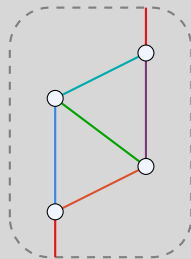
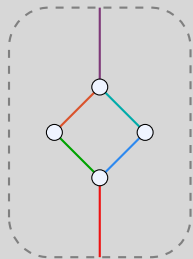
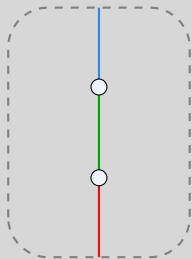


Computing Renormalization Invariant Properties of Levin-Wen Models

Daniel Barter (ANU)

Jacob Bridgeman (PI)

Corey Jones (OSU)



2D, non-chiral, long range entangled topological phases

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- ▶ When $C = \mathbf{Vec}(G)$, the category of G -graded vector spaces, the Levin-Wen model and the Kitaev model define the same phase of matter.

What is a fusion category?

generated by trivalent vertices



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
generated by trivalent vertices



We impose the relation

A diagrammatic equation showing a relation between two trivalent vertices. The left vertex has edges a , b , and c at the bottom, and d at the top. The edges are labeled with α (left), β (right), and e (middle). The right vertex has edges a , b , and c at the bottom, and d at the top. The edges are labeled with ν (left), μ (right), and f (middle). The two vertices are connected by an equals sign and a summation symbol $\sum_{\mu, \nu}$. The summation is over μ, ν and the coefficient is $F_{\alpha, \beta}^{\mu, \nu}$.

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We impose the relation

$$\begin{array}{c} & & d & & \\ & \beta & / & & \\ \alpha & / & e & \backslash & \\ a & / & & \backslash & c \\ & & b & & \end{array} = \sum_{\mu, \nu} F_{\alpha, \beta}^{\mu, \nu} \begin{array}{c} & & d & & \\ & & \backslash & & \\ & \nu & / & & \\ & f & / & \backslash & \mu \\ a & / & & \backslash & c \\ & & b & & \end{array}$$


The tensor F must satisfy the pentagon equation, also known as the 3-2 Pachner equation.

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Let G be a finite group and $g, h, k \in G$.


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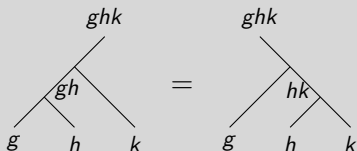
$\mathbf{Vec}(G)$ is generated by 

Example: $\mathbf{Vec}(G)$: G -graded vector spaces

Let G be a finite group and $g, h, k \in G$.

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and the generators satisfy the relation


$$\begin{array}{c} ghk \\ / \quad \backslash \\ gh \quad k \\ / \quad \backslash \\ g \quad h \end{array} = \begin{array}{c} ghk \\ / \quad \backslash \\ g \quad hk \\ / \quad \backslash \\ h \quad k \end{array}$$

The Guiding Principle

Let C be a fusion category and Σ an **orientable** 2-dimensional surface possibly with boundary.

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In mathematics, Z_C is called a fully extended (2+1)D topological quantum field theory. It captures the renormalization invariant properties of the Levin-Wen model.

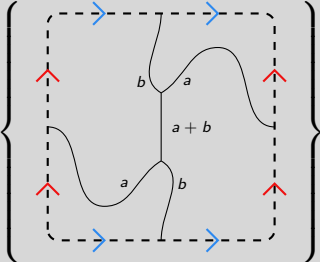
Ground states on the torus

Let A be an Abelian group and T the torus.

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$$Z_{\mathbf{vec}(A)}(T) = \mathbb{C} \left\{ \begin{array}{c} \text{Diagram of a torus with paths } a, b, a+b \end{array} \right\}$$


The diagram shows a square representing a torus, enclosed in a dashed black border. Blue arrows on the top and bottom edges point to the right, and red arrows on the left and right edges point upwards. A central vertical line is labeled $a+b$. Two paths, labeled a and b , branch off from this central line. Path a starts at the top, goes right, then down, then left, then up, and finally right to the right edge. Path b starts at the top, goes right, then down, then left, then up, and finally right to the right edge. The paths a and b are shown as solid black lines.

Ground states on the torus

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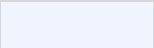

The diagram shows a square representing a torus with dashed lines. Blue arrows on the top and bottom edges point to the right, representing the horizontal cycle. Red arrows on the left and right edges point upwards, representing the vertical cycle. A central vertical line is labeled $a+b$. From the top of this line, a path labeled a goes right and then down to the right edge. From the bottom of this line, a path labeled b goes left and then down to the left edge. A path labeled b also goes from the top of the central line down to the bottom of the central line. A path labeled a goes from the bottom of the central line left to the left edge. The paths a and b are shown as wavy lines.

The phase corresponding to A has $|A|^2$ ground states on the torus.

Question

How many domain walls are there between two chunks of the $\mathbb{Z}/2\mathbb{Z}$ phase?

At least 6.

Domain wall	Action on particles
	Condenses e on both sides
	Condenses m on left and e on right
	Condenses e on left and m on right
	Condenses m on both sides
	trivial domain wall
	$e \leftrightarrow m$

Kitaev, Kong and Ostrick

- ▶ In *Models for gapped boundaries and domain walls*, Kitaev and Kong demonstrated that domain walls between the Levin-Wen phases associated to C and D correspond to simple C – D bimodule categories.

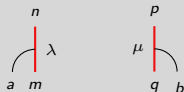
Kitaev, Kong and Ostrik

- ▶ In *Models for gapped boundaries and domain walls*, Kitaev and Kong demonstrated that domain walls between the Levin-Wen phases associated to C and D correspond to simple C – D bimodule categories.
- ▶ **Ostrik's Theorem:**

$$\left\{ \begin{array}{l} \text{simple } \mathbf{Vec}(G)\text{--}\mathbf{Vec}(H) \\ \text{bimodule categories} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{subgroups } P \text{ of } G \times H^{\text{op}} \\ \text{and cohomology classes} \\ \omega \in H^2(P, U(1)) \text{ mod-} \\ \text{ulo conjugation} \end{array} \right\}$$

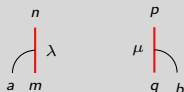
What is a bimodule category?

Generated by trivalent vertices



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We impose the relations

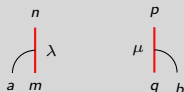
$$\begin{array}{c} \mu \\ \lambda \end{array} \left| \begin{array}{c} \text{arc} \\ \text{arc} \end{array} \right. = \sum_{\alpha, \nu} L_{\lambda, \mu}^{\alpha, \nu} \begin{array}{c} \alpha \\ \nu \end{array} \left| \begin{array}{c} \text{arc} \\ \text{arc} \end{array} \right.$$

$$\begin{array}{c} \mu \\ \lambda \end{array} \left| \begin{array}{c} \text{arc} \\ \text{arc} \end{array} \right. = \sum_{\alpha, \nu} R_{\lambda, \mu}^{\alpha, \nu} \begin{array}{c} \nu \\ \alpha \end{array} \left| \begin{array}{c} \text{arc} \\ \text{arc} \end{array} \right.$$

$$\begin{array}{c} \mu \\ \lambda \end{array} \left| \begin{array}{c} \text{arc} \\ \text{arc} \end{array} \right. = \sum_{\nu, \kappa} C_{\lambda, \mu}^{\nu, \kappa} \begin{array}{c} \nu \\ \kappa \end{array} \left| \begin{array}{c} \text{arc} \\ \text{arc} \end{array} \right.$$

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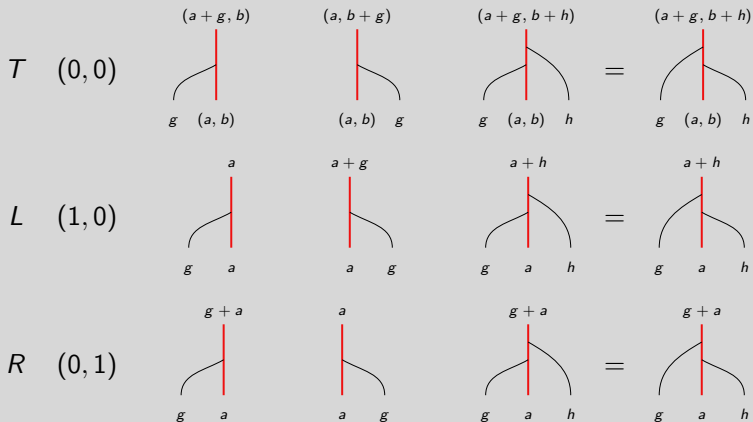


We impose the relations

$$\begin{aligned} \begin{array}{c} \mu \\ | \\ \lambda \end{array} &= \sum_{\alpha, \nu} L_{\lambda, \mu}^{\alpha, \nu} \begin{array}{c} \alpha \\ | \\ \nu \end{array} \\ \begin{array}{c} \mu \\ | \\ \lambda \end{array} &= \sum_{\alpha, \nu} R_{\lambda, \mu}^{\alpha, \nu} \begin{array}{c} \nu \\ | \\ \alpha \end{array} \\ \begin{array}{c} \mu \\ | \\ \lambda \end{array} &= \sum_{\nu, \kappa} C_{\lambda, \mu}^{\nu, \kappa} \begin{array}{c} \nu \\ | \\ \kappa \end{array} \end{aligned}$$

The tensors L, R, C satisfy a dizzying number of coherence equations.




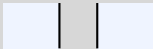


$\text{Vec}(\mathbb{Z}/2\mathbb{Z})\text{-Vec}(\mathbb{Z}/2\mathbb{Z})$ bimodules



$\mathbf{Vec}(\mathbb{Z}/2\mathbb{Z})\text{-Vec}(\mathbb{Z}/2\mathbb{Z})$ bimodules

F_0	$(0, 1), (1, 0)$	
X	$(1, 1)$	
F_1	$(0, 1), (1, 0)$	

The Correspondence

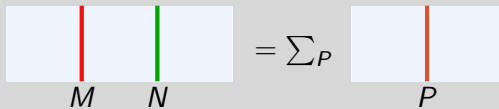
Bimodule	Domain Wall
T	
L	
R	
F_0	
X	
F_1	

Proof:

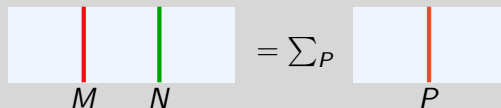
We can compute the following table in two ways:

$\otimes_{\mathbb{Z}/2\mathbb{Z}}$	T	L	R	F_0	X	F_1
T	$2 \cdot T$	T	$2 \cdot R$	R	T	R
L	$2 \cdot L$	L	$2 \cdot F_0$	F_0	L	F_0
R	T	$2 \cdot T$	R	$2 \cdot R$	R	T
F_0	L	$2 \cdot L$	F_0	$2 \cdot F_0$	F_0	L
X	T	L	R	F_0	X	F_1
F_1	L	T	F_0	R	F_1	X

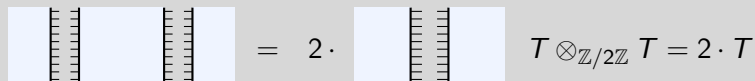
Method 1: Domain Walls



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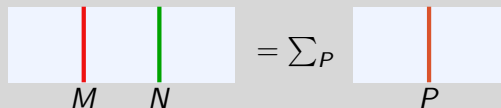


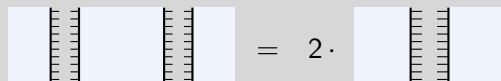
M N $= \sum_P$ P



$= 2 \cdot$ $T \otimes_{\mathbb{Z}/2\mathbb{Z}} T = 2 \cdot T$

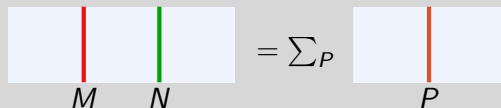
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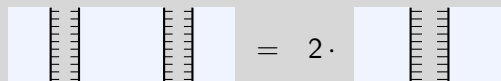

$$\begin{array}{|c|} \hline \\ \hline \end{array} = 2 \cdot \begin{array}{|c|} \hline \\ \hline \end{array} \quad T \otimes_{\mathbb{Z}/2\mathbb{Z}} T = 2 \cdot T$$


$$\begin{array}{|c|} \hline \\ \hline \end{array} = \begin{array}{|c|} \hline \\ \hline \end{array} \quad T \otimes_{\mathbb{Z}/2\mathbb{Z}} L = T$$

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M N $= \sum_P$ P



$= 2 \cdot$ $T \otimes_{\mathbb{Z}/2\mathbb{Z}} T = 2 \cdot T$



$=$ $T \otimes_{\mathbb{Z}/2\mathbb{Z}} L = T$



$=$ $T \otimes_{\mathbb{Z}/2\mathbb{Z}} F_1 = R$

Method 2: Bimodules

- ▶ Explicitly compute all the relative tensor products $M \otimes_{\mathbb{Z}/2\mathbb{Z}} N$

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- ▶ See *Domain walls in topological phases and the Brauer-Picard ring for $\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})$* , [arXiv:1806.01279](https://arxiv.org/abs/1806.01279) for details.

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Let Σ an orientable 2-dimensional surface possibly with boundary, **decorated with bimodule strings**.

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The Guiding Principle Again

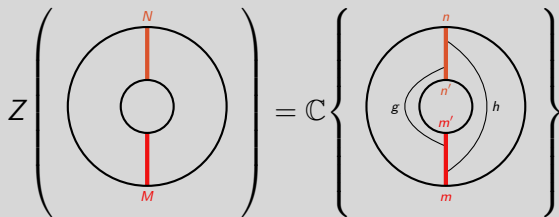
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In mathematics, Z is called a defect topological quantum field theory.

Binary Interface Defects

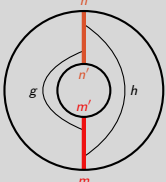
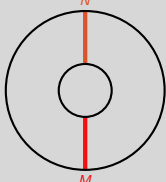


Binary Interface Defects

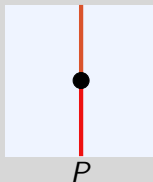
$$Z \left(\begin{array}{c} \text{---} N \\ \bigcirc \\ \bigcirc \\ \text{---} M \end{array} \right) = \mathbb{C} \left\{ \begin{array}{c} \text{---} n \\ \bigcirc \\ \bigcirc \text{---} n' \\ \bigcirc \text{---} m' \\ \text{---} m \end{array} \right\}$$

The right hand side is a category. Simple representations of this category parameterize **binary interface defects**.

Binary Interface Defects

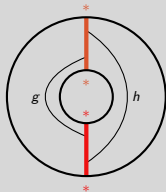
$$Z \left(\text{Diagram 1} \right) = \mathbb{C} \left\{ \text{Diagram 2} \right\}$$


The right hand side is a category. Simple representations of this category parameterize **binary interface defects**.



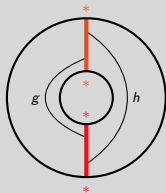
Example 1: $\frac{F_0}{F_0}$

Define $M_{g,h} =$



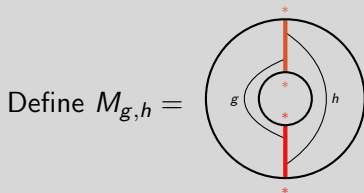
Example 1: $\left. \begin{matrix} F_0 \\ F_0 \end{matrix} \right|$

Define $M_{g,h} =$



Then $M_{g_0, h_0} M_{g_1, h_1} = M_{g_0 + g_1, h_0 + h_1}$, so the algebra is isomorphic to $\mathbb{C}\{\mathbb{Z}/2 \times \mathbb{Z}/2\}$. Therefore, there are 4 binary interface defects.

Example 1: $F_0|_{F_0}$

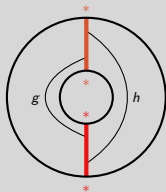


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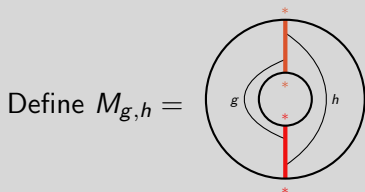
$$F_0|_{F_0}(x,y) =$$

Example 2: $\left. \begin{array}{l} F_0 \\ F_1 \end{array} \right|$

Define $M_{g,h} =$

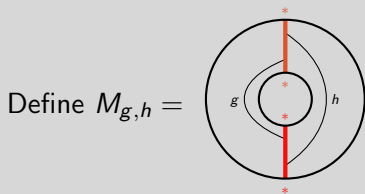


Example 2: $\left. \begin{matrix} F_0 \\ F_1 \end{matrix} \right|$



Then $M_{g_0, h_0} M_{g_1, h_1} = (-1)^{h_0 g_1} M_{g_0 + g_1, h_0 + h_1}$. This algebra is isomorphic to the 2×2 matrix algebra via $M_{g,h} \mapsto X^g Z^h$ (Pauli matrices). The 2×2 matrix algebra only has one simple representation.

Example 2: $\left. \begin{matrix} F_0 \\ F_1 \end{matrix} \right|$



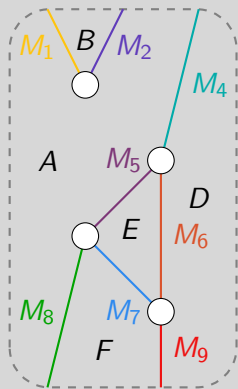
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$$\left. \begin{matrix} F_0 \\ F_1 \end{matrix} \right| = \begin{array}{c} \text{Y} \\ \vdots \end{array}$$

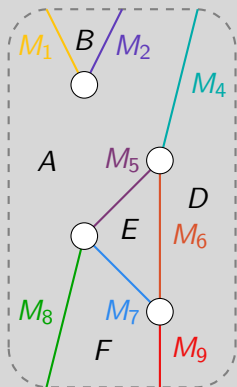
Fusing Binary Interface Defects

- ▶ In *Fusing binary interface defects in topological phases: The $\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})$ case*, [arXiv:1810.09469](https://arxiv.org/abs/1810.09469), we give physical interpretations for all binary interface defects in the $\mathbb{Z}/p\mathbb{Z}$ -model.
- ▶ We also compute all possible vertical and horizontal fusions involving binary interface defects.

Domain Wall Structures

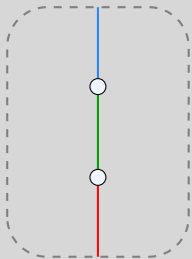


Domain Wall Structures



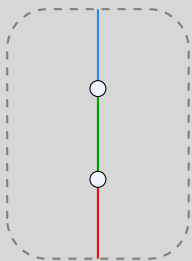
Domain Wall Structure algorithm: Assign defects to all the internal holes. The DWS algorithm computes the resulting compound defect.

Defect Fusion

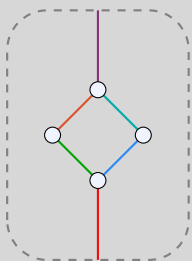


vertical defect fusion.

Defect Fusion



vertical defect fusion.



horizontal defect fusion.

Thank you for Listening!

See *Computing defects associated to bounded domain wall structures: The $\mathbb{Z}/p\mathbb{Z}$ case*, [arXiv:1901.08069](https://arxiv.org/abs/1901.08069) for details.