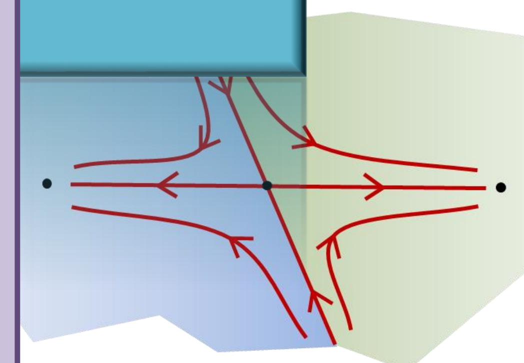

work with

GLEN EVENBLY

(UC Irvine)

PERIMETER ... ETICAL PHYSICS

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Motivation:

- Field theory ⟷ space-time symmetries

  e.g. translation invariance $x \to x' = x + \epsilon$

  at criticality scale invariance (augmented to global/local conformal group?)

  $x \to x' = (1 + \epsilon)x$

- On the lattice (genuine physics or UV cut-off)

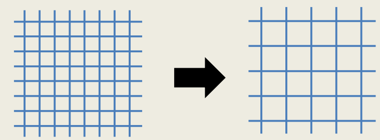
  translation ? $x \to x' = x + a$ discrete

  $a \equiv$ lattice spacing

  scale transformation?

Goals:

1) Define a ***global* scale transformation** on the lattice



Hamiltonian (Hilbert space)

Entanglement renormalization / MERA

PRL 2007 (arXiv:cond-mat/0512165)

PRL 2008 (arXiv:quant-ph/0610099)

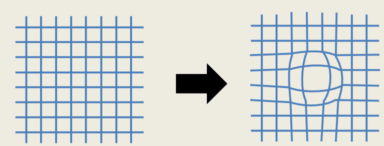
Lagrangian (Euclidean path integral)

Tensor network renormalization

Evenbly, Vidal PRL 2015 (arXiv:1412.0732)

Evenbly, arXiv:1509.07484

2) Define a ***local* scale transformation** on the lattice



Hamiltonian (Hilbert space)

Entanglement renormalization / MERA

Czech, Evenbly, Lamprou, McCandlish, Qi, Sully, Vidal arXiv:1510.07637

on-going discussions with Tobias Osborne and Vaughan Jones

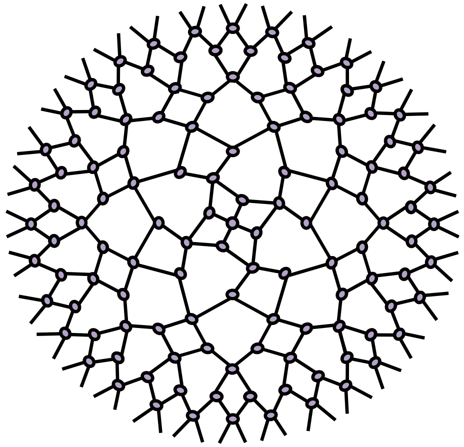
Lagrangian (Euclidean path integral)

Tensor netw
renormalization

Evenbly, Vidal, PRL 2015 arXiv:1510.00689

Evenbly, Vidal, PRL 2015 arXiv:1502.05385

Outline:



wave-functions / Hamiltonians

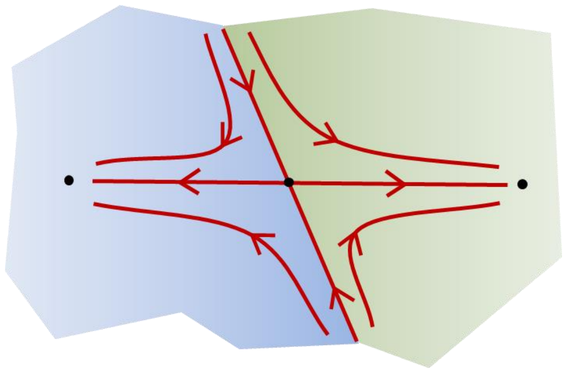
global scale transformation (RG transformation)

local scale transformations

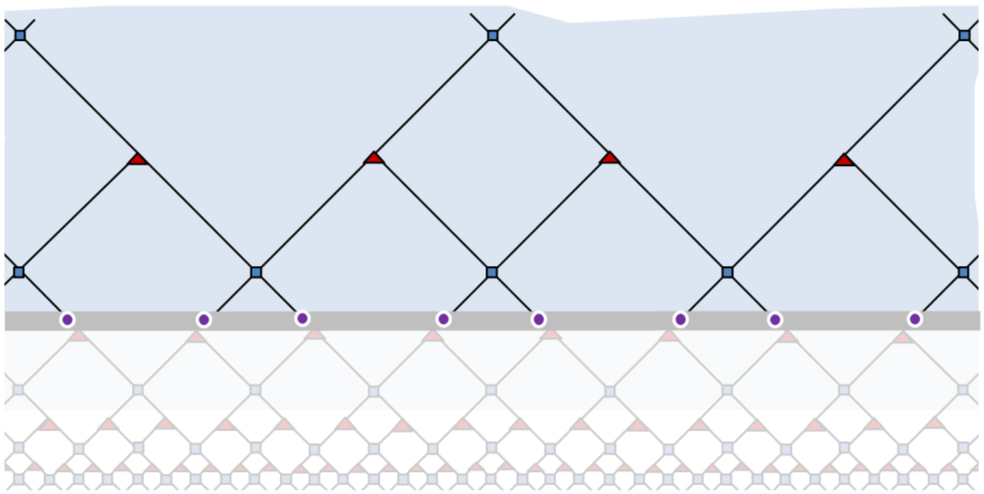
Euclidean path integrals / classical partition functions

global scale transformation (RG transformation)

local scale transformations

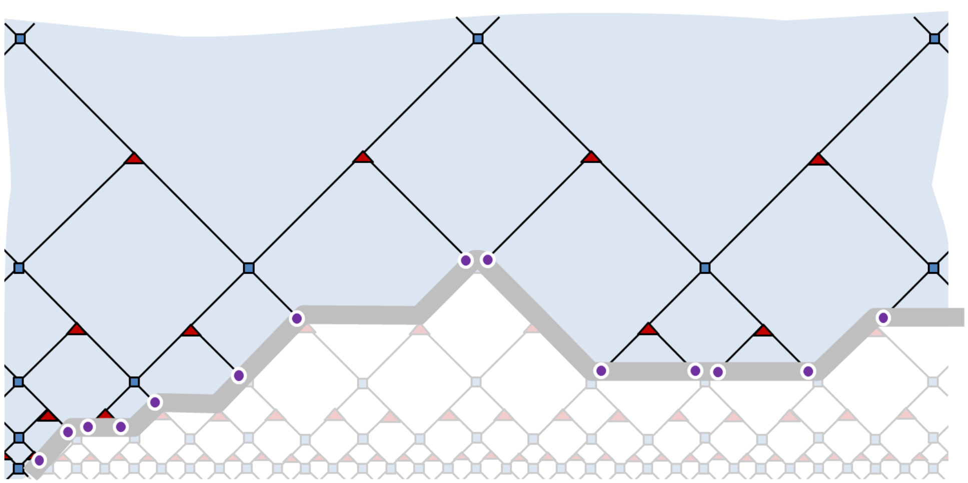


Claim 1:

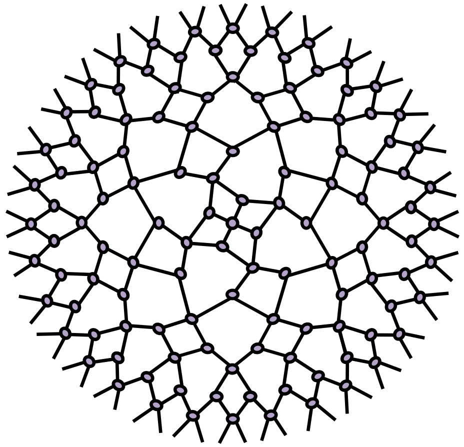


global scale transformation

Claim 2:



local scale transformation



wave-functions / Hamiltonians

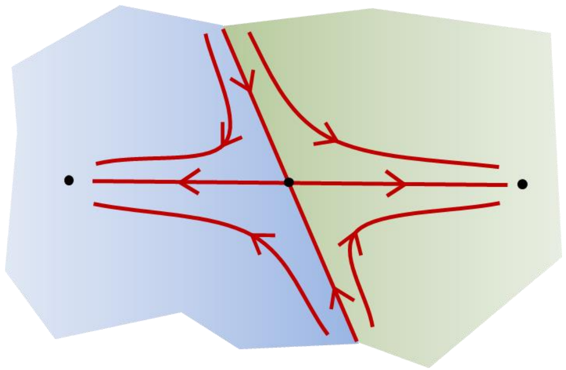
global scale transformation (RG transformation)

local scale transformations

Euclidean path integrals / classical partition functions

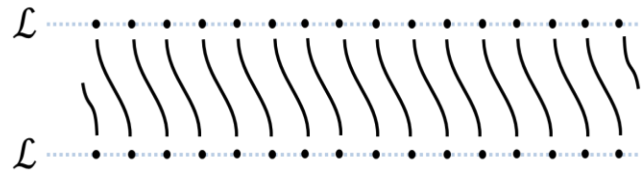
global scale transformation (RG transformation)

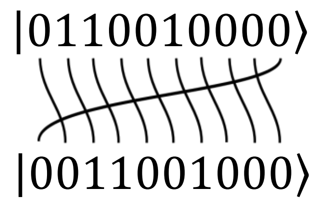
local scale transformations



How do we define a translation on the lattice?

- unitary map $\mathbb{V}^{\otimes N} \rightarrow \mathbb{V}^{\otimes N}$

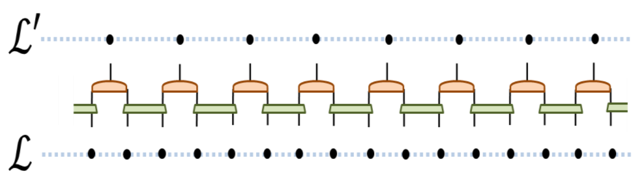




How do we define a scale transformation on the lattice?

- many possibilities

- here, isometric map $\mathbb{V}^{\otimes N/2} \rightarrow \mathbb{V}^{\otimes N}$



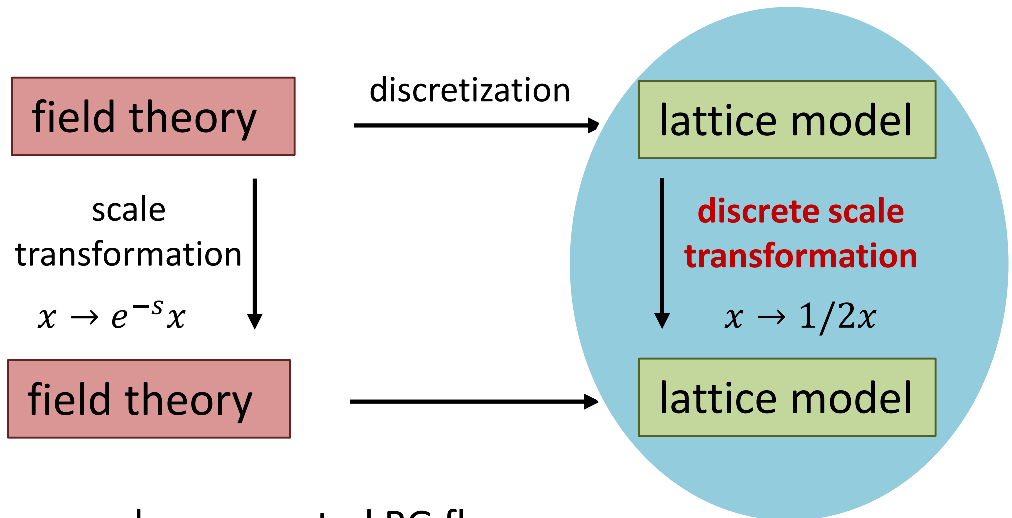
variational

locality

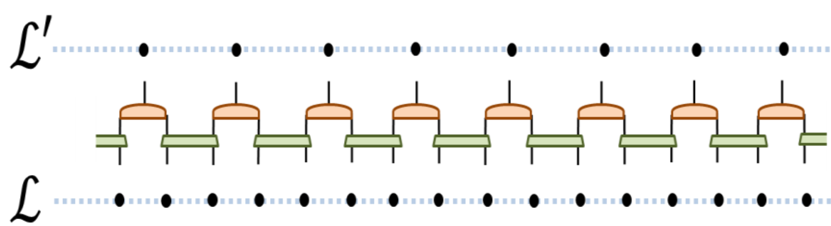
entanglement

- natural requirement:

consistency with the QFT / *continuum*

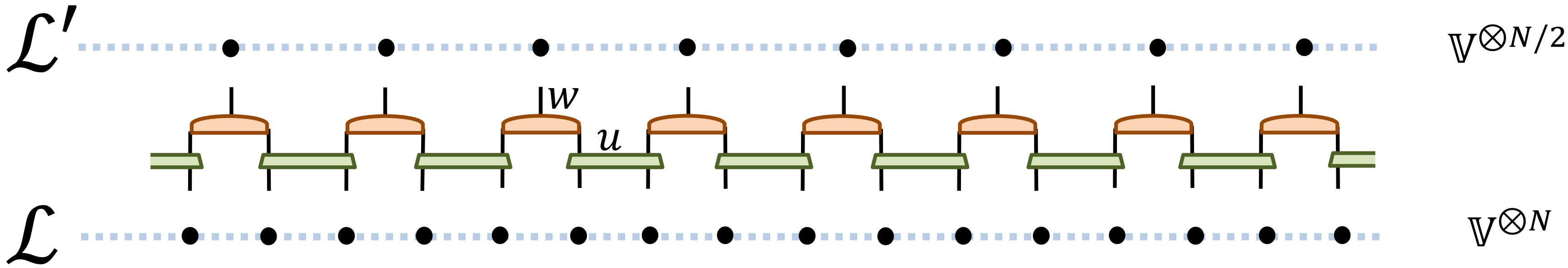


reproduce expected RG flow,
including *explicit scale invariance* at RG fixed-points



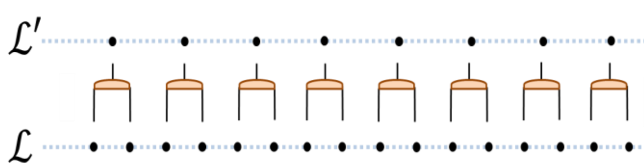
PRL 2007
(arXiv:cond-mat/0512165)

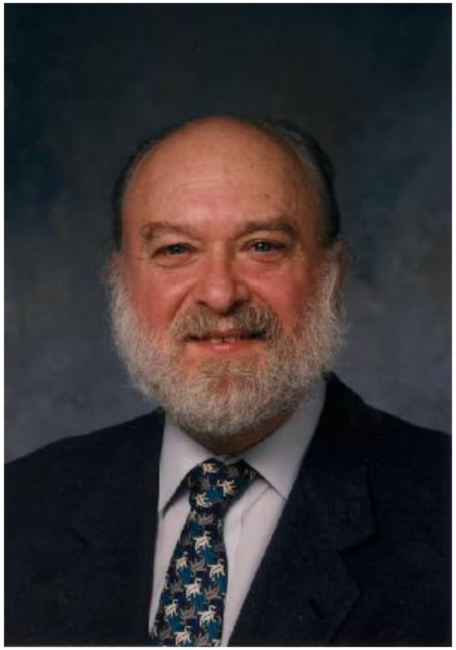
Entanglement renormalization



1

spin blocking
(as Kadanoff 1966)

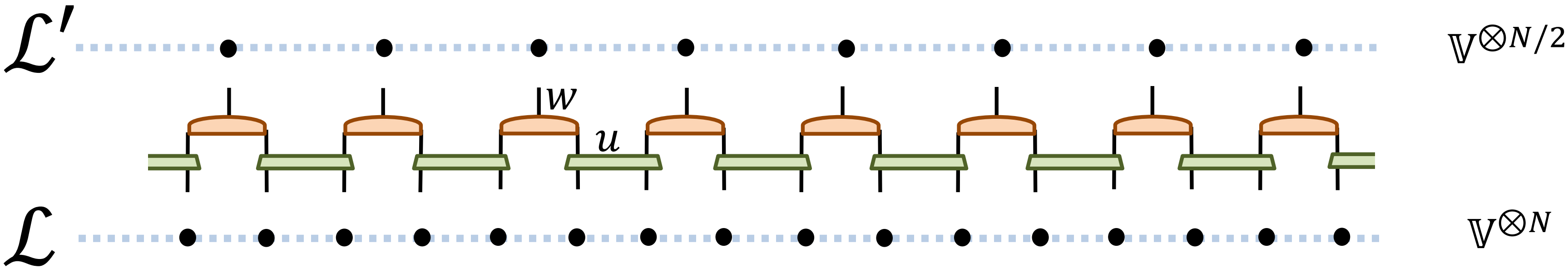




Leo Kadanoff
1937 - 2015

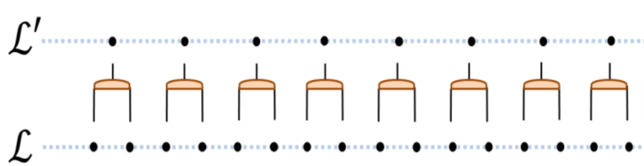
PRL 2007
(arXiv:cond-mat/0512165)

Entanglement renormalization



1

spin blocking
(as Kadanoff 1966)



2

variational optimization
(as DMRG, White 1992)

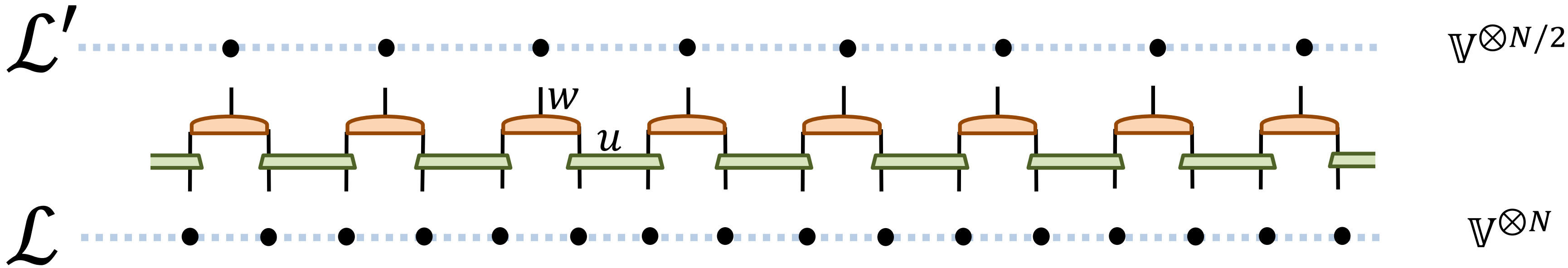
only structural ansatz
ask the Hamiltonian H !
(low energy)

$$W\colon \mathbb{V}^{\otimes N/2} \rightarrow \mathbb{V}^{\otimes N}$$

$$H \rightarrow H' \equiv W^{\dagger} H W$$

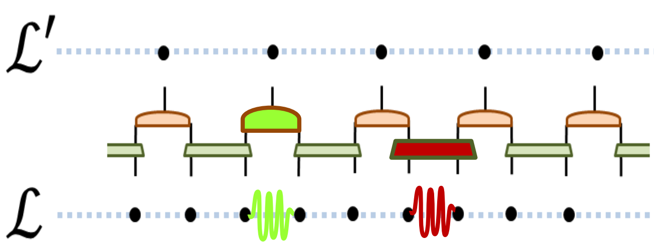
PRL 2007
(arXiv:cond-mat/0512165)

Entanglement renormalization



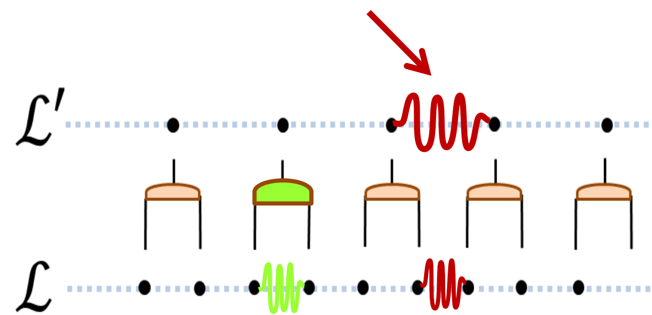
3

entanglement



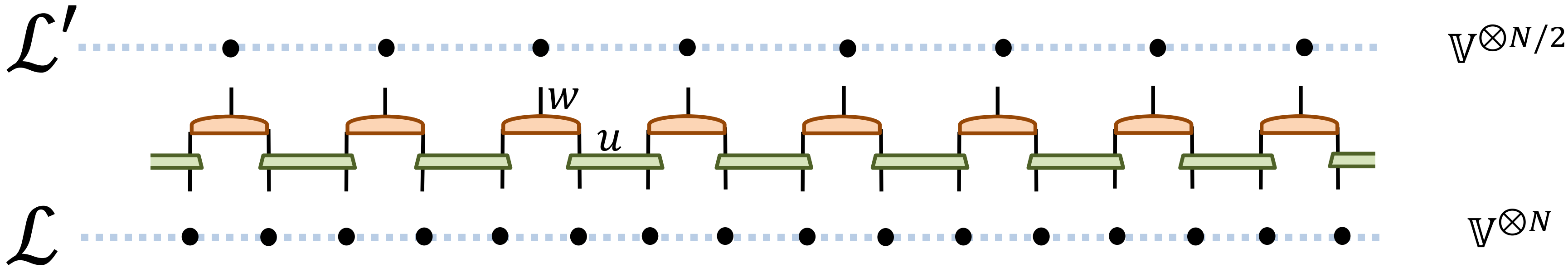
removal of all
short-range
entanglement !

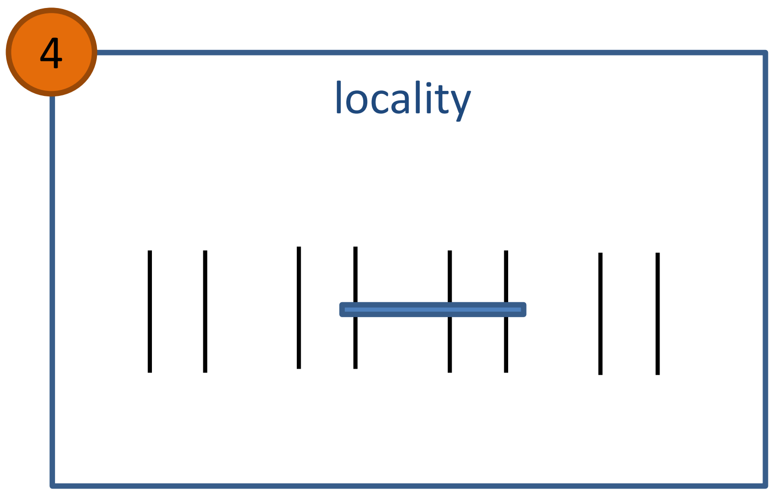
failure to remove
some short-range
entanglement !

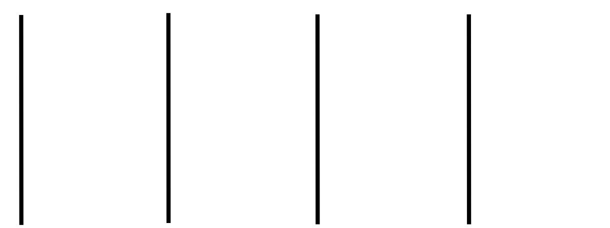


Entanglement renormalization

PRL 2007
(arXiv:cond-mat/0512165)







Entanglement renormalization

PRL 2007
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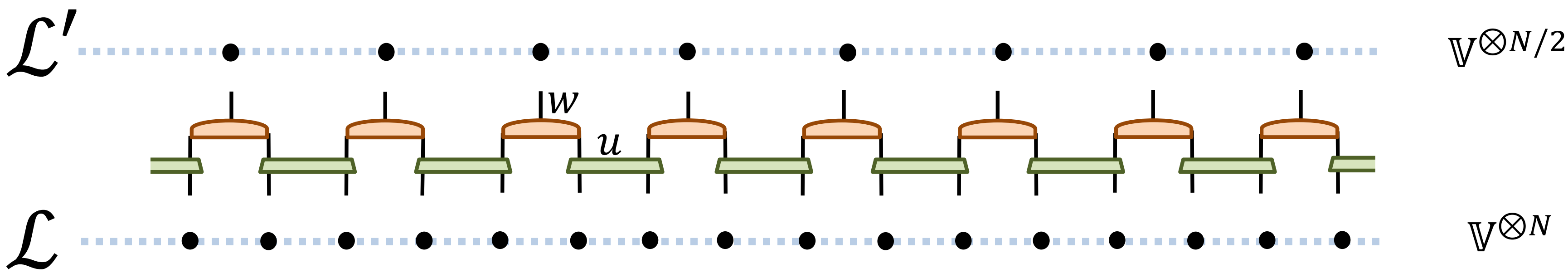

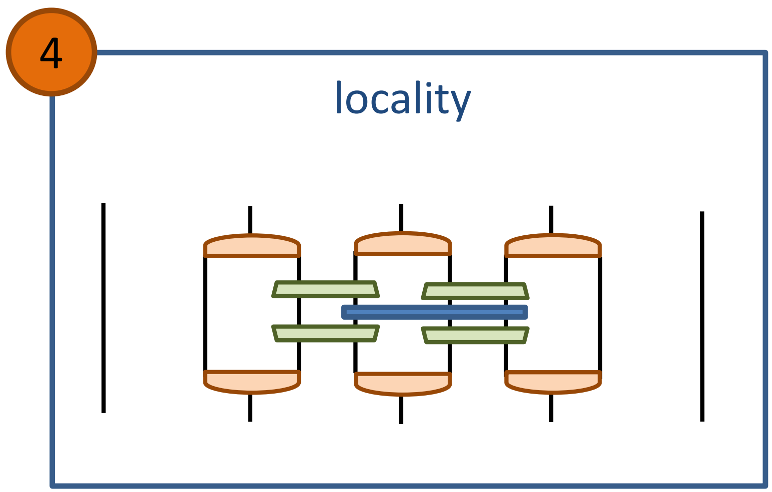

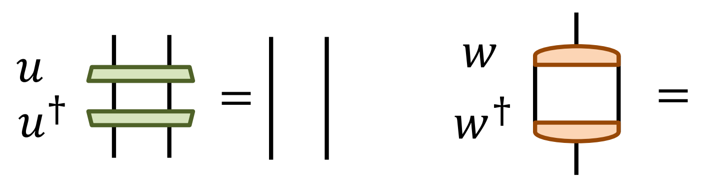

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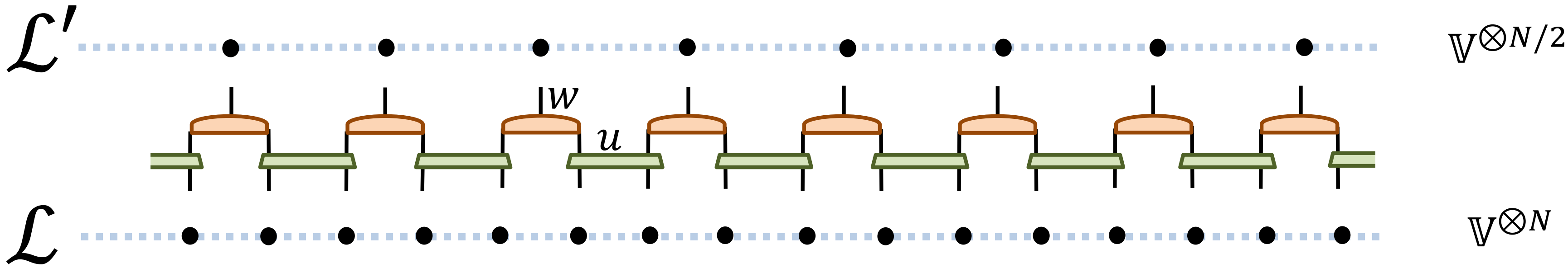
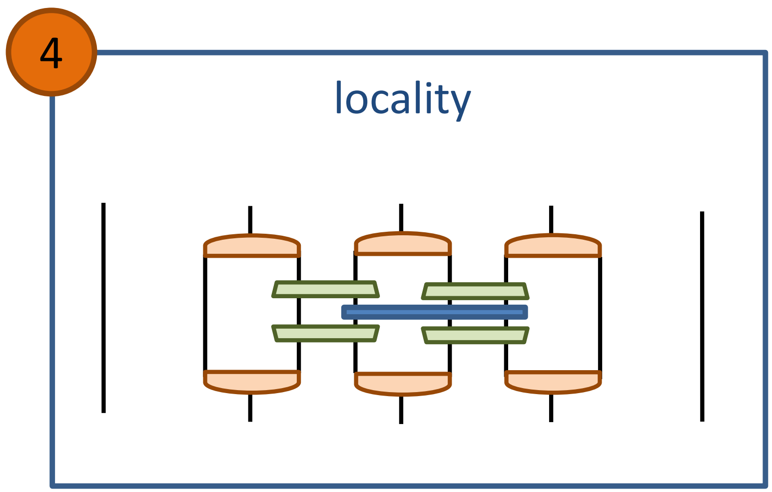
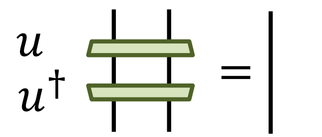
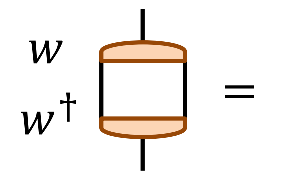
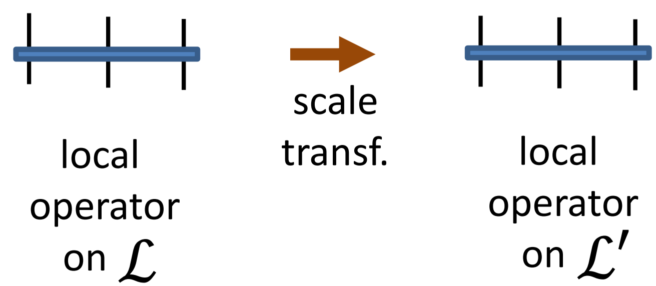










Claim: Entanglement renormalization defines a *proper* scale transformation on the lattice

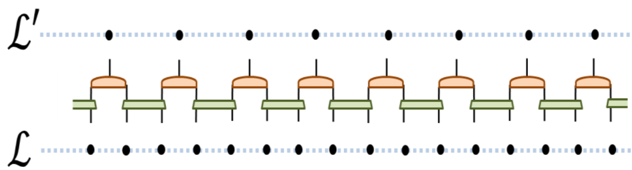
- Explicit scale invariance at criticality !

input

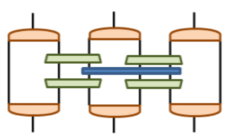
1D quantum Hamiltonian on the lattice

- **at a critical point**

1 - optimization



2 - diagonalization



→ scaling operators

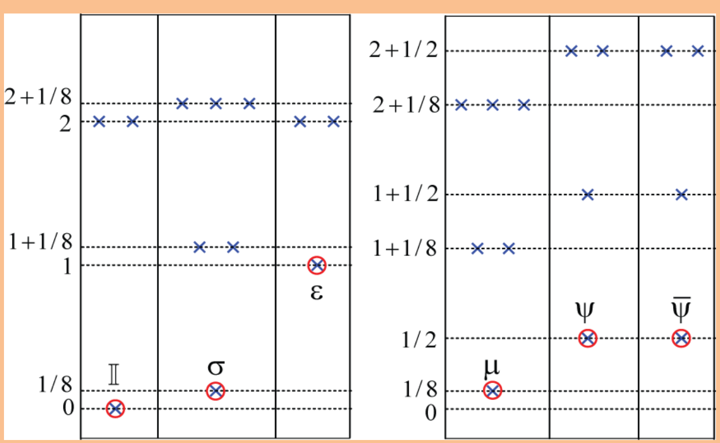
output

Numerical extraction of conformal data of underlying CFT:

- **central charge** c
- **scaling dimensions** $\Delta_\alpha \equiv h_\alpha + \bar{h}_\alpha$ **and conformal spins** $s_\alpha \equiv h_\alpha - \bar{h}_\alpha$
- **OPE coefficients** $C_{\alpha\beta\gamma}$

e.g. critical Ising model (approx. an hour on your laptop)

Pfeifer, Evenbly, Vidal 08



$(\Delta_{\mathbb{I}} = 0)$

$\Delta_\sigma \approx 0.124997$

$\Delta_\varepsilon \approx 0.99993$

$\Delta_\mu \approx 0.125002$

$\Delta_\psi \approx 0.500001$

$\Delta_{\bar{\psi}} \approx 0.500001$

$C_{\epsilon\sigma\sigma} = \frac{1}{2}$ $\quad C_{\epsilon\mu\mu} = -\frac{1}{2}$

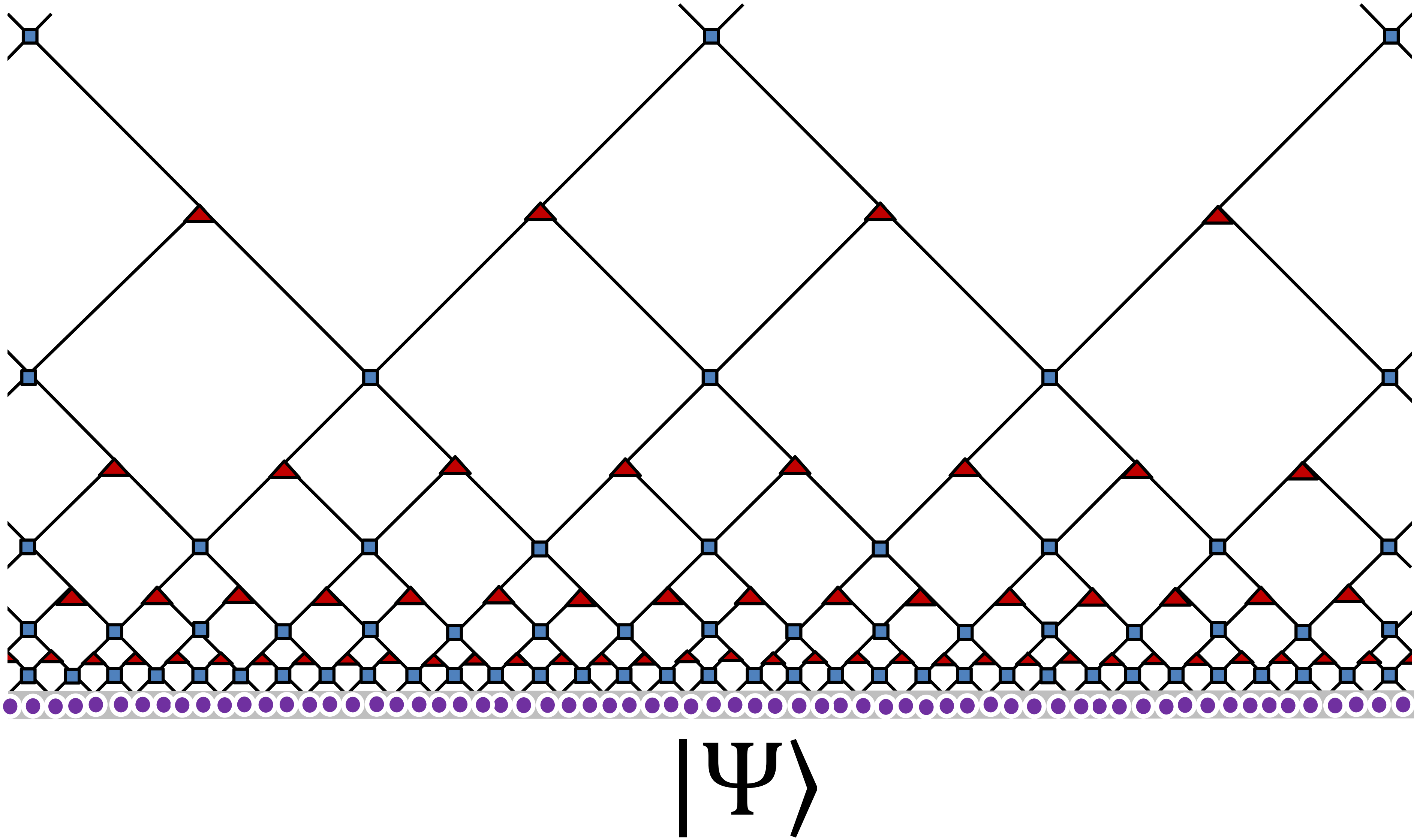
$C_{\epsilon\psi\bar{\psi}} = i$ $\quad C_{\epsilon\bar{\psi}\psi} = -i$

$C_{\psi\mu\sigma} = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}$ $\quad C_{\bar{\psi}\mu\sigma} = \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}$

$(\pm 6 \times 10^{-4})$

multi-scale entanglement renormalization ansatz
(MERA)

PRL 2008
(arXiv:quant-ph/0610099)



multi-scale entanglement renormalization ansatz
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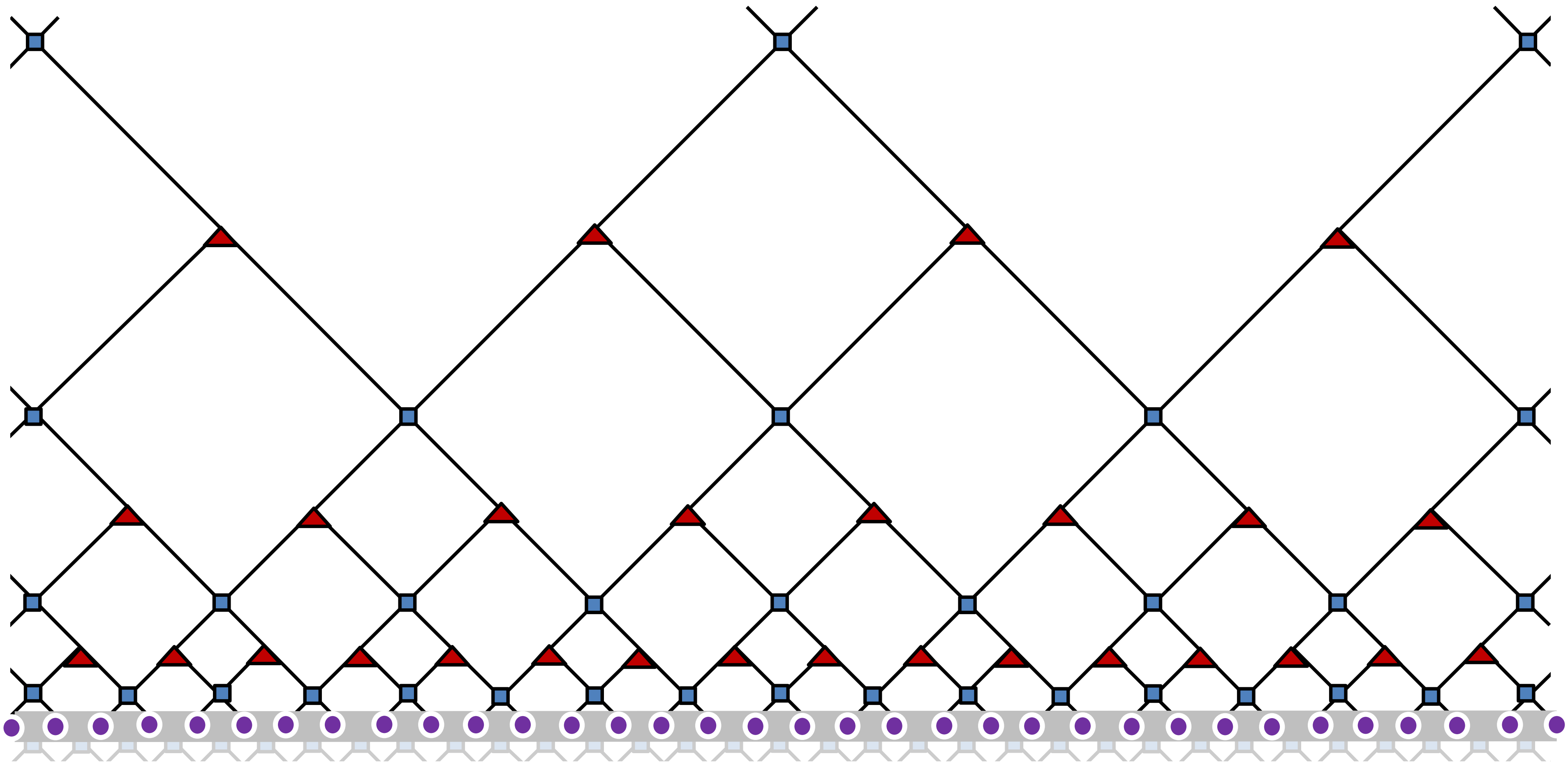
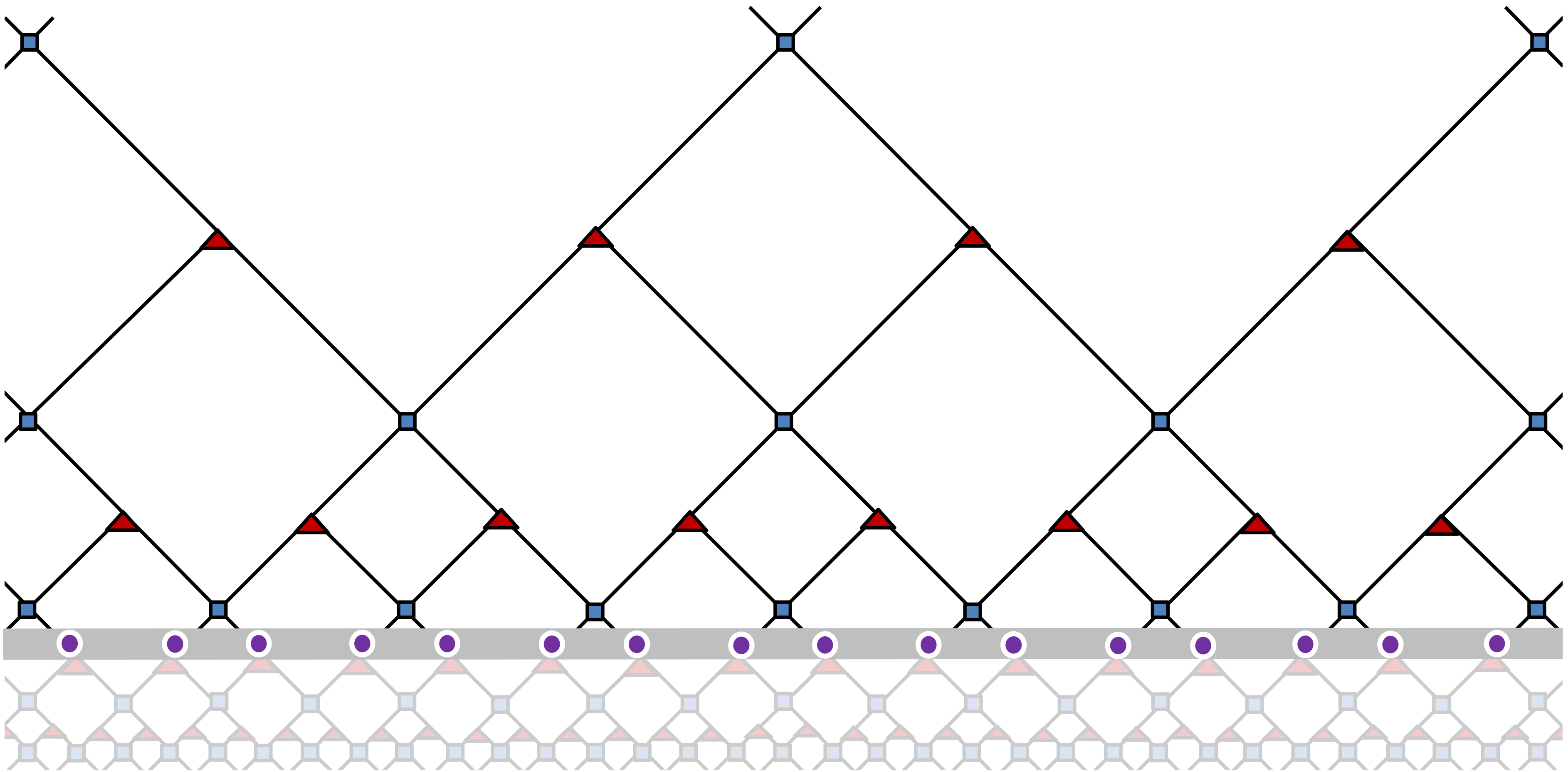


$|\Psi'\rangle$

multi-scale entanglement renormalization ansatz
(MERA)

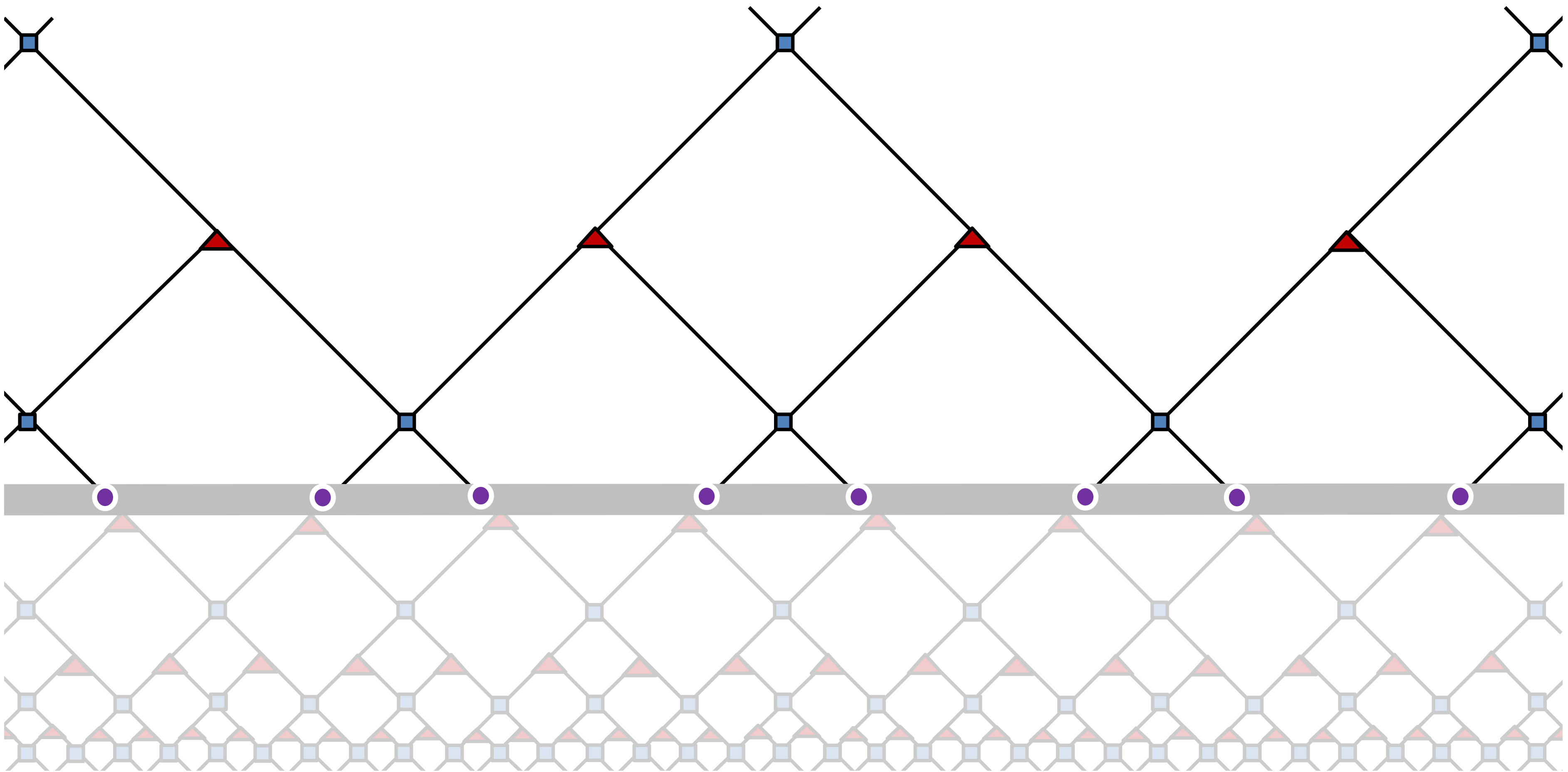
PRL 2008
(arXiv:quant-ph/0610099)



$$|\Psi''\rangle$$

multi-scale entanglement renormalization ansatz (MERA)

PRL 2008 (arXiv:quant-ph/0610099)

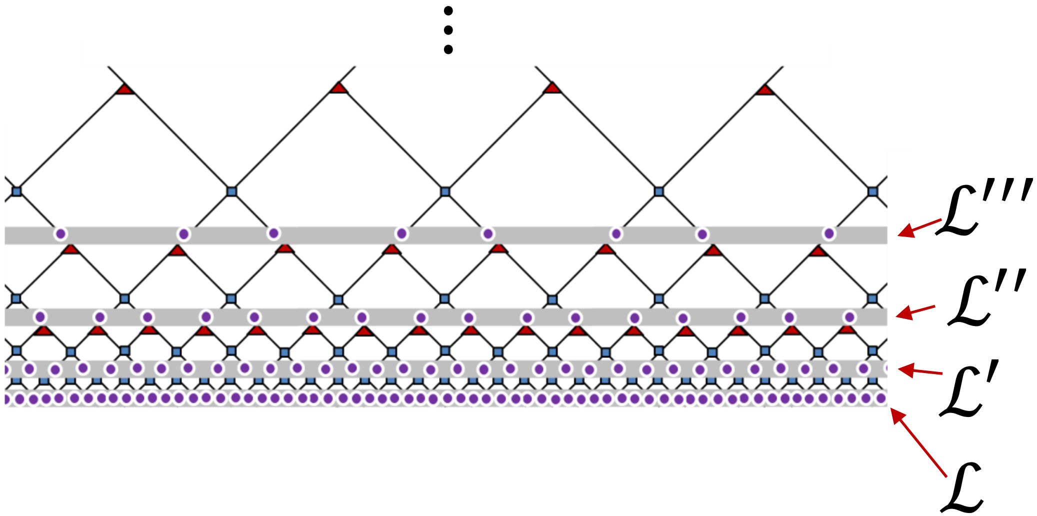


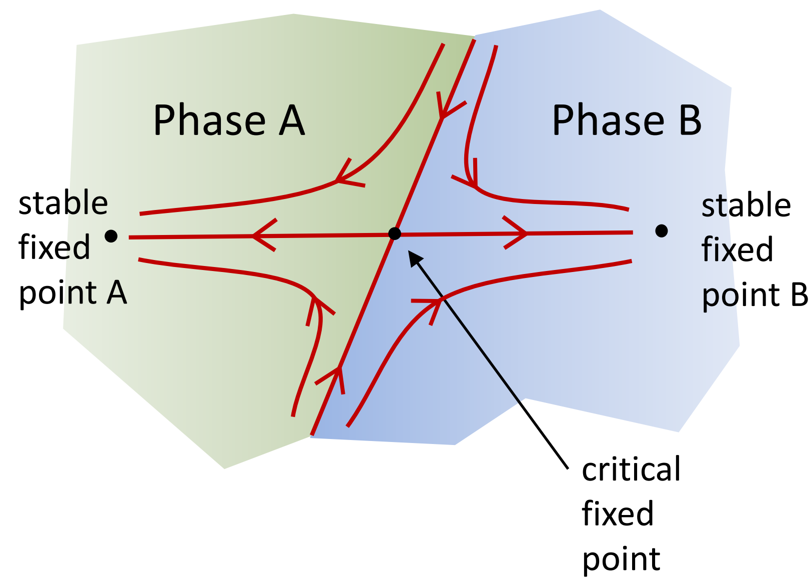
$$|\Psi'''\rangle$$

PRL 2008
(arXiv:quant-ph/0610099)

MERA defines an RG flow in the space of wave-functions

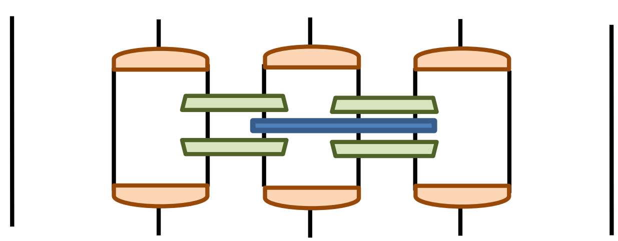
$$|\Psi\rangle \rightarrow |\Psi'\rangle \rightarrow |\Psi''\rangle \rightarrow |\Psi'''\rangle \cdots$$

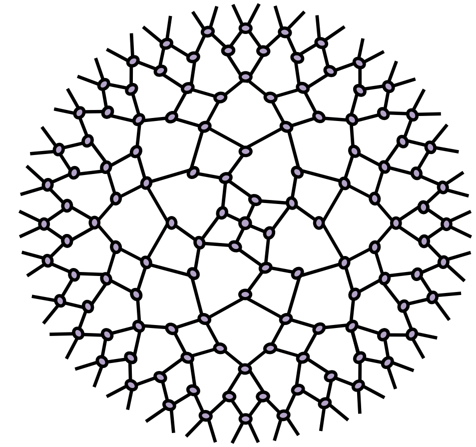




... and in the space of Hamiltonians

$$H \rightarrow H' \rightarrow H'' \rightarrow H''' \cdots$$





wave-functions / Hamiltonians

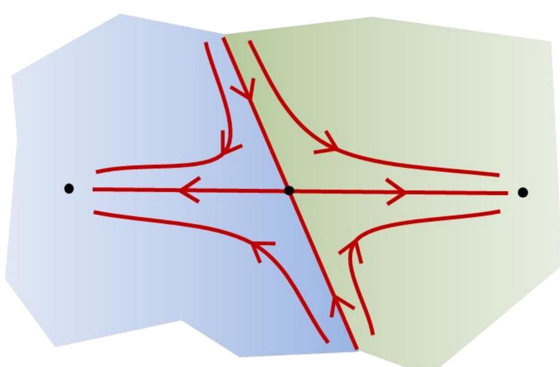
global scale transformation (RG transformation)

local scale transformations

Euclidean path integrals / classical partition functions

global scale transformation (RG transformation)

local scale transformations

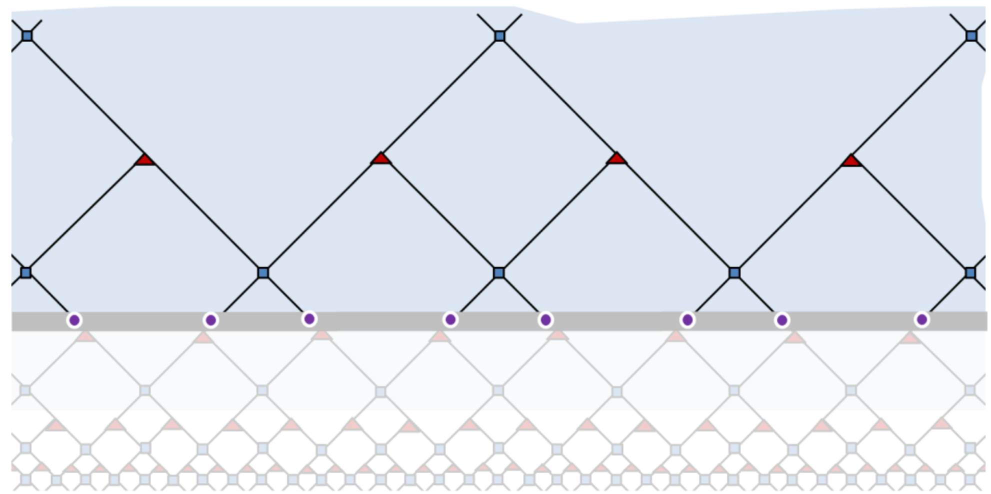


Claim 1:

MERA defines a ***global*** scale transformation on the lattice

discrete RG flow with
expected structure, including

scale invariance at criticality



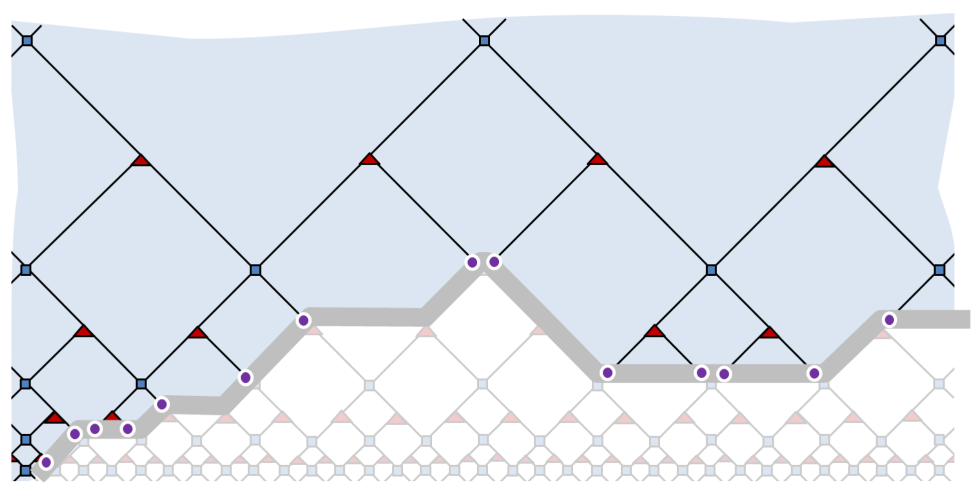
Czech, Evenbly, Lamprou,
McCandlish, Qi, Sully, Vidal
arXiv:1510.07637

Claim 2:

MERA also defines ***local*** scale transformations on the lattice

Expected behaviour
from continuum (CFT), including

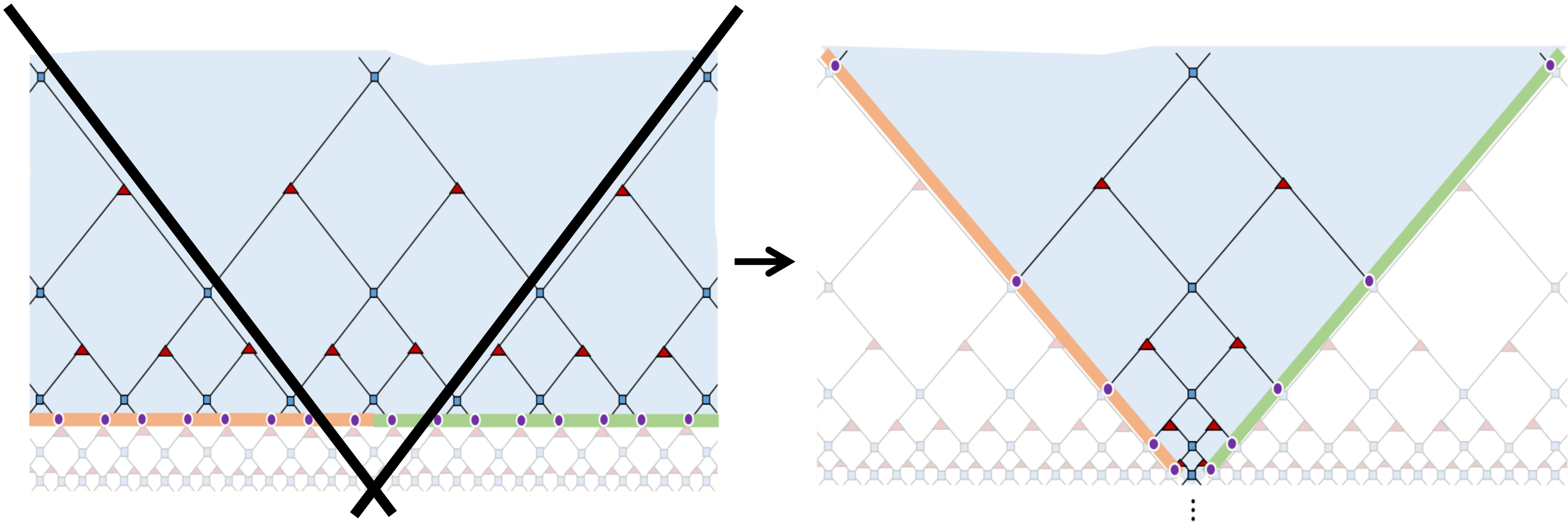
local scale invariance
(covariance) at criticality



Czech, Evenbly, Lamprou,
McCandlish, Qi, Sully, Vidal
arXiv:1510.07637

Example of local scale transformation on the lattice:

(involving both *coarse-graining* and *fine-graining*)



ground state

$|\Psi\rangle$

on infinite (discrete) line

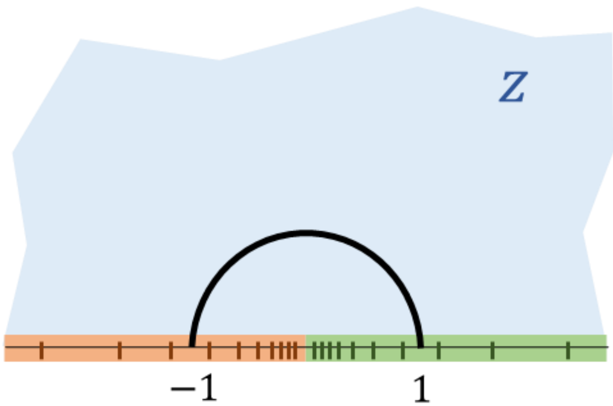
?

Test: compare with CFT in the continuum

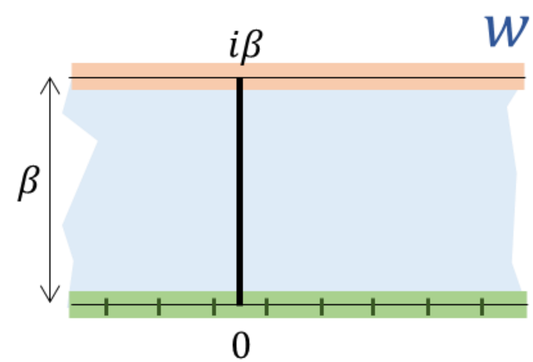
Czech, Evenbly, Lamprou, McCandlish, Qi, Sully, Vidal
arXiv:1510.07637

conformal transformation

$$z \rightarrow w = \frac{\beta}{\pi}\log(z)$$



→



Euclidean path integral
on half upper plane
prepares the ground state

$$|\Psi\rangle$$

on the infinite line

Euclidean path integral
on infinite strip
prepares thermal state

$$\rho_\beta$$

[*comment to experts: TFD]

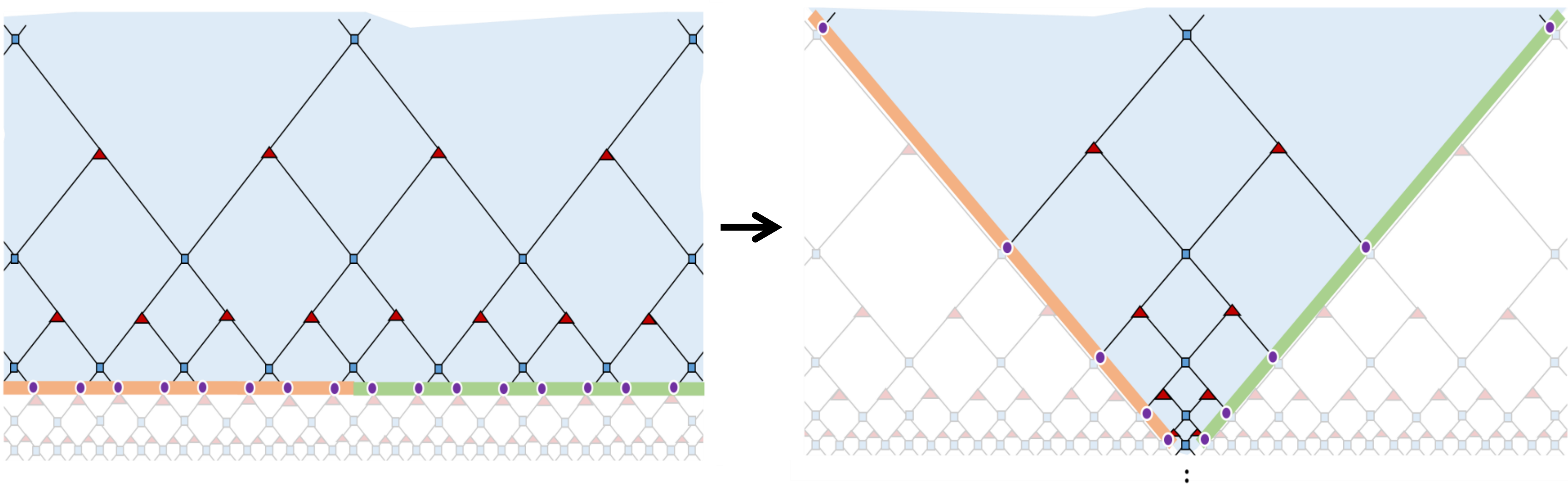
on the infinite line

Can we do the same on the lattice?

Czech, Evenbly, Lamprou, McCandlish, Qi, Sully, Vidal arXiv:1510.07637

conformal transformation

$$z \to w = \frac{\beta}{\pi} \log(z)$$



ground state

$$|\Psi\rangle$$

on infinite (discrete) line

thermal state

$$\rho_\beta$$

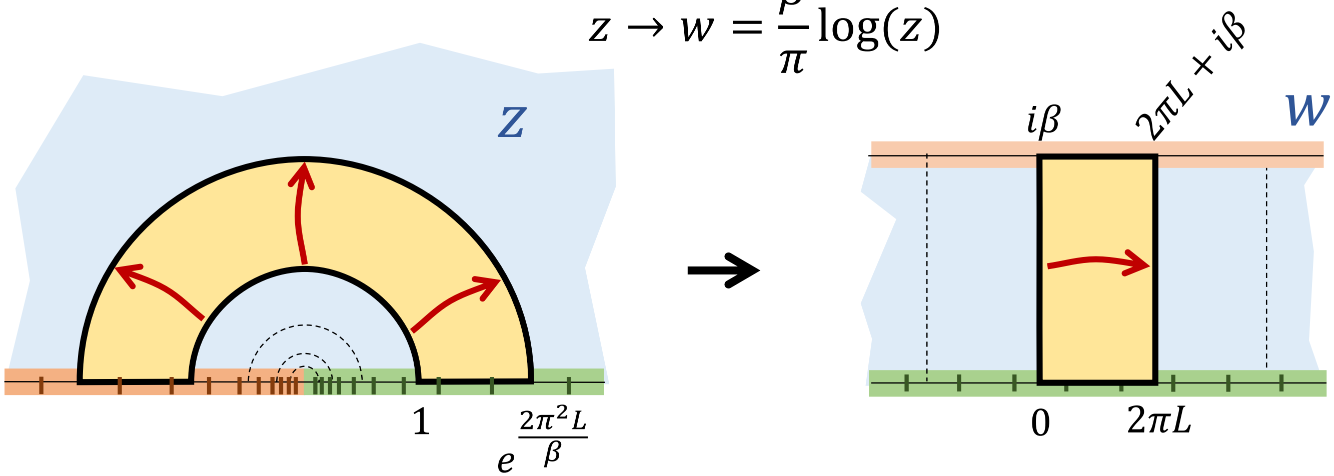
[*comment to experts: TFD]

on infinite (discrete) line ?

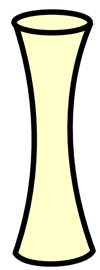
Yes!

(but first, let us put it in a finite geometry) conformal transformation

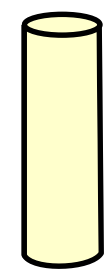
$$z \to w = \frac{\beta}{\pi}\log(z)$$



quotient:
half upper plane / scaling
= topological cylinder



quotient:
infinite strip / translation
= flat cylinder

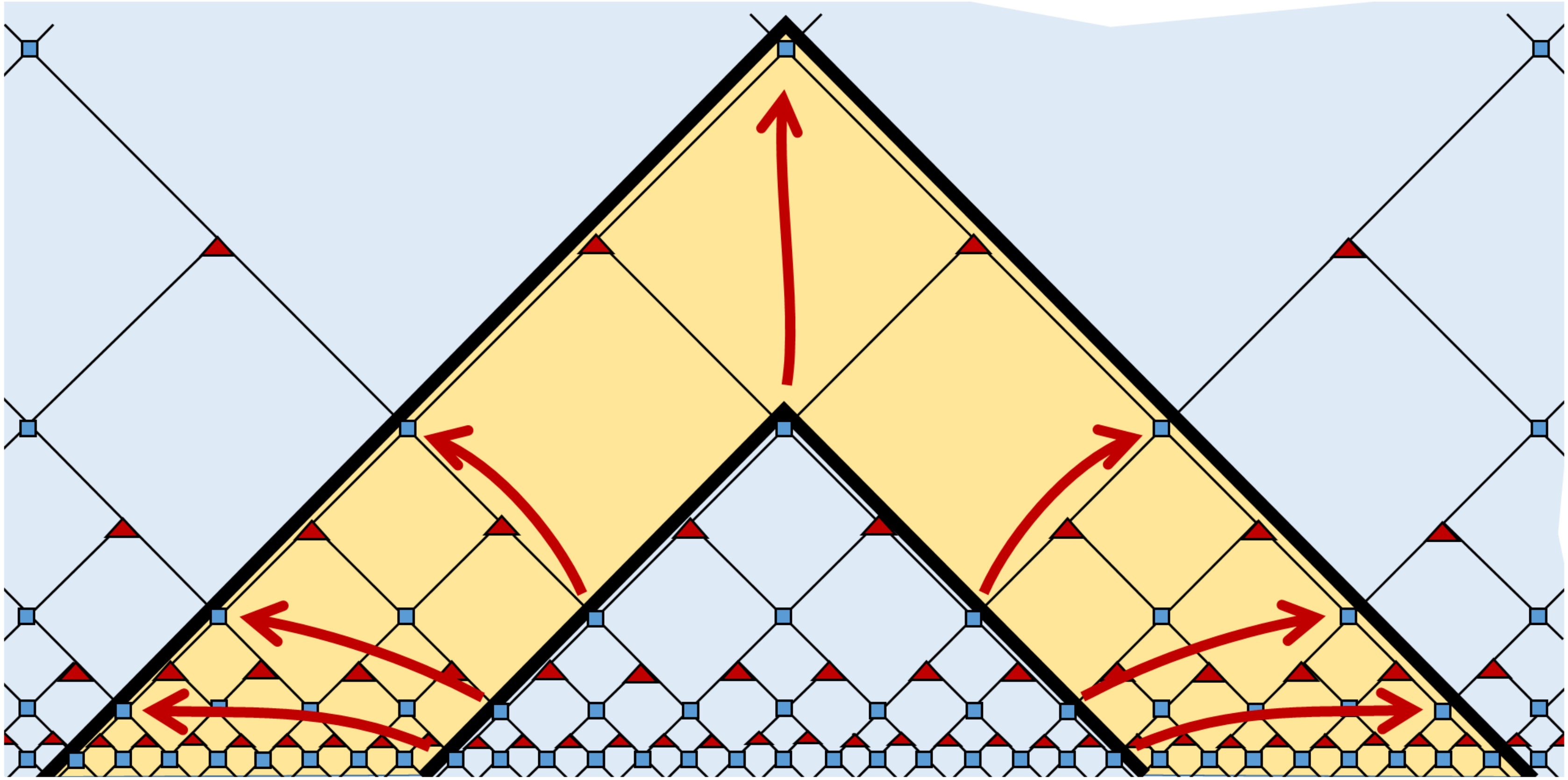


thermal state

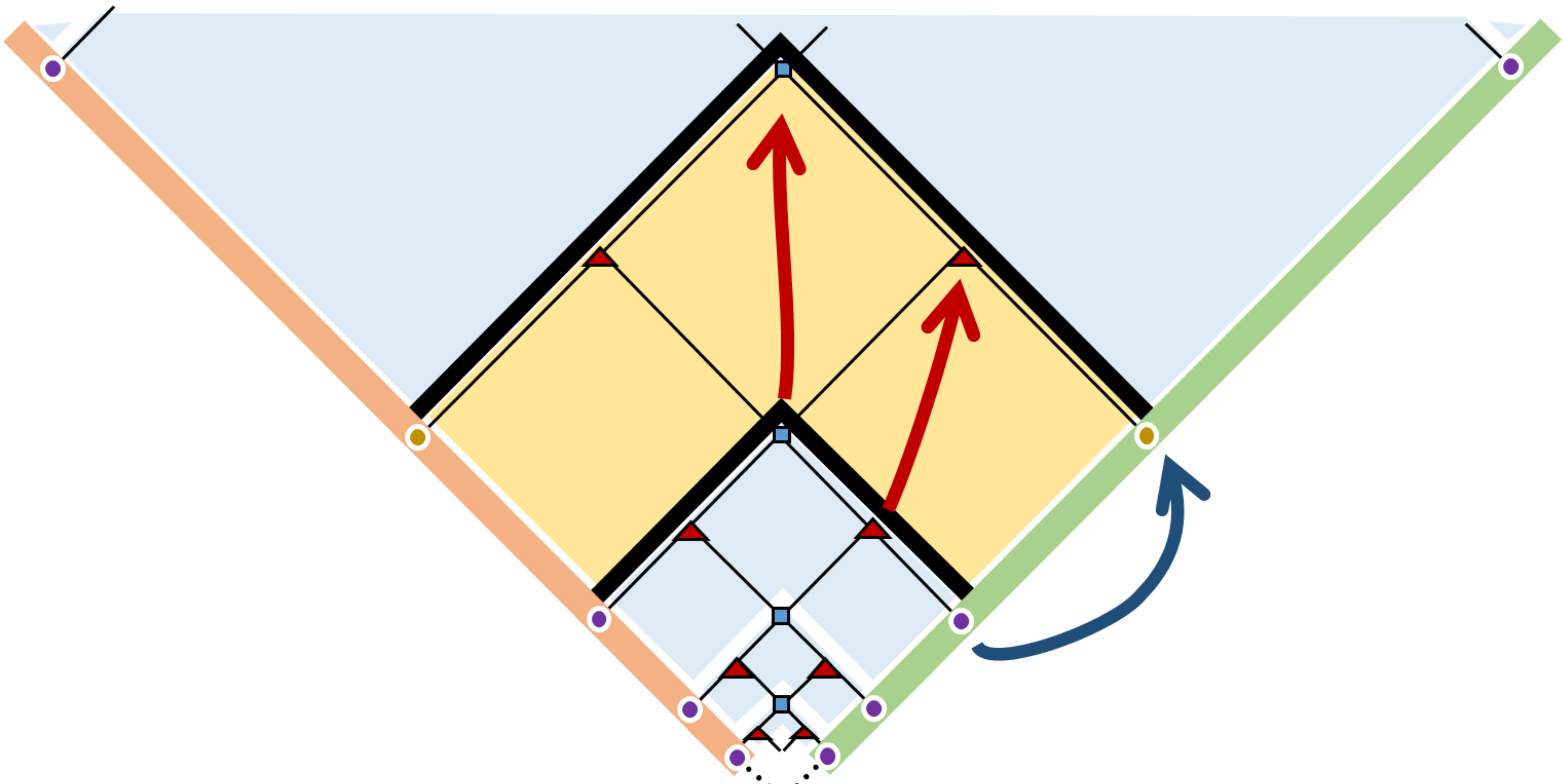
$$\rho_\beta$$

on finite circle

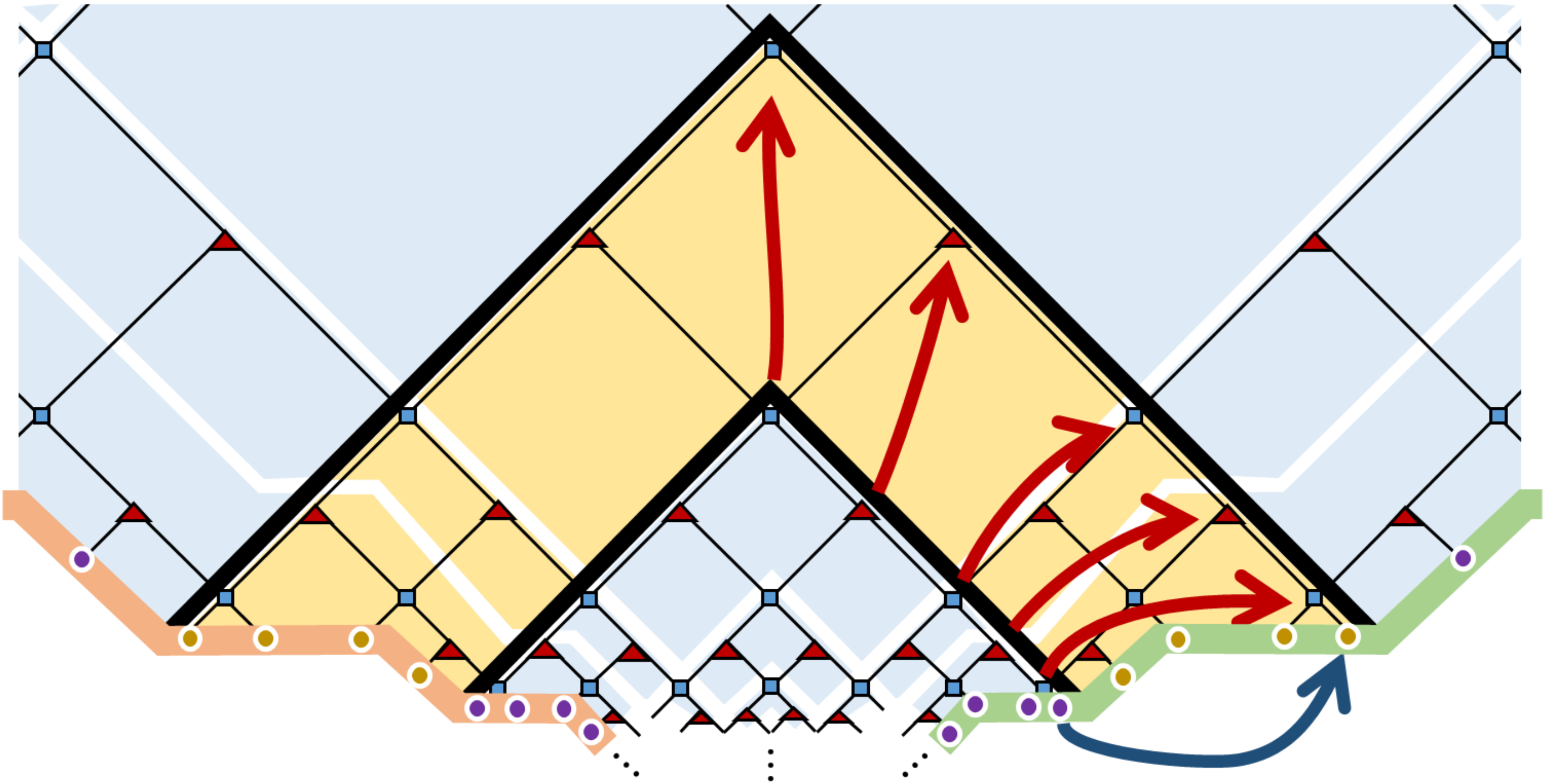
Luckily, this scaling is an exact symmetry of the MERA



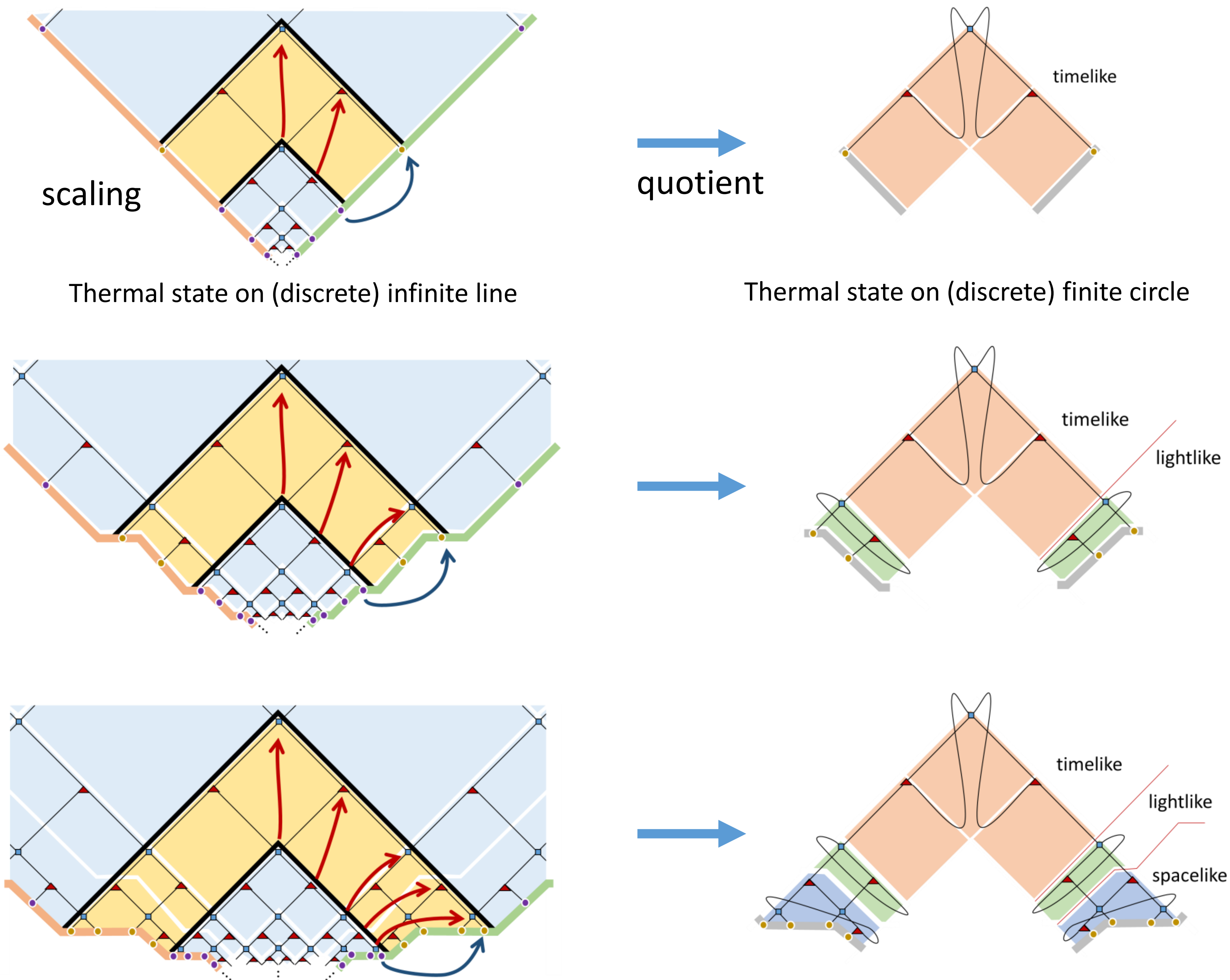
Scaling on a regular MERA without local scale transformation

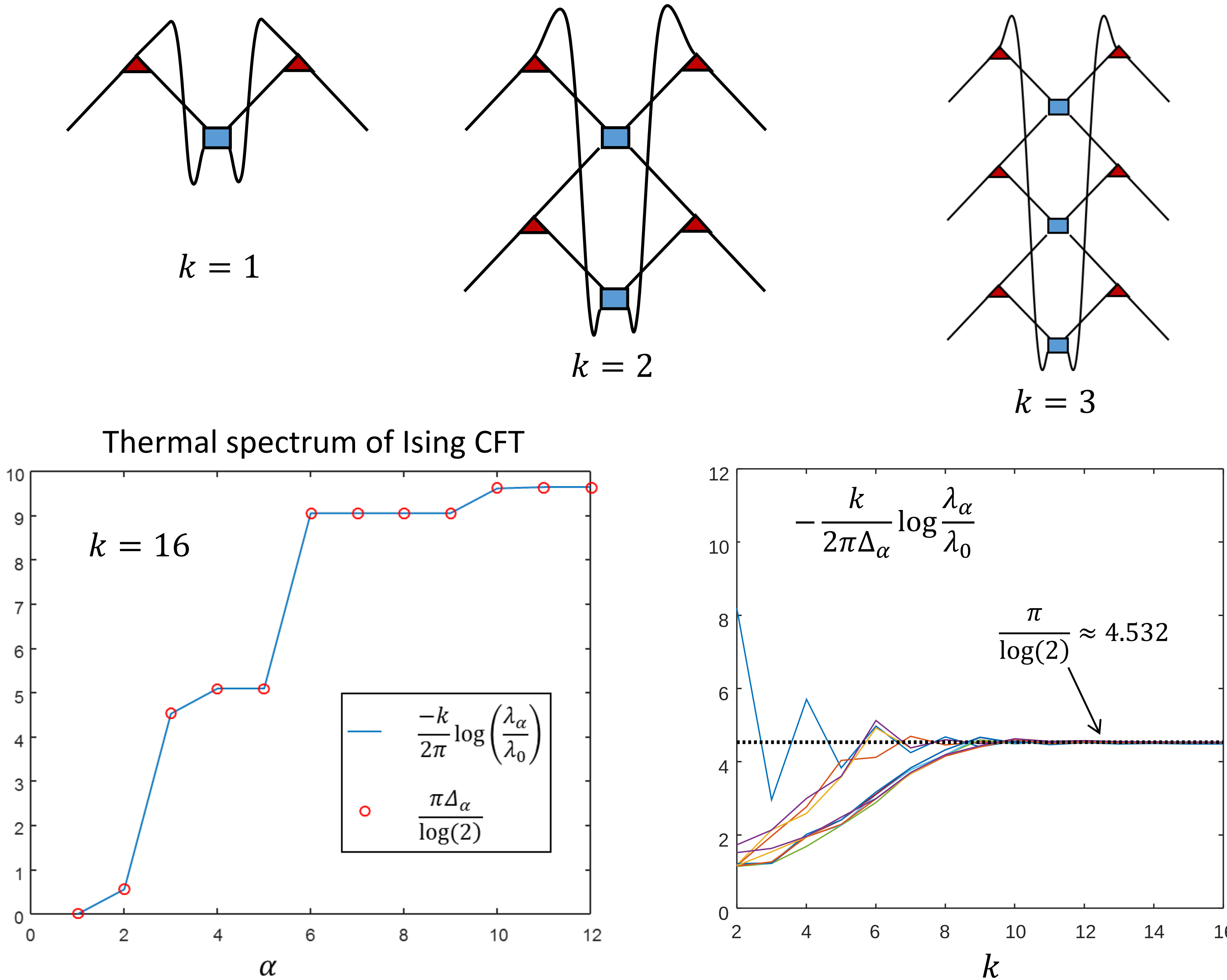


Scaling on a MERA after a local scale transformation



Scaling on a MERA after a local scale transformation


scaling
quotient
Thermal state on (discrete) infinite line
Thermal state on (discrete) finite circle
timelike
timelike
lightlike
timelike
lightlike
spacelike

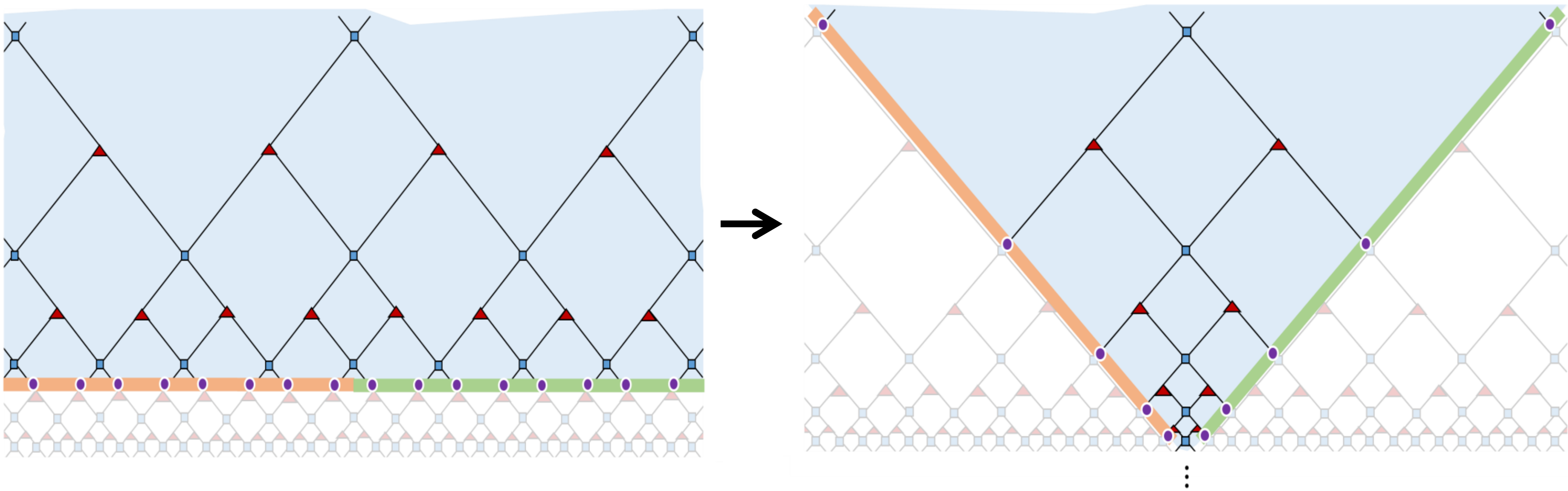


$k = 1$
$k = 2$
$k = 3$
Thermal spectrum of Ising CFT
$k = 16$
$\frac{-k}{2\pi}\log\left(\frac{\lambda_\alpha}{\lambda_0}\right)$
$\frac{\pi \Delta_\alpha}{\log(2)}$
α
$-\frac{k}{2\pi\Delta_\alpha}\log\frac{\lambda_\alpha}{\lambda_0}$
$\frac{\pi}{\log(2)} \approx 4.532$
k

Czech, Evenbly, Lamprou, McCandlish, Qi, Sully, Vidal
arXiv:1510.07637

conformal transformation

$$z \to w = \frac{\beta}{\pi}\log(z)$$



ground state

$$|\Psi\rangle$$

on infinite (discrete) line

thermal state

$$\rho_\beta$$

on infinite (discrete) line

Thus, under "our" *local scale transformation* on the lattice, the ground state transformed as it would under a local scale transformation in the continuum.

This is evidence that

disentanglers and isometries in MERA → local scale transformations on the lattice

Official theme of Coogee'16:

This year, the workshop will have a special focus on the connections between topology, quantum many-body physics, and quantum information...

Illegal sub-theme of Coogee'16:

"semi-continuous limit" of tensor networks / discrete analogue of conformal group

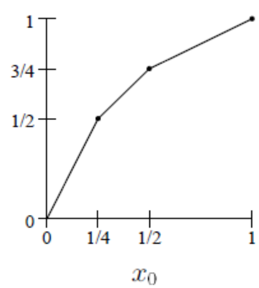
Tobias Osborne: "*Effective conformal field theories for tensor network states*"

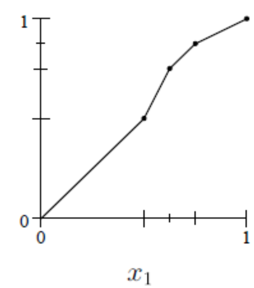
Vaughan Jones: "*Quantum spin chains, block spin renormalization, scale invariance and Thompson's groups F and T*"

dyadic rationals !

Thompson's group F, T !!

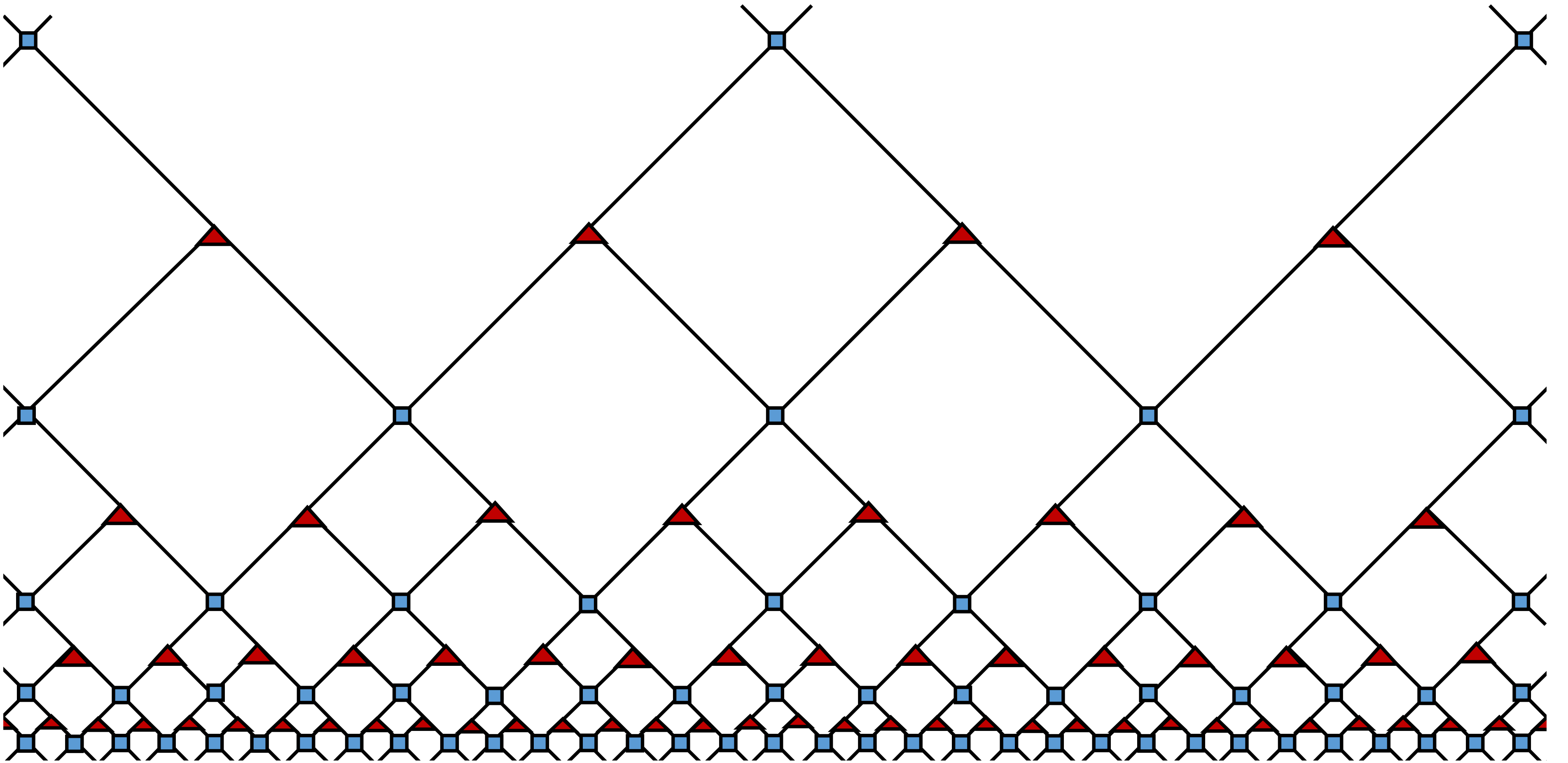
piece-wise linear maps!



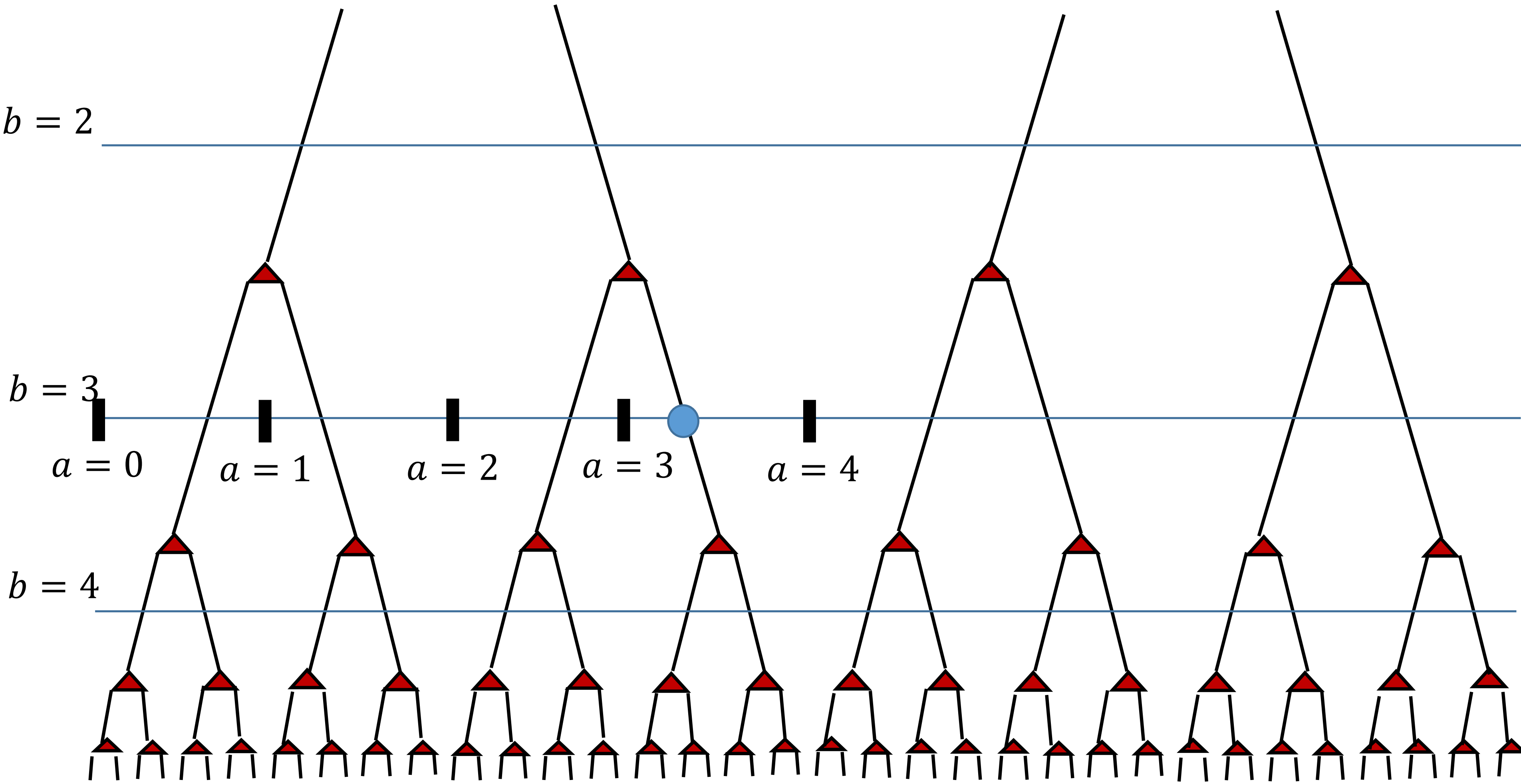


$$\mathbf{Conf}_{1+1} = \mathbf{Diff}(S^1) \times \mathbf{Diff}(S^1) \;!!$$

What local scale transformations can we implement “naturally”?

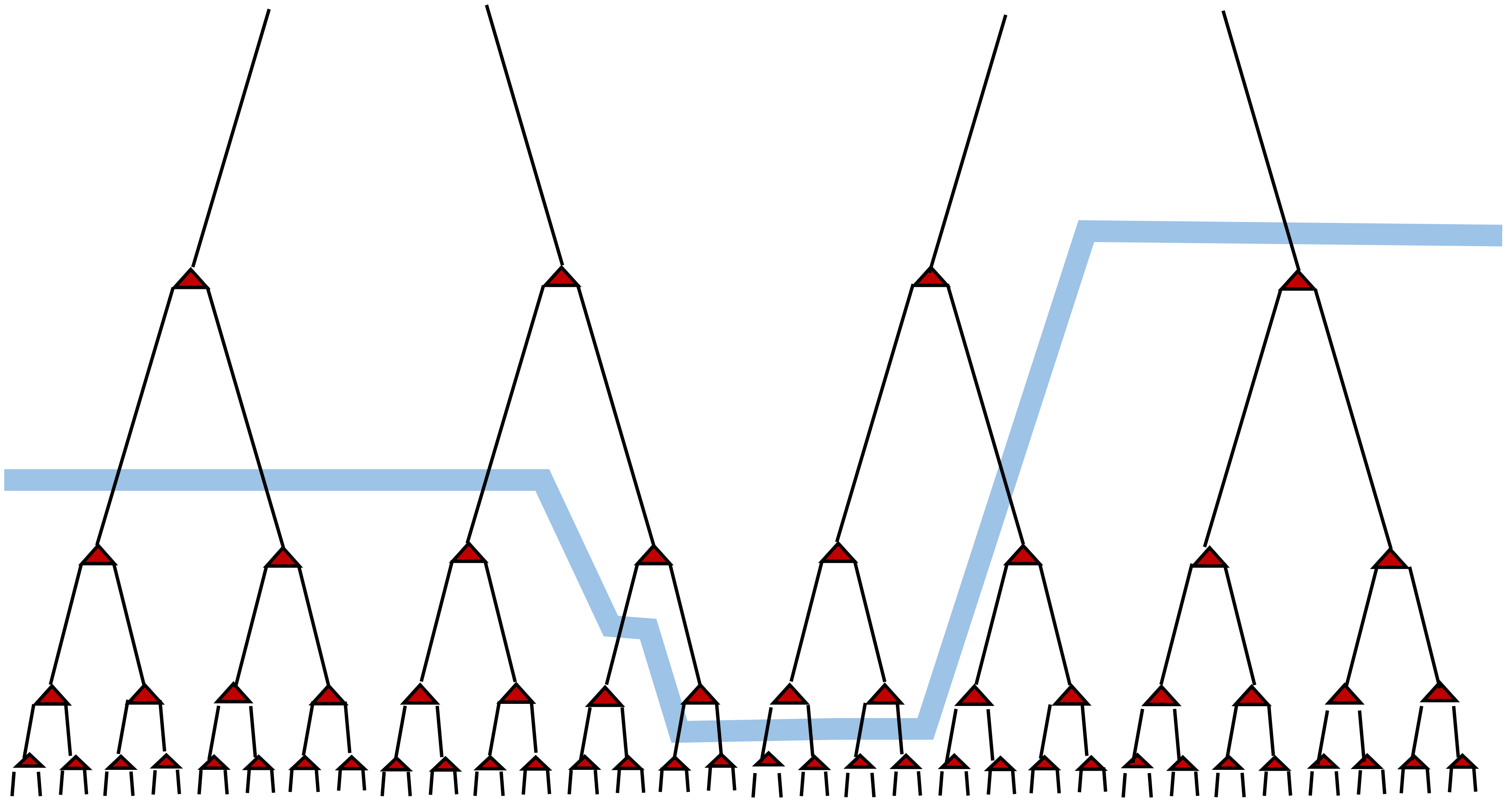


What local scale transformations can we implement "naturally"?



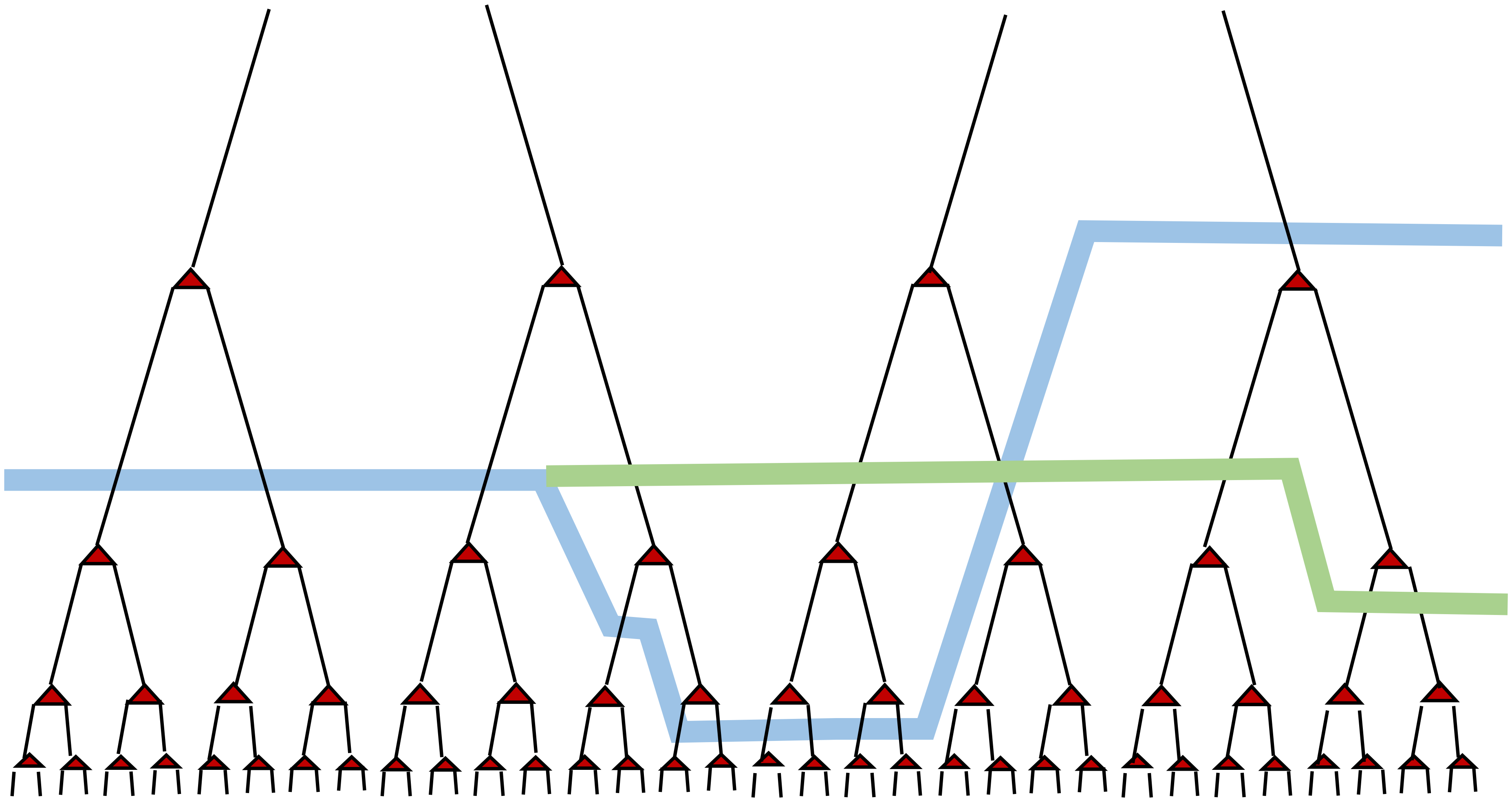
- dyadic rational $\frac{a}{2^b}$
- site: standard dyadic interval $\left[\frac{a}{2^b}, \frac{a+1}{2^b}\right]$

What local scale transformations can we implement “naturally”?

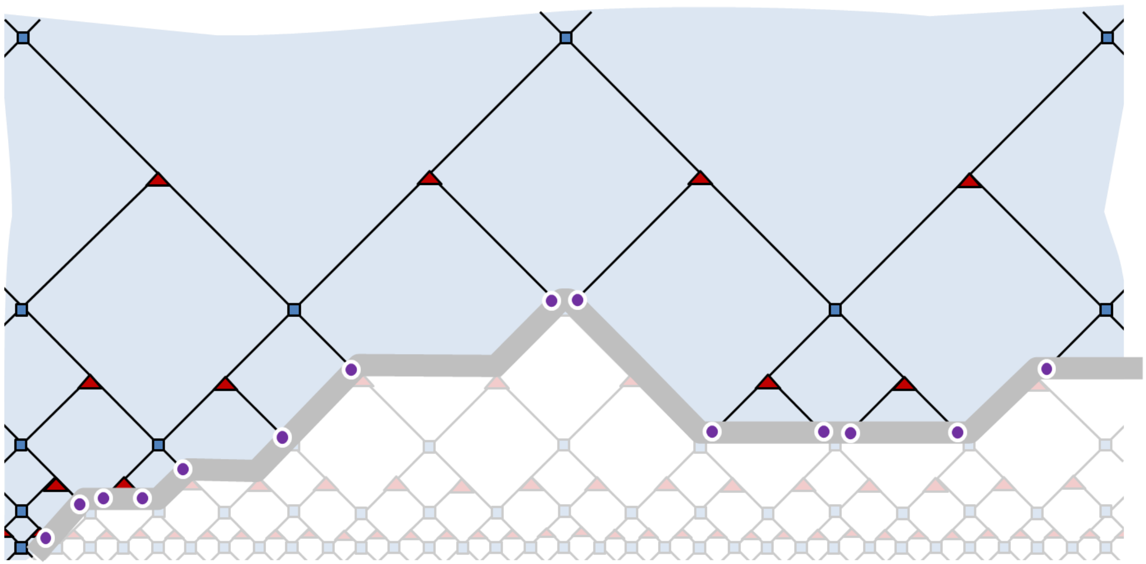


- dyadic subdivision

What local scale transformations can we implement “naturally”?

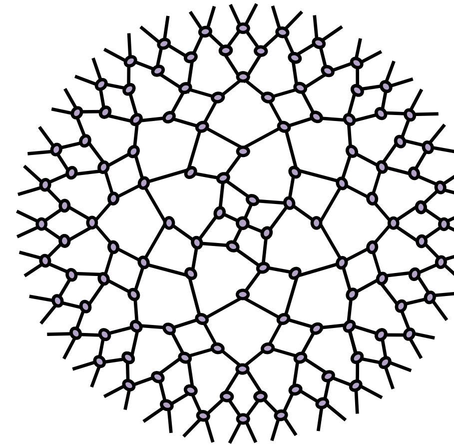


- dyadic rearrangement



(Perhaps): Local scale transformations in MERA are Thompson's group F

+ translations: Thompson's group T



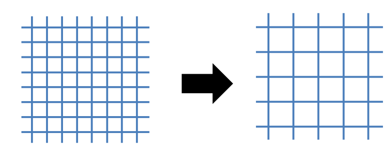
wave-functions / Hamiltonians

global scale transformation (RG transformation)

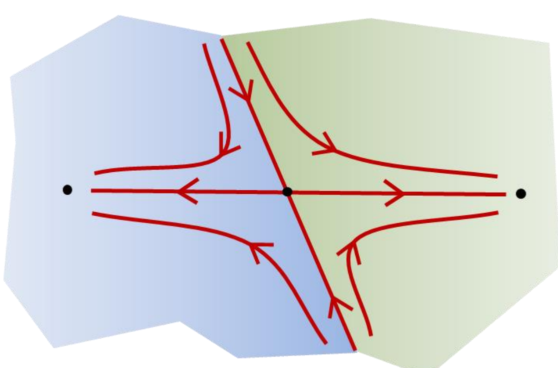
local scale transformations

Euclidean path integrals / classical partition functions

global scale transformation (RG transformation)



local scale transformations



Euclidean path integral

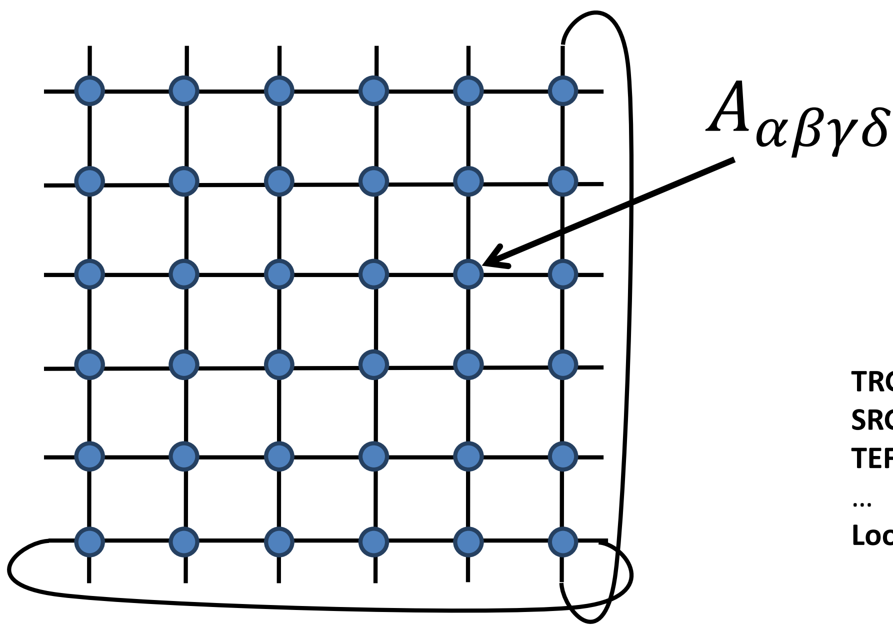
$$Z(\lambda) = tr\, e^{-\beta H_q^{1d}}$$

Classical partition function

$$Z(T) = \sum_{\{s\}} e^{-\frac{1}{T} H_{cl}^{2d}}$$

as a tensor network

$$Z =$$

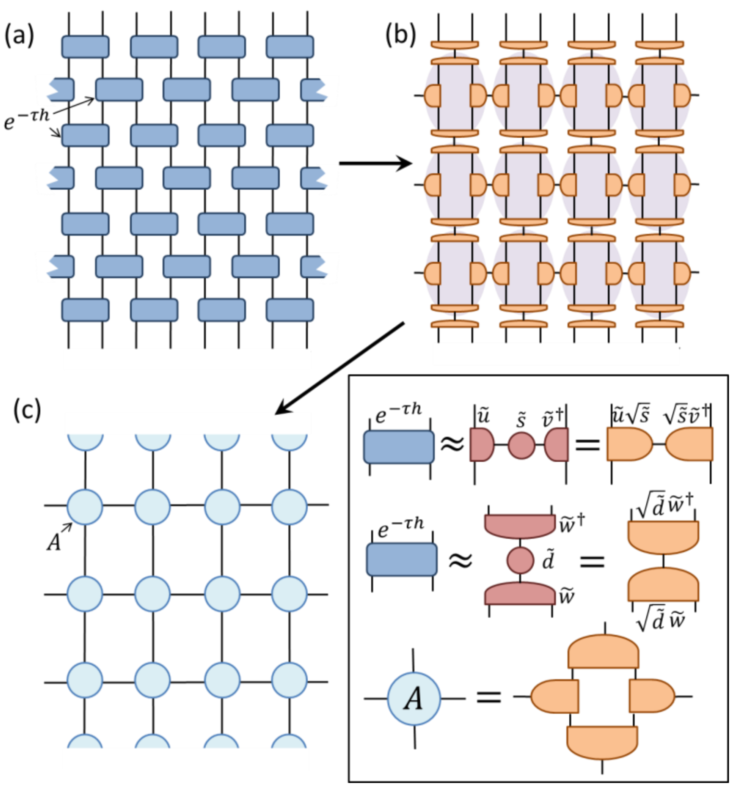


TRG Levin, Nave (2006)
SRG Xiang et al (2009)
TEFR Gu, Wen (2009)
...
Loop-TNR Yang, Gu, Wen (Dec2015)

Euclidean path integral

$$Z(\lambda) = tr\ e^{-\beta H_q^{1d}}$$

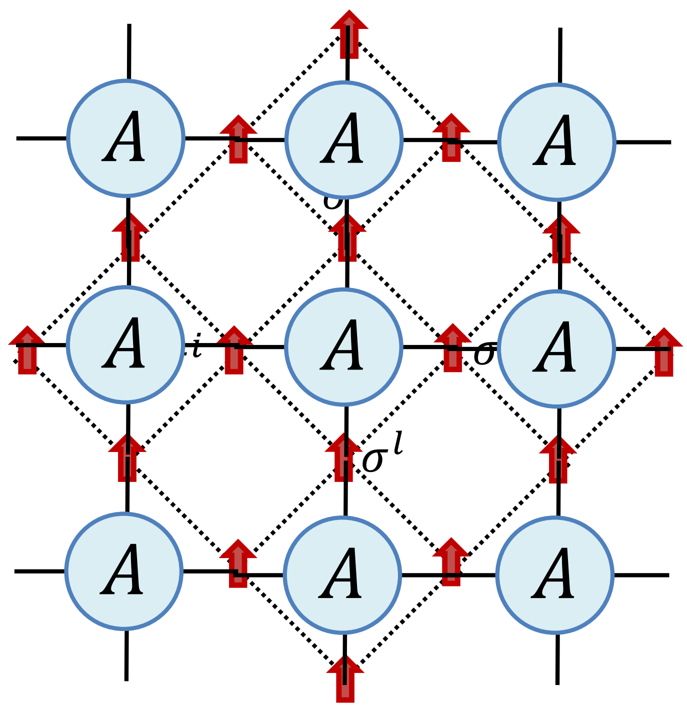
$$H_q^{1d} = \sum_i (\sigma_z^i + \sigma_x^i \sigma_x^{i+1})$$



Classical partition function

$$Z(T) = \sum_{\{s\}} e^{-\frac{1}{T} H_{cl}^{2d}}$$

$$H_{cl}^{2d} = \sum_{<i,j>} \sigma^i \sigma^j$$



$$A_{ijkl} = e^{-(\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_l + \sigma_l\sigma_i)/T}$$

Euclidean path integral

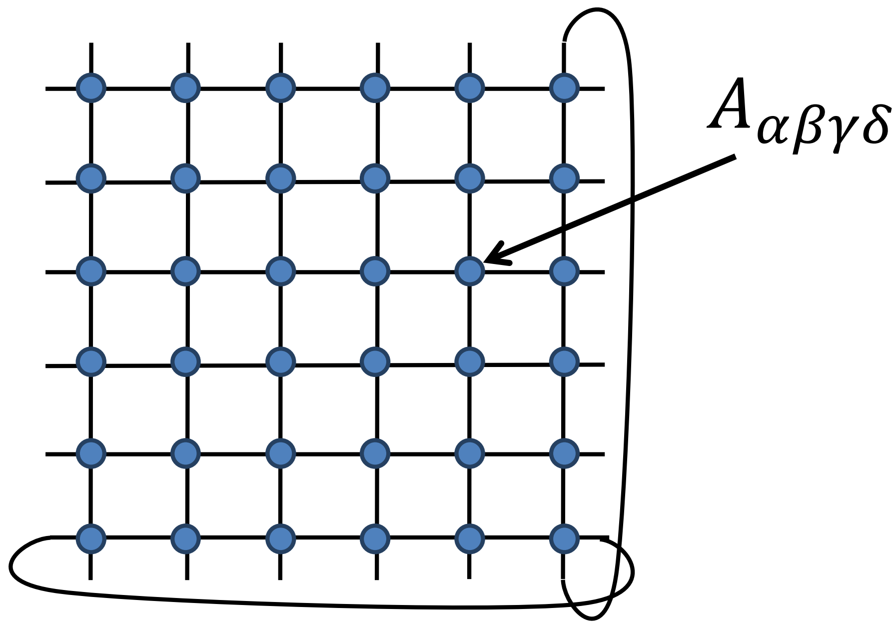
$$Z(\lambda) = tr\ e^{-\beta H_q^{1d}}$$

Statistical partition function

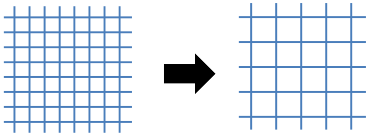
$$Z(T) = \sum_{\{s\}} e^{-\frac{1}{T} H_{cl}^{2d}}$$

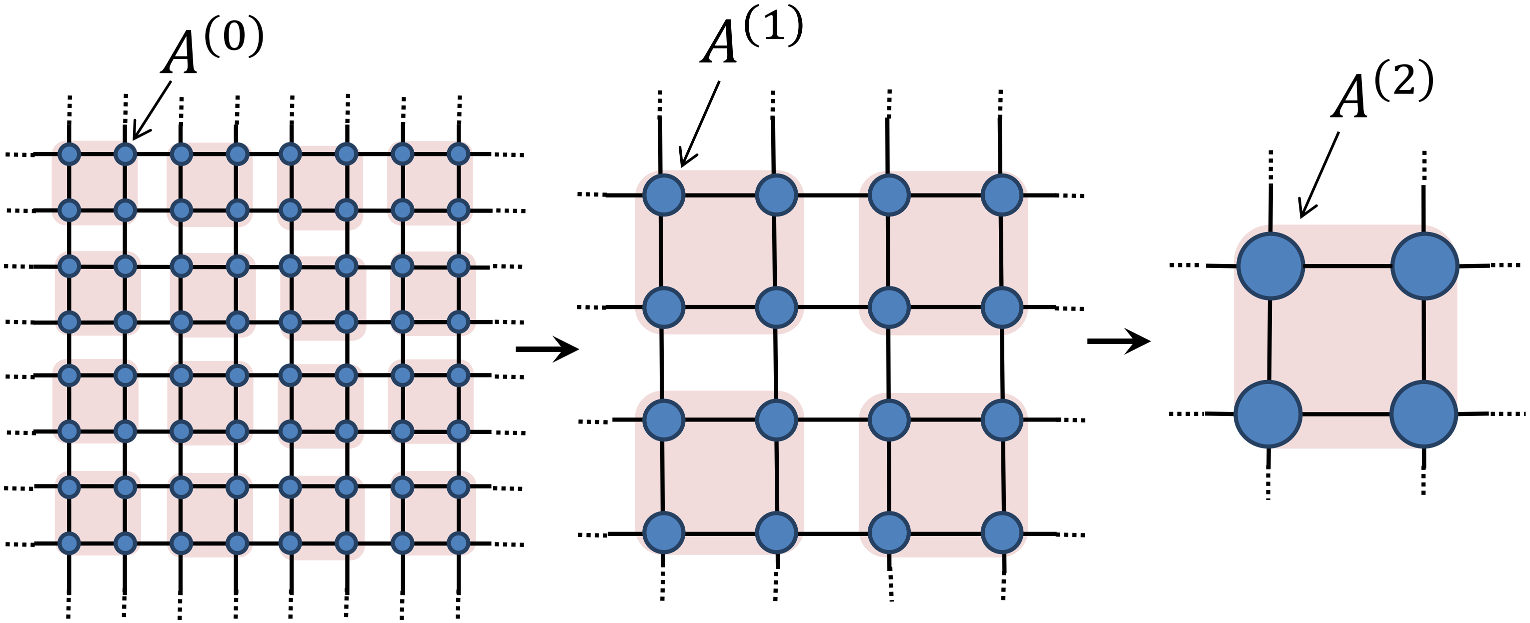
as a tensor network

$$Z =$$



How do we define a global scale transformation on the lattice?

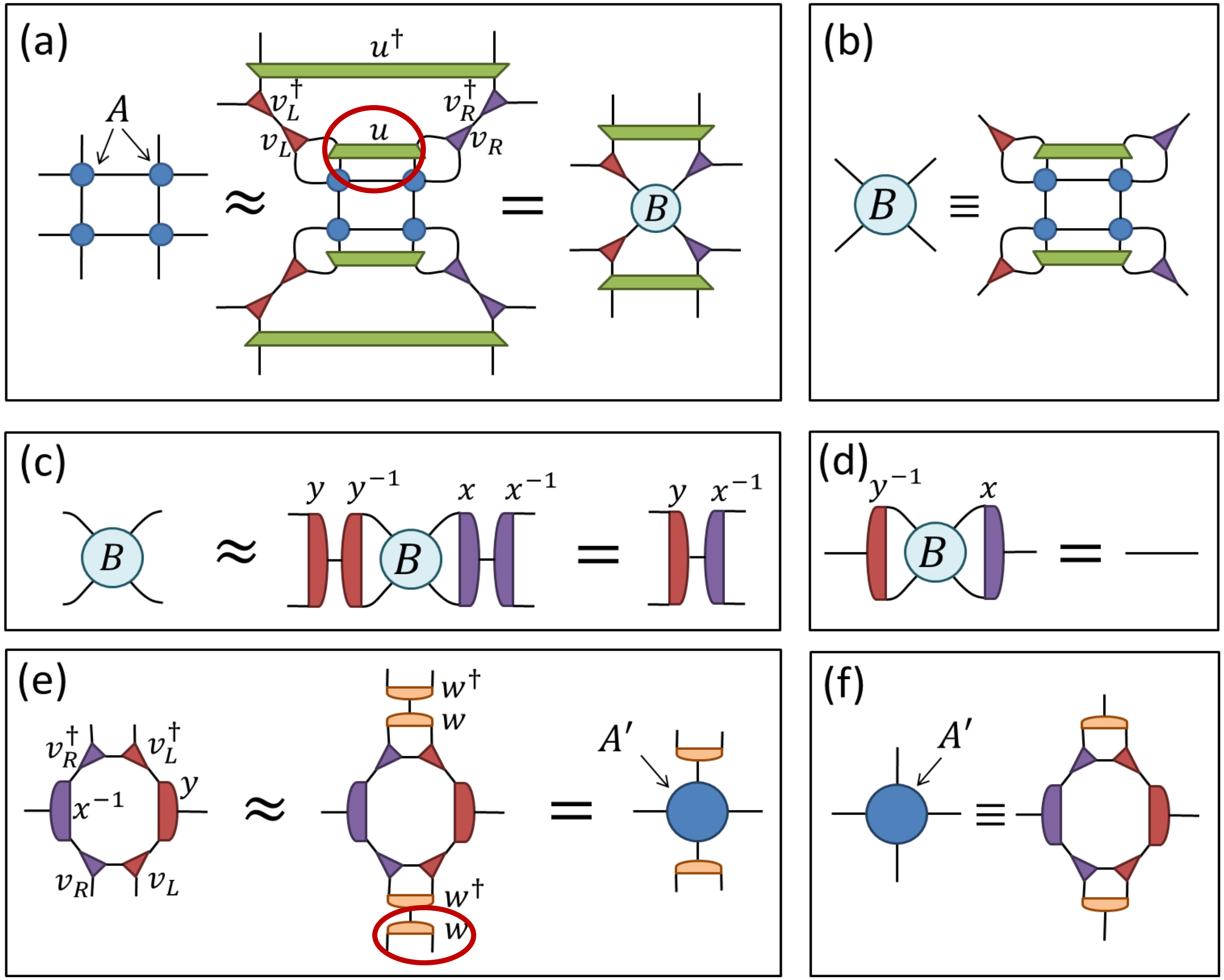




Tensor Network Renormalization (TNR)

Evenbly, Vidal PRL 2015
(arXiv:1412.0732)

Evenbly, arXiv:1509.07484



Tensor Network Renormalization (TNR)

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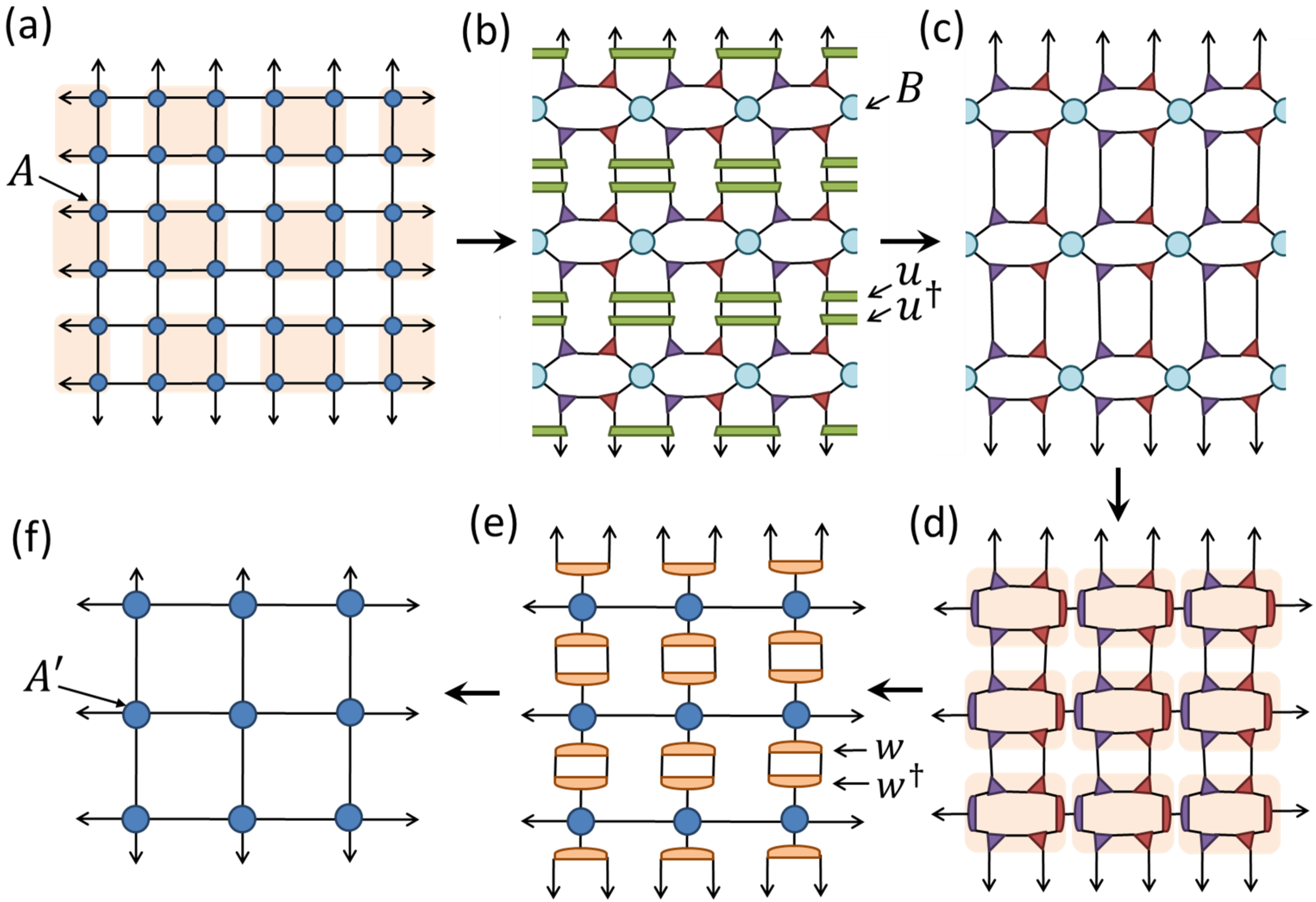
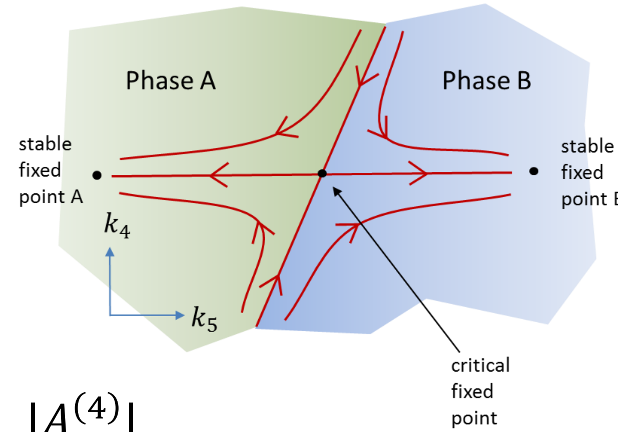


Tensor Network Renormalization (TNR) defines a proper RG flow in the space of tensors

$$A \to A' \to A'' \to \cdots \to A^{fp}$$



Example: 2D classical Ising model

$|A^{(1)}|$ $|A^{(2)}|$ $|A^{(3)}|$ $|A^{(4)}|$

below critical

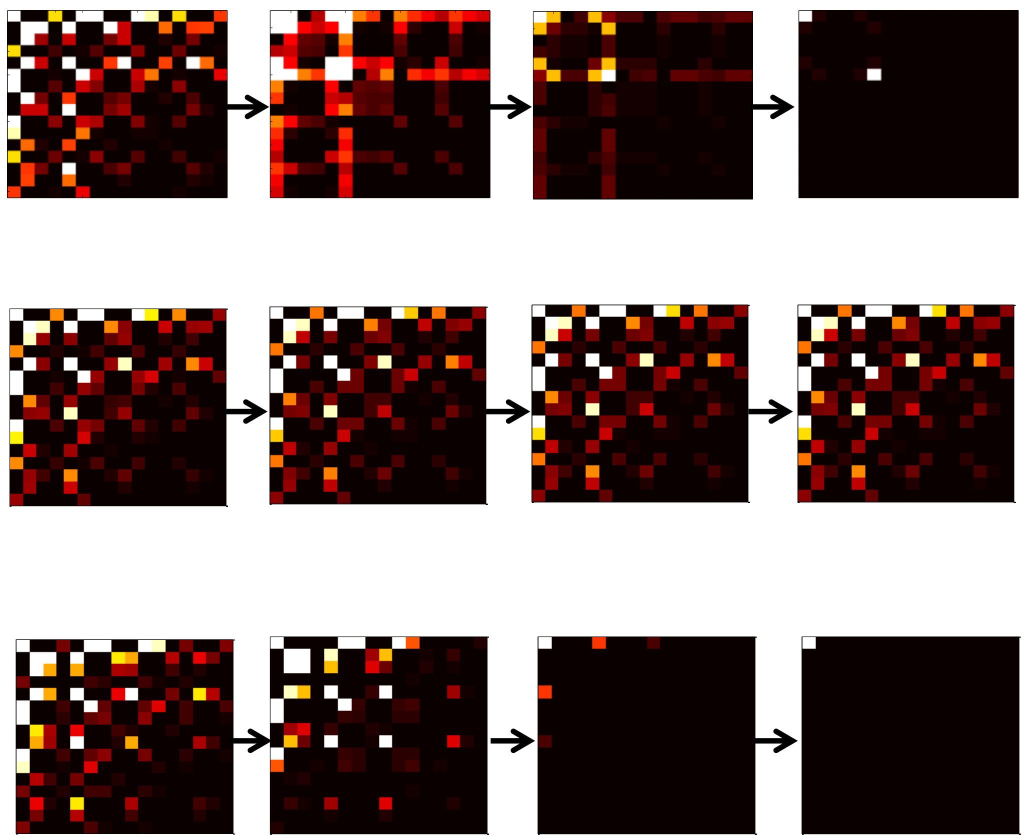
$T = 0.9\, T_c$

critical

$T = T_c$

above critical

$T = 1.1\, T_c$



ordered (Z2) fixed point

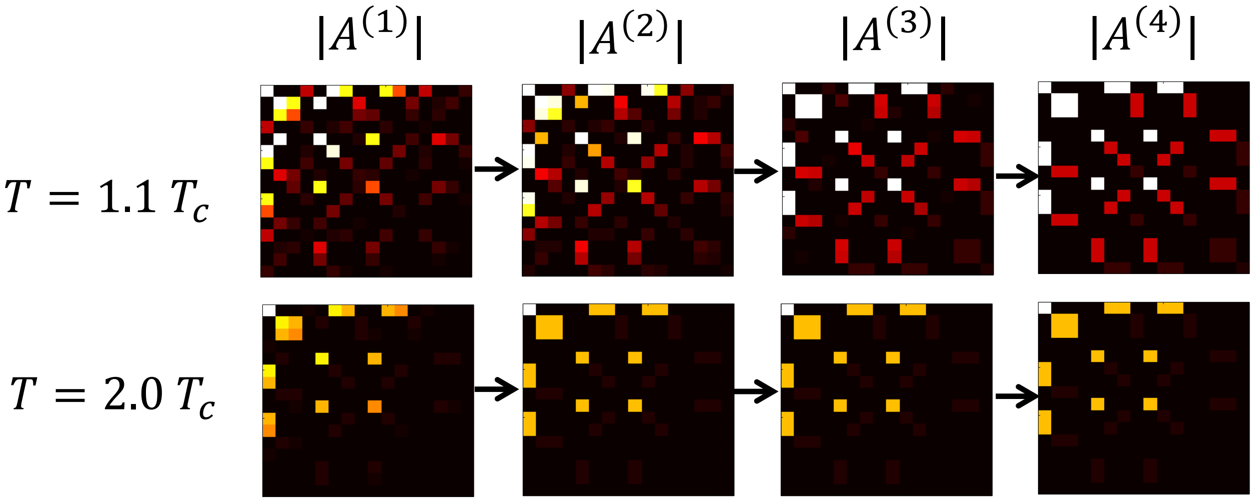
also TEFR of Gu, Wen (2009)

critical fixed point

disordered (trivial) fixed point

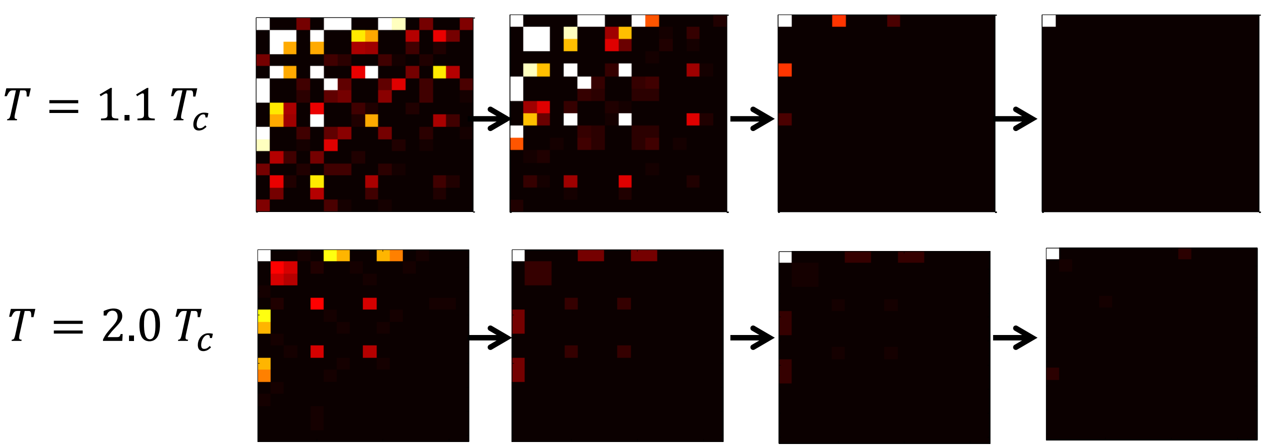
also TEFR of Gu, Wen (2009)

Without disentanglers: (TRG, Levin Nave 2006)



different "fixed-points"
for every T in same phase

With disentanglers: (TNR 2014)

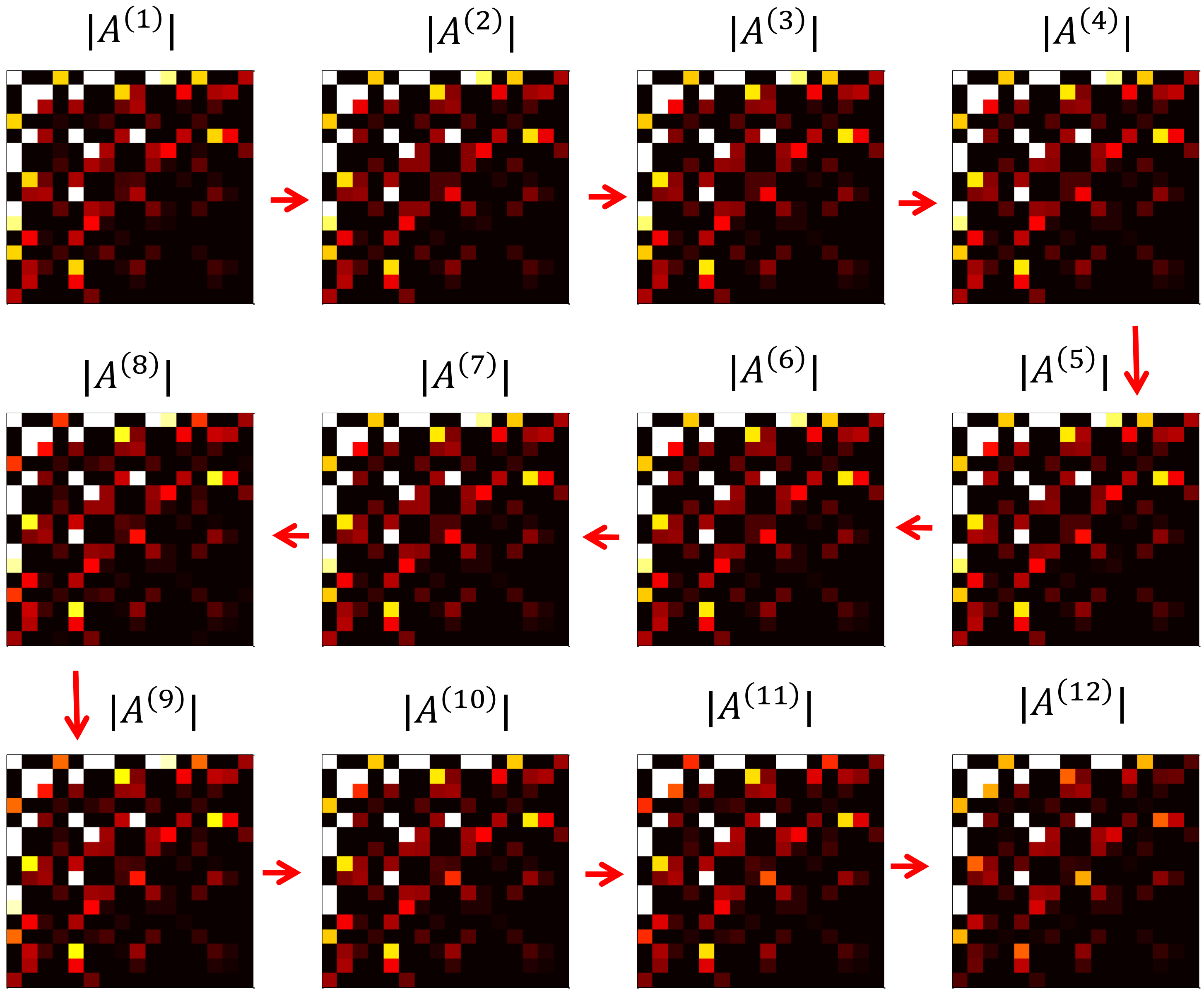


unique fixed-point
for any T in same phase

- at criticality, *approximate* fixed-point

critical point:
$T = T_c$

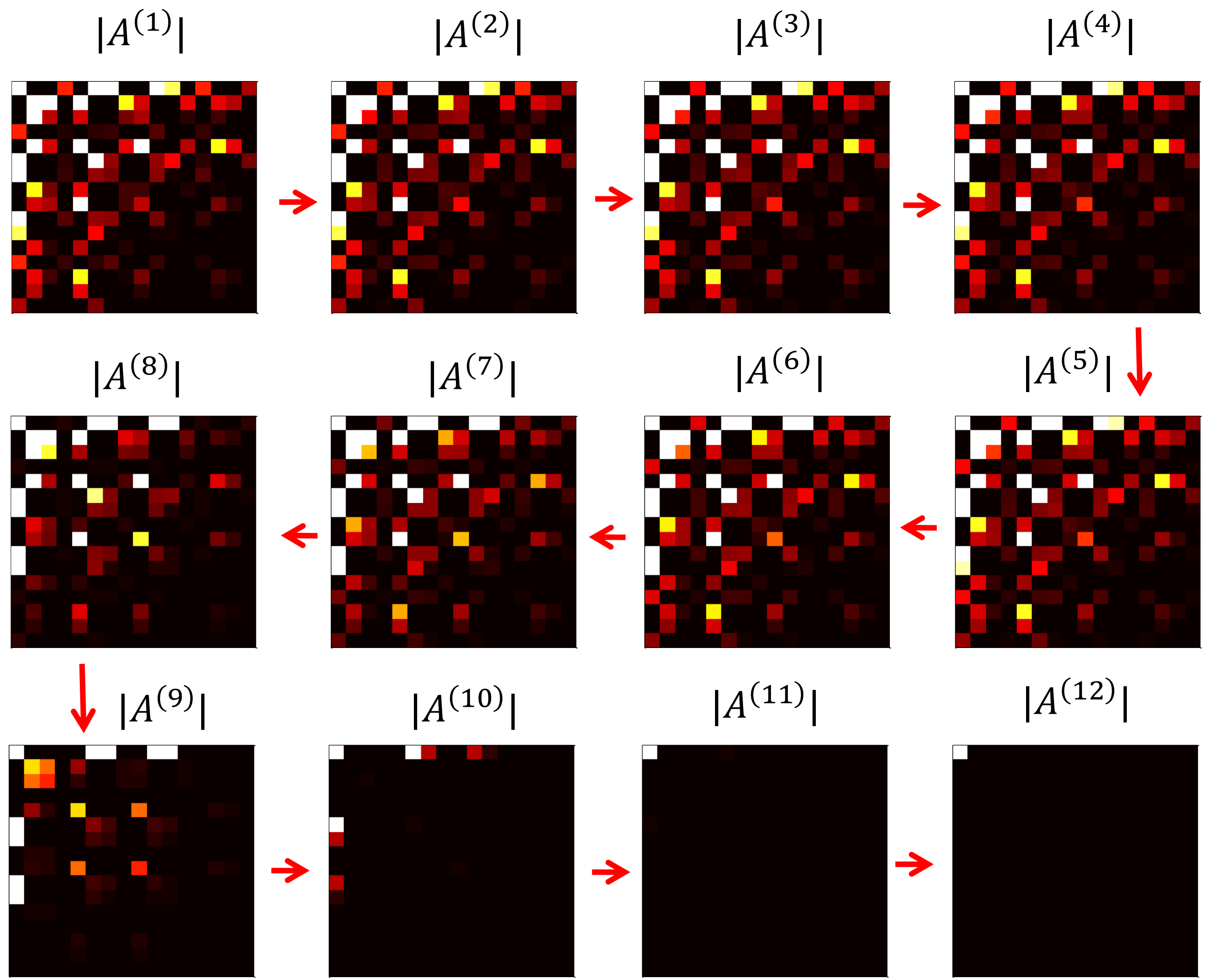
TNR with
bond dimension:
$\chi = 4$

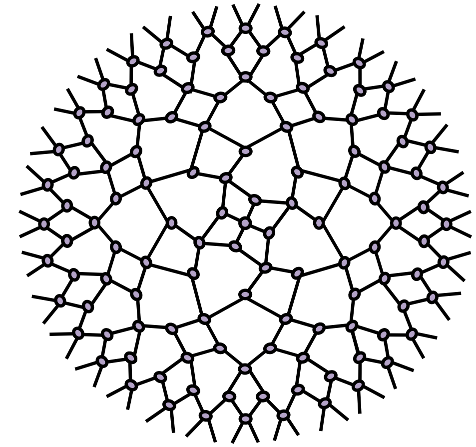


- near criticality…

more difficult!
$T = 1.002\,T_c$

TNR with bond dimension:
$\chi = 4$





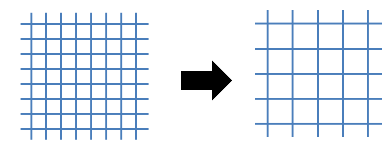
wave-functions /
Hamiltonians

global scale
transformation
(RG transformation)

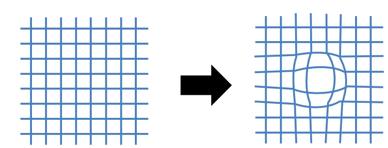
local scale
transformations

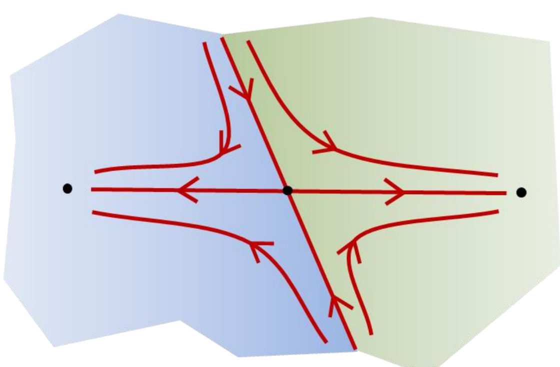
Euclidean path integrals /
classical partition functions

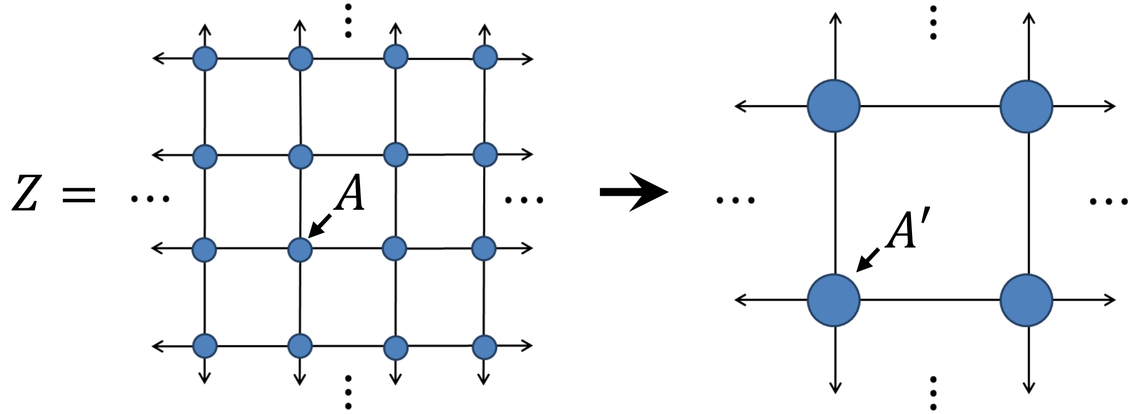
global scale
transformation
(RG transformation)

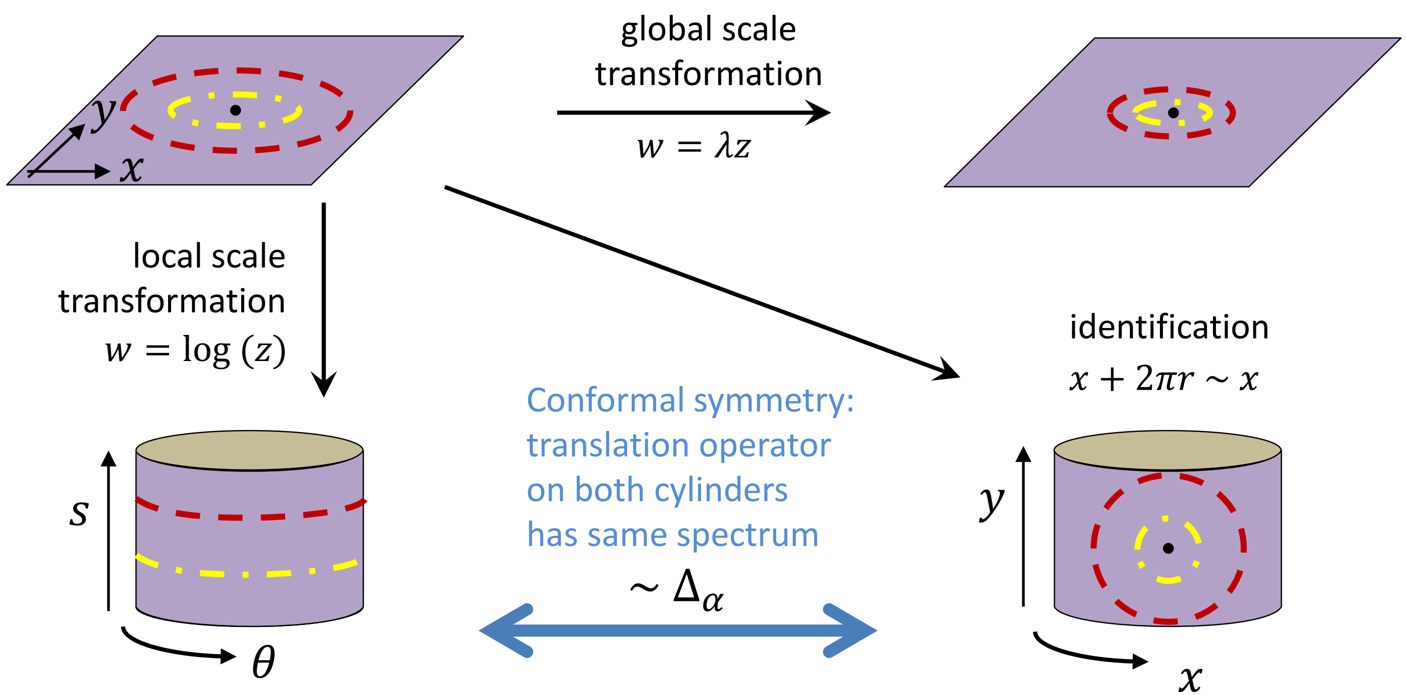


local scale
transformations





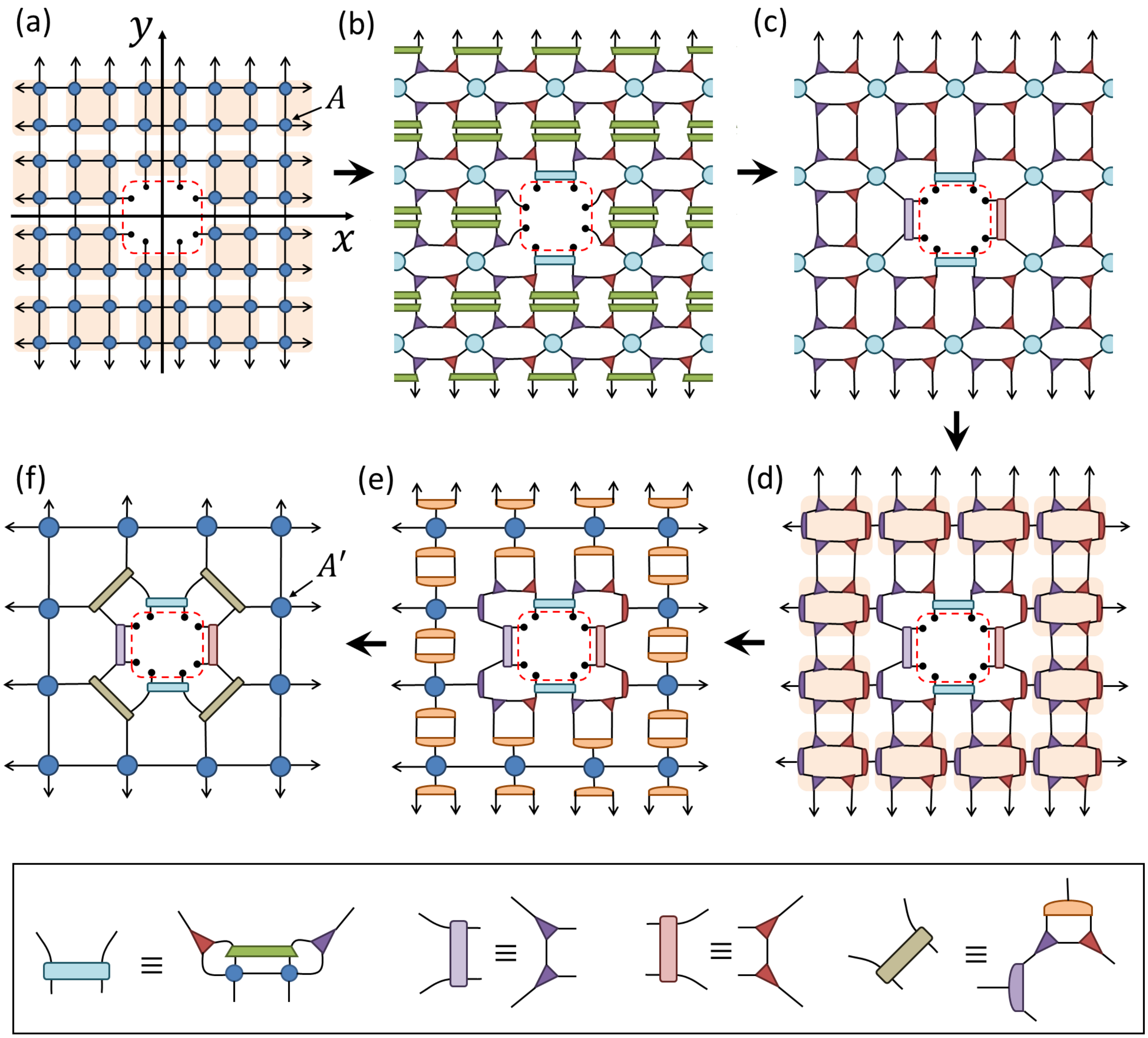




Can we do the same on the lattice?

Plane to cylinder

Evenbly, Vidal
arXiv:1510.00689



Plane to cylinder

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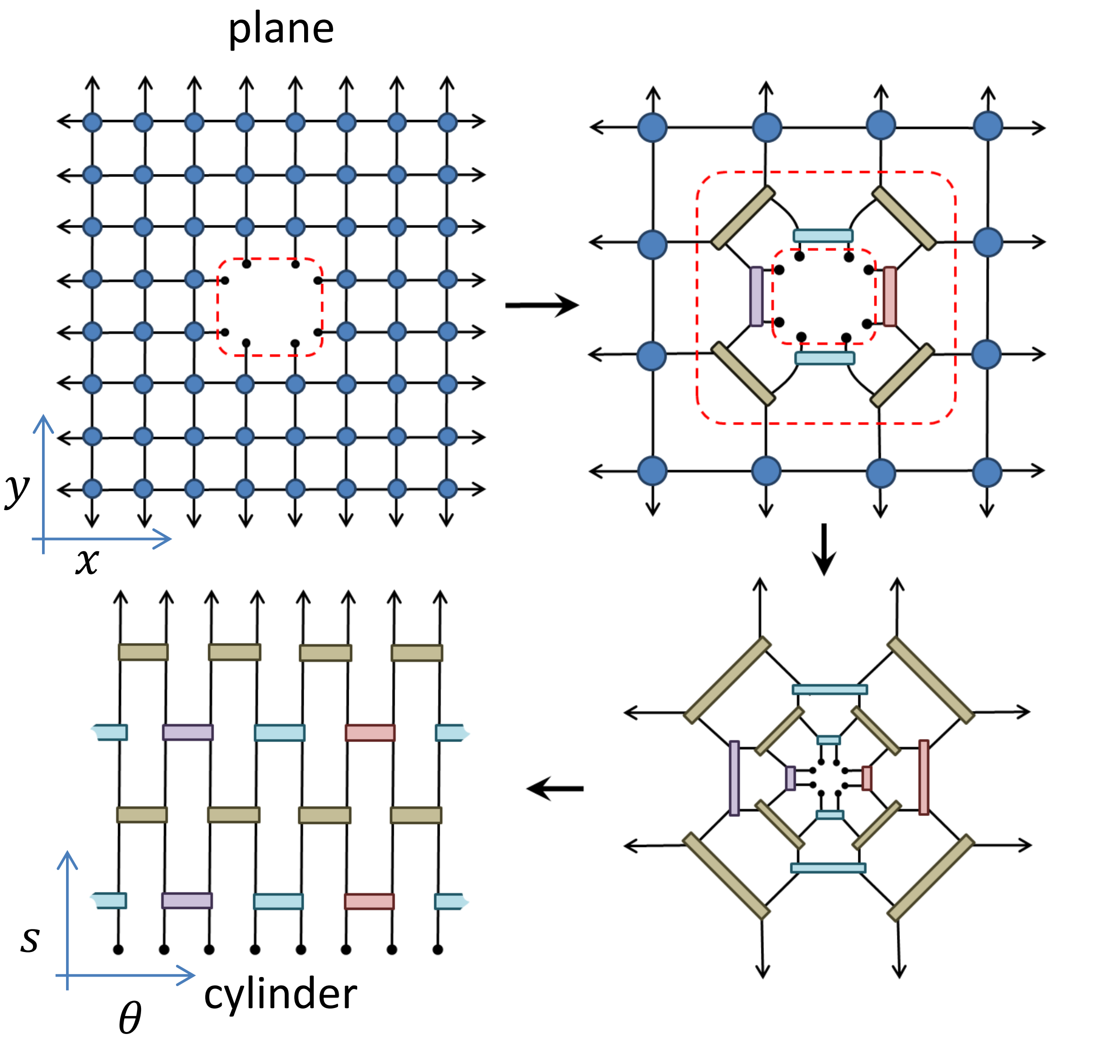


- radial quantization in CFT

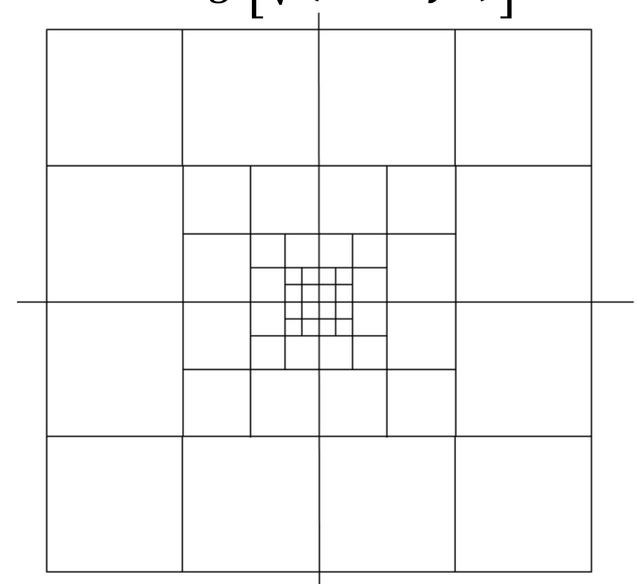
$$z \equiv x + iy \quad \text{(plane)}$$

conformal transformation

$$z \rightarrow w = \log(z)$$

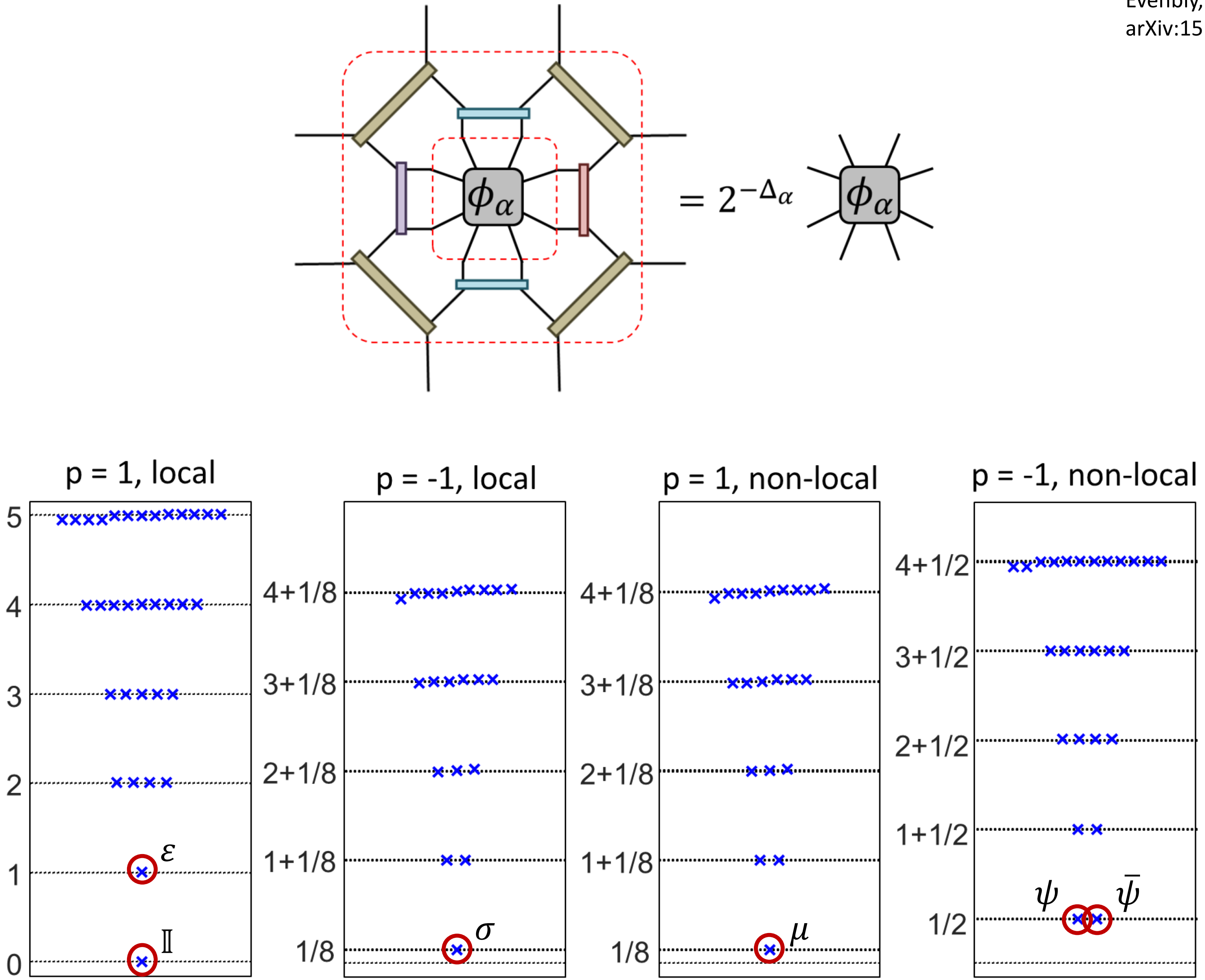
$$w \equiv s + i\theta \quad \text{(cylinder)}$$

$$s \equiv log\left[\sqrt{(x^2 + y^2)}\right]$$



- Extraction of scaling dimensions, OPE

Evenbly, Vidal
arXiv:1510.00689

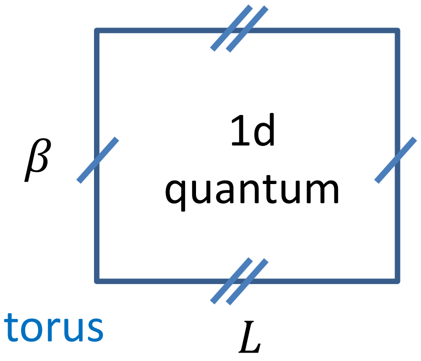


Evenbly, Vidal
arXiv:1510.00689

exact	TNR(6)	TNR(16)	TNR(24)	TRG(80)
0.125	0.125679	0.124941	0.124997	0.124989
1	1.001499	1.000071	1.000009	1.000256
1.125	1.125552	1.125011	1.124991	1.125532
1.125	1.127024	1.125201	1.125027	1.125641
2	2.003355	2.000087	2.000010	2.002235
2	2.003365	2.000133	2.000017	2.002367
2	2.003374	2.000279	2.000022	2.002607
2	2.003525	2.000319	2.000060	2.003926
2.125	2.114545	2.124944	2.124985	2.127266
2.125	2.129043	2.125290	2.125038	2.130337
2.125	2.142611	2.125670	2.125096	2.131299
3	3.005045	3.000524	3.000052	3.007488
3	3.005092	3.000777	3.000061	3.017253
3	3.005259	3.000887	3.000073	3.017316
3	3.005318	3.001010	3.000105	3.020581
3	3.005805	3.001261	3.000206	3.023023
3.125	3.109661	3.124866	3.124889	3.132764
3.125	3.116466	3.125201	3.125019	3.132890
3.125	3.118175	3.125319	3.125059	3.136725
3.125	3.144798	3.126159	3.125099	3.137217
3.125	3.145661	3.126163	3.125158	3.141363
3.125	3.146323	3.126315	3.125172	3.146419
max err.	0.83%	0.046%	0.0069%	0.76%

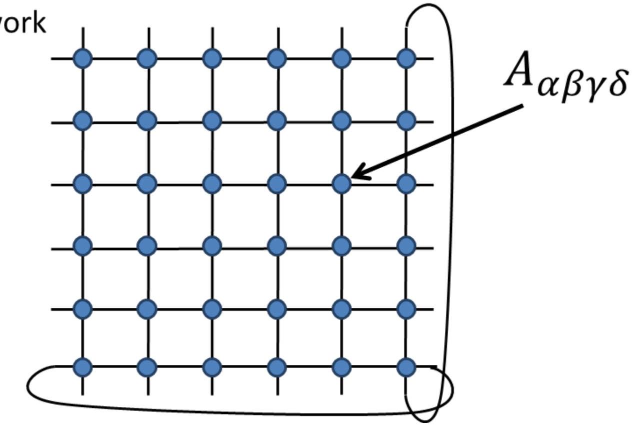
Euclidean path integral

$$Z(\lambda) = tr\ e^{-\beta H_q^{1d}}$$



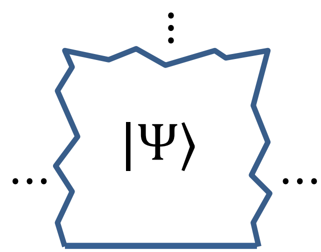
as a tensor network

$$Z =$$



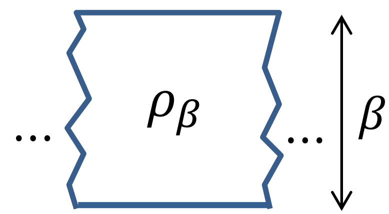
Euclidean time evolution on different geometries

upper half-plane



ground state

infinite strip

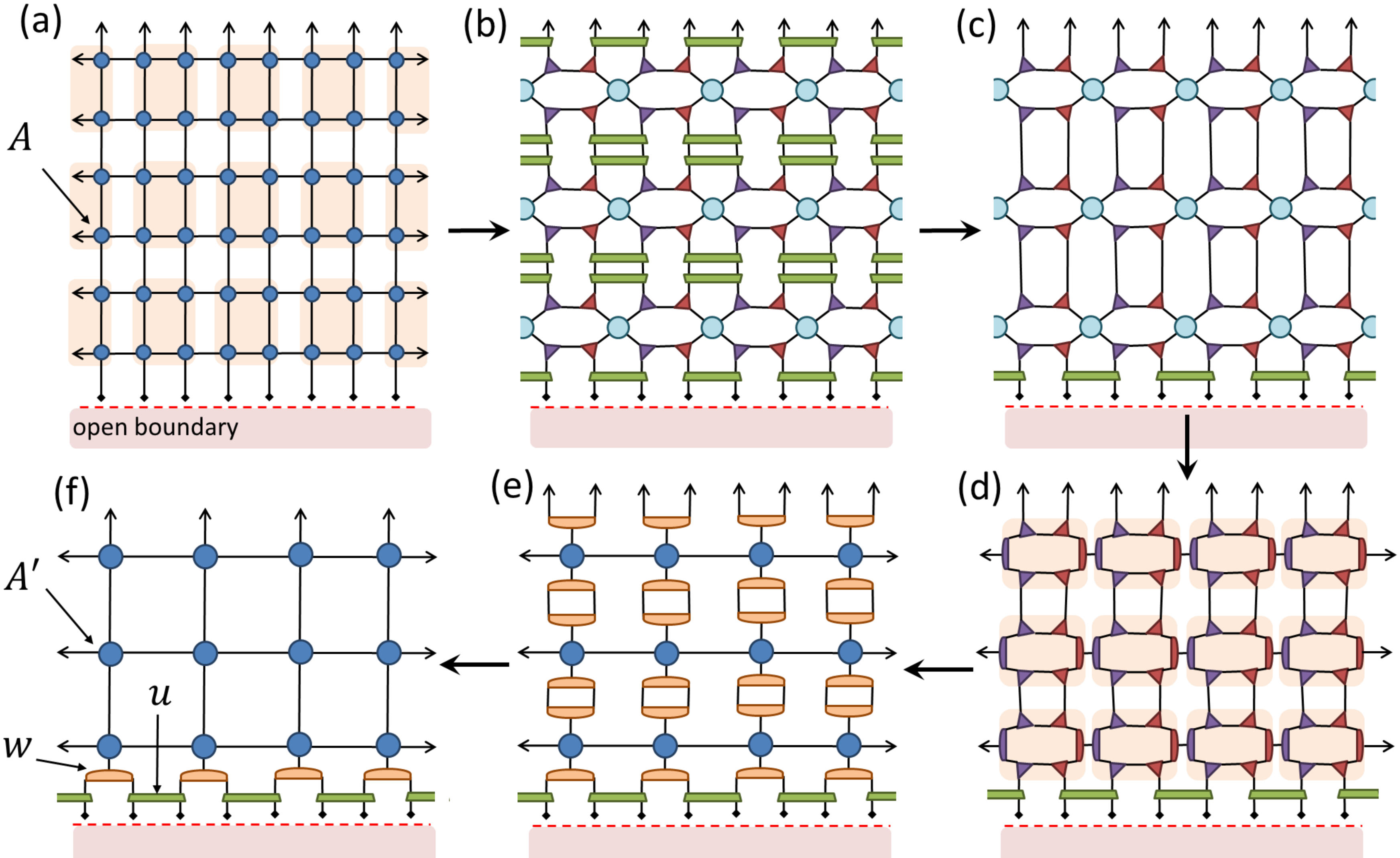


thermal state

Upper half plane

Evenbly, Vidal, PRL 2015
arXiv:1502.05385

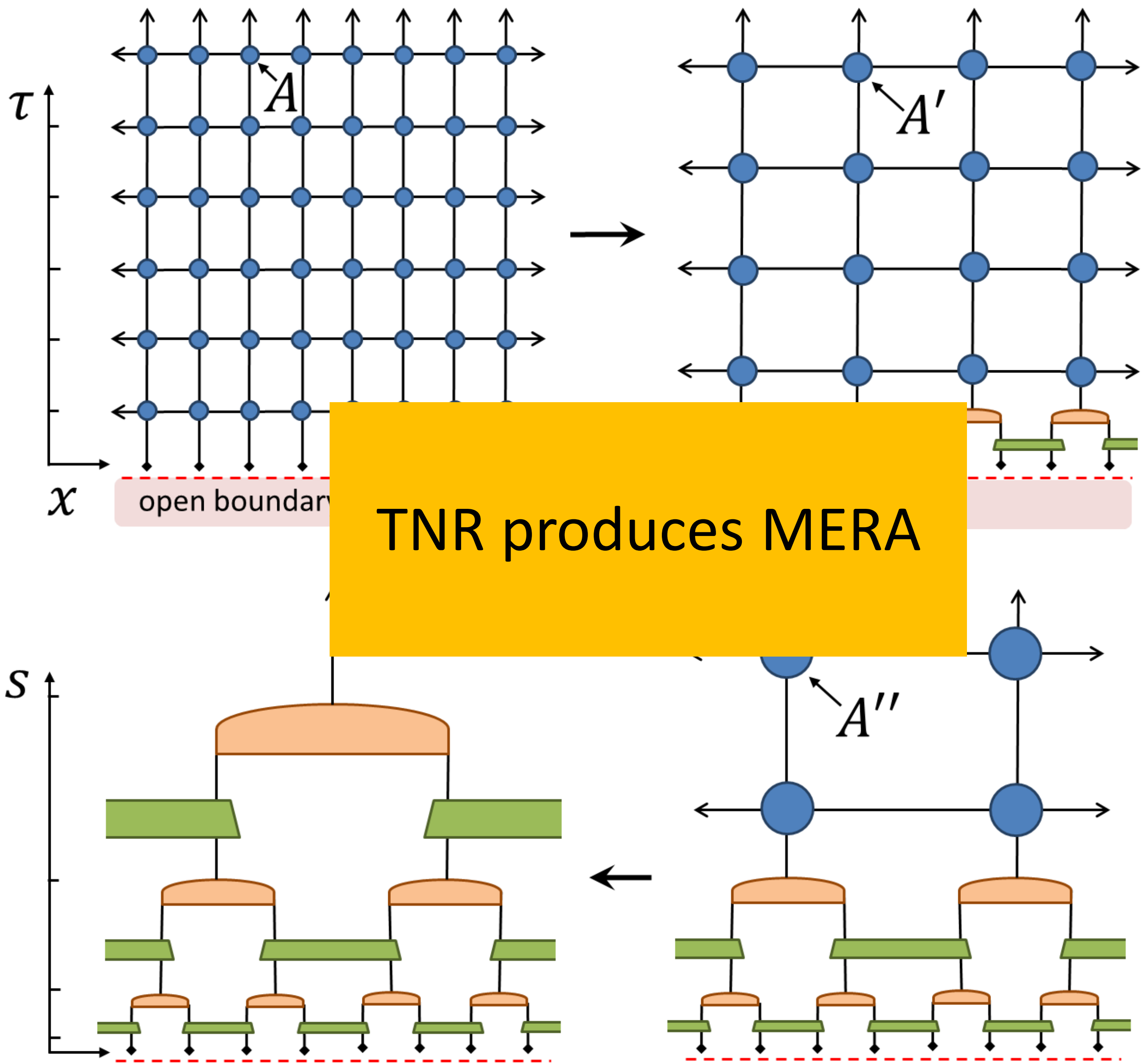
$$|\Psi\rangle \sim e^{-\tau H}|\phi_0\rangle$$



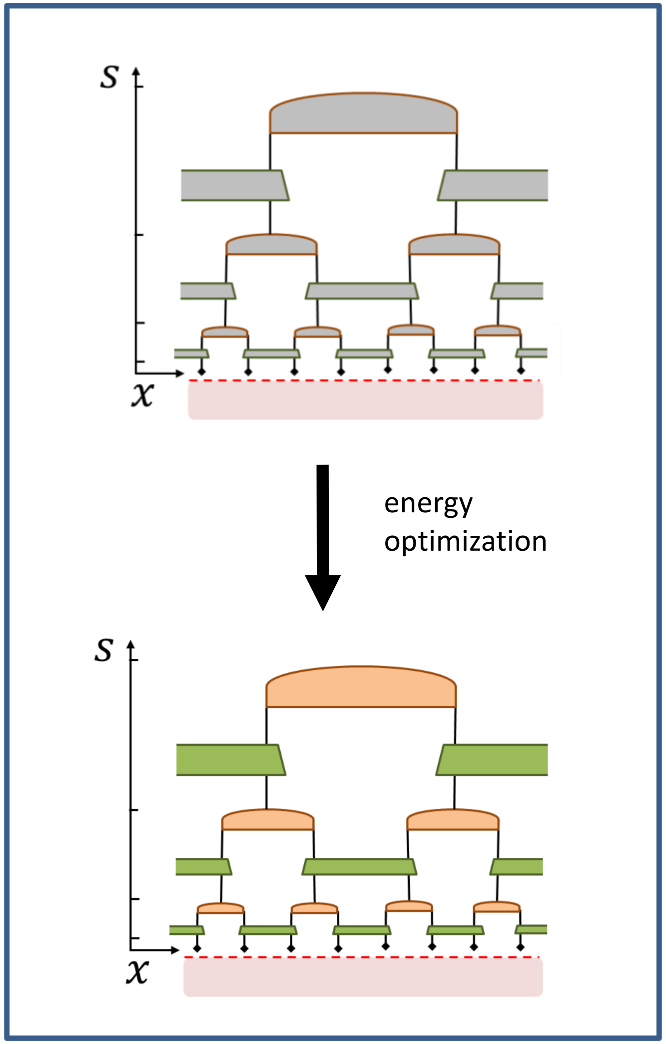
Upper half plane

Evenbly, Vidal, PRL 2015
arXiv:1502.05385

$|\Psi\rangle \sim e^{-\tau H}|\phi_0\rangle$

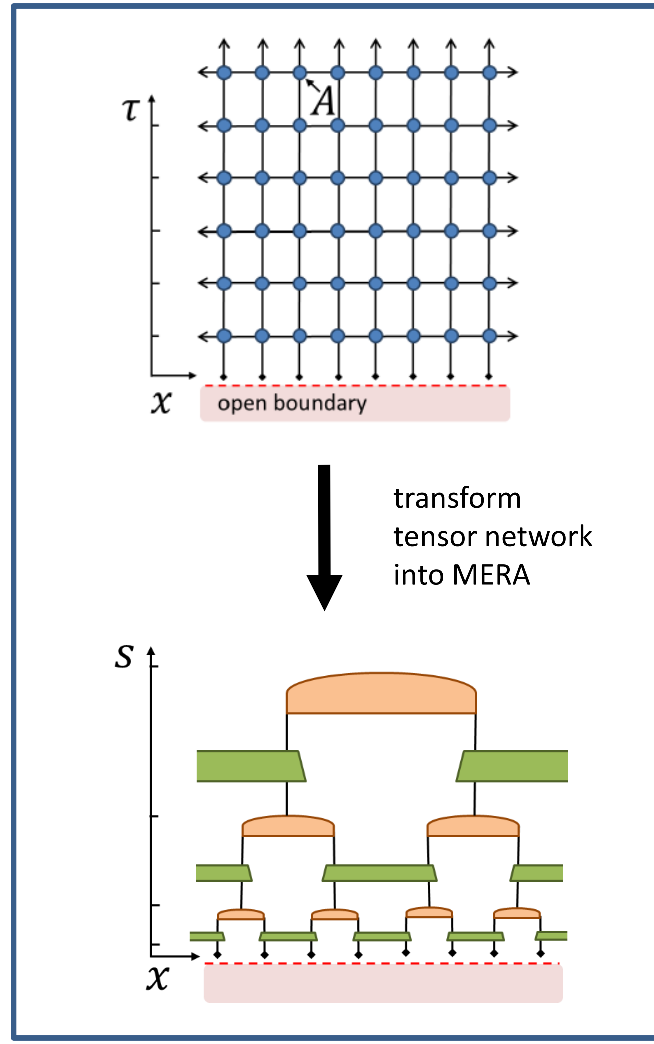


MERA = variational ansatz ⇒ MERA = by-product of TNR



energy minimization

- 1000s of iterations over scale
- local minima
- correct ground ?



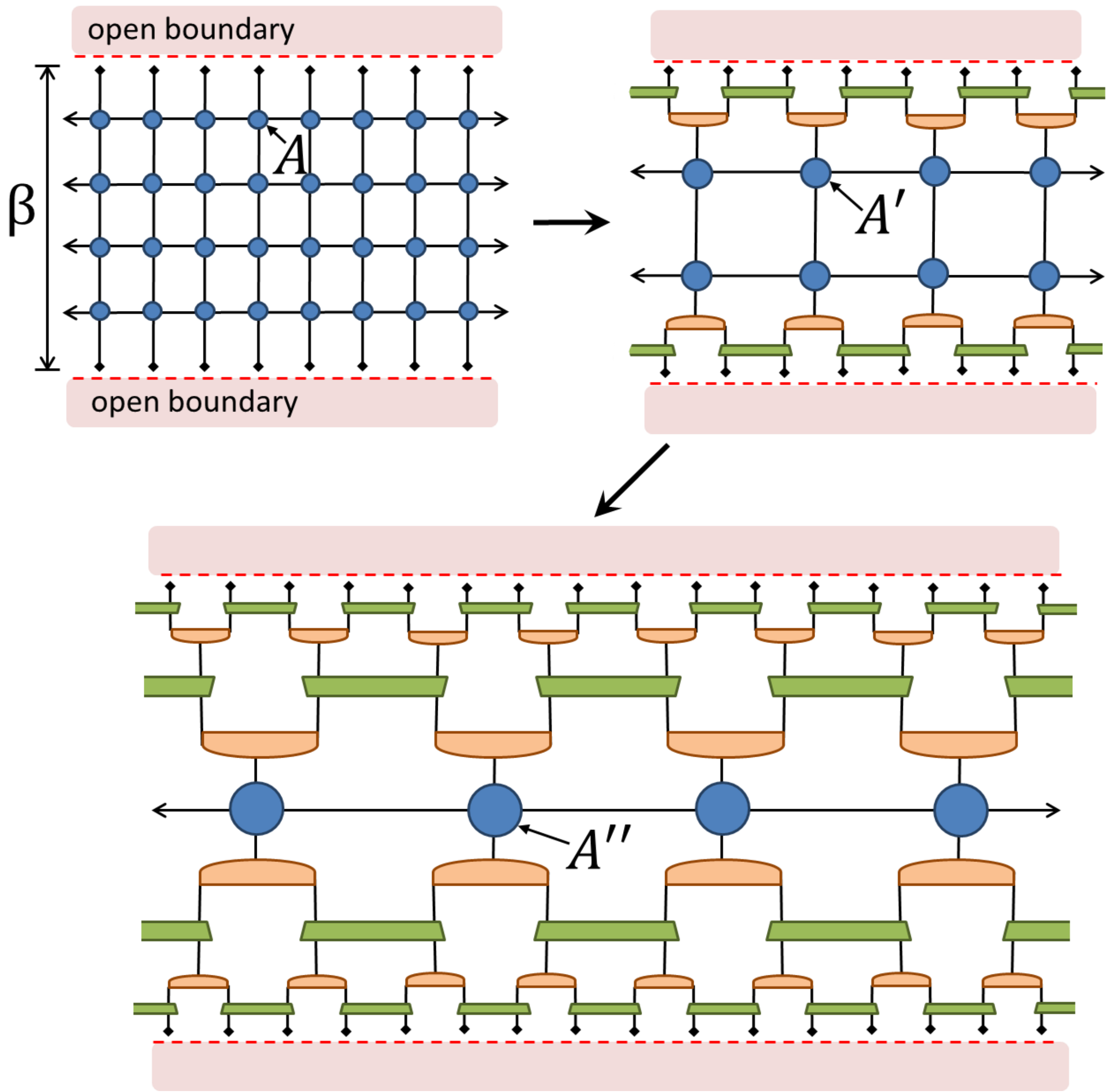
TNR -> MERA

- single iteration over scale
- rewrite tensor network for ground state
- certificate of accuracy

infinite strip of finite width β

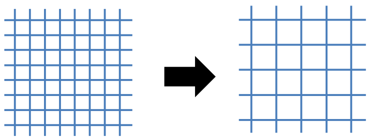
Evenbly, Vidal, PRL 2015
arXiv:1502.05385

$$\rho_\beta \sim e^{-\beta H}$$

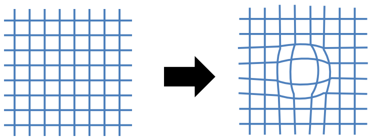


Summary:

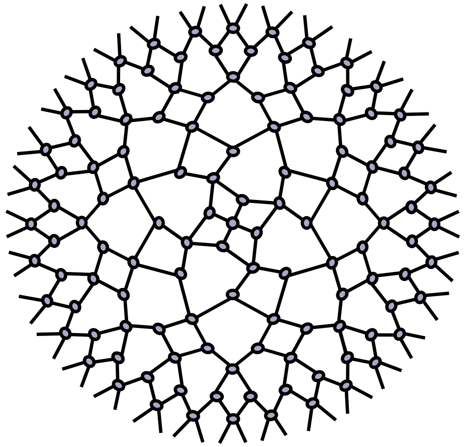
global scale transformation (RG transformation)



local scale transformation



on the lattice !



Frontier:

What about 2+1 dimensions? (3+1 …?)

(QCD?)

What about diffeomorphism invariance on the lattice?

(quantum gravity?)

What about tensor networks in the continuum?

(space-time symmetries)