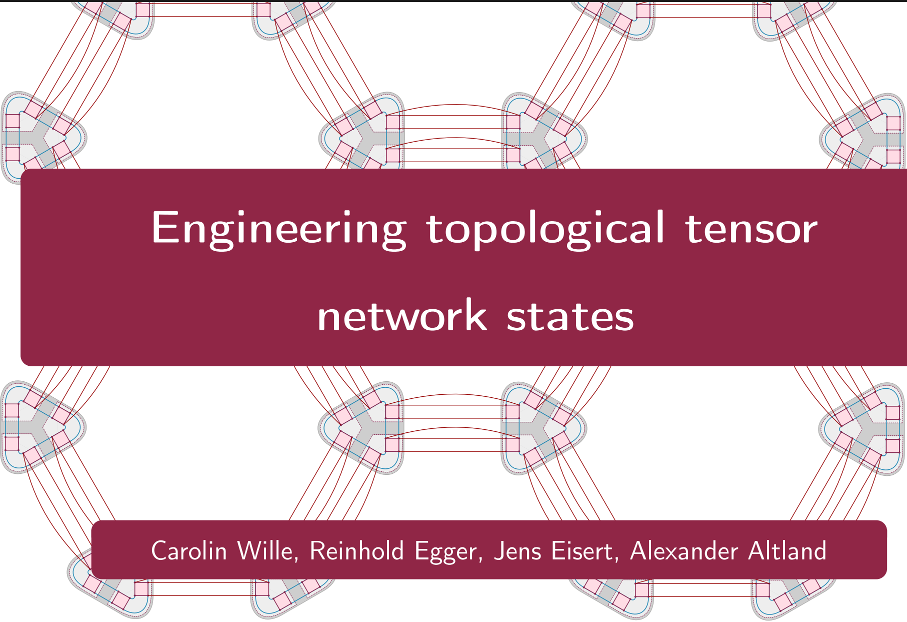


Engineering topological tensor network states

Carolin Wille, Reinhold Egger, Jens Eisert, Alexander Altland

Condensed matter

Quantum Information

Topological quantum order

- Topological quantum memories
- Topological quantum computation
- Tensor network phase classification

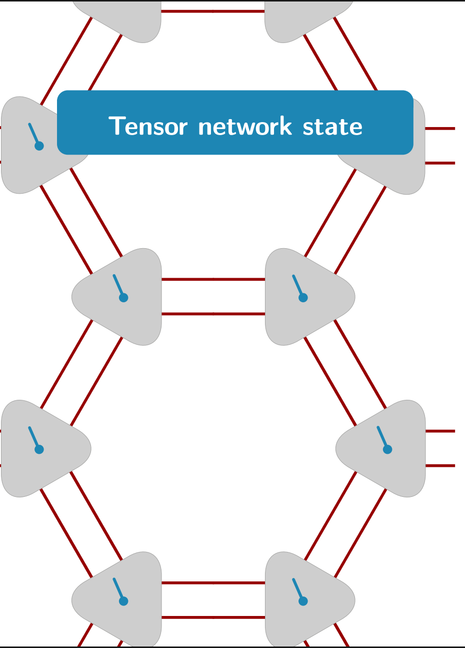
How can topological matter be synthesized?

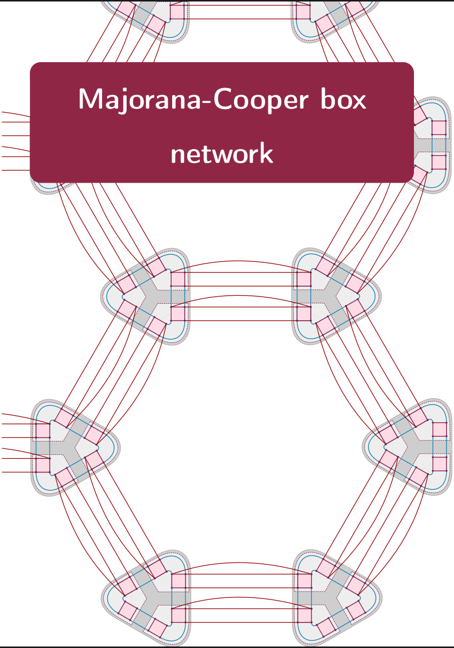
(topological ordered spin Hamiltonians in 2D)

Network of tunnel coupled Majorana Cooper boxes

+

Tensor networks and Hamiltonian gadgets


Tensor network state


Majorana-Cooper box network

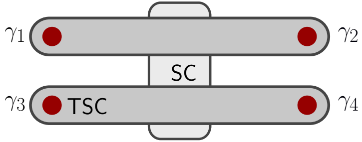
Outline

1) Effective spin Hamiltonians from Majorana Cooper box networks
2) Levin-Wen string net models
3) Hamiltonian gadgets
4) Tensor network ground states and perturbative parent Hamiltonians
5) Synthesis and Results

Majorana Cooper Box

Parity constraint

$$\gamma_1\gamma_2\gamma_3\gamma_4 = 1$$



Béri & Cooper, PRL (2012)

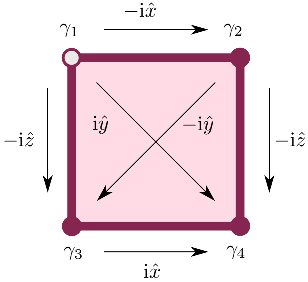
Altland & Egger, PRL (2013)

Effective qubit

$$\gamma_1\gamma_2 = -\mathrm{i}\hat{x}$$

$$\gamma_2\gamma_3 = -\mathrm{i}\hat{y}$$

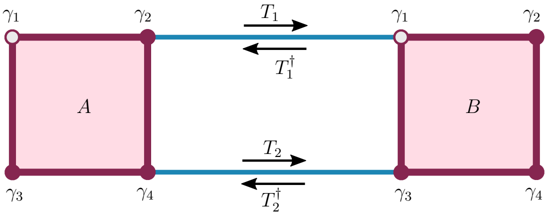
$$\gamma_1\gamma_3 = -\mathrm{i}\hat{z}$$



Tunnel coupled MCBs

Tunneling $\hat{T}_1 = \lambda\, \gamma_2 \gamma_1 e^{\mathrm{i}(\hat{\varphi}_A - \hat{\varphi}_B)}$

Charge transfer $e^{\mathrm{i}\hat{\varphi}_A} |N_A\rangle = |N_A + 1\rangle$

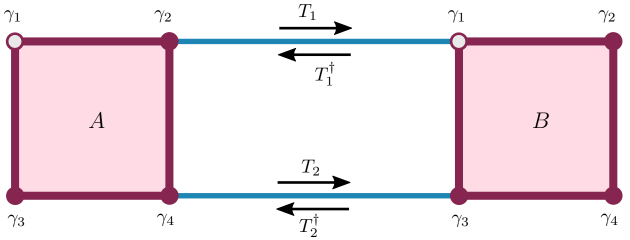


A. Zazunov, A. Levy Yeyati, R. Egger *Phys. Rev. B 84* 165440 (2011)

Low-energy effective theory

H_0: charge Hamiltonians, H_t: tunneling,

$$H = H_0 + H_t, \qquad E_c \gg |\lambda|$$

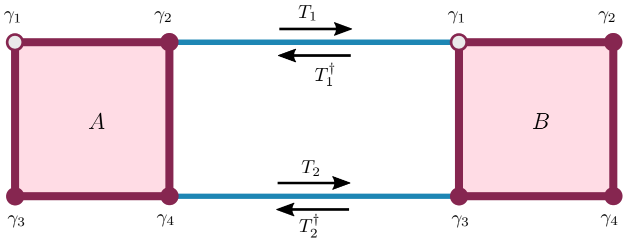


S. Bravyi, D. P. DiVincenzo, and D. Loss, *Ann. Phys.* 326, 2793 (2011)

Low-energy effective theory

P_0: charge free subspace

$$H_{\text{eff}} = \sum_{k=1}^{\infty} H^{(k)} \,, \quad H^{(k)} = P_0 \left(H_t \frac{1}{-H_0} \right)^{k-1} H_t P_0$$



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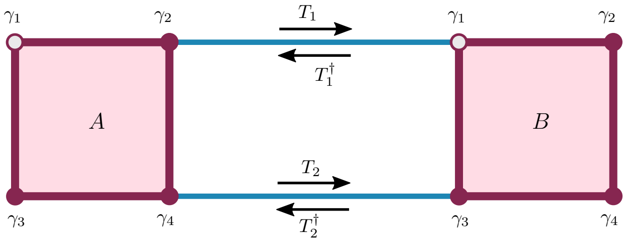
Example $k = 2$

$$O_{12,+} = P_0 T_2^\dagger \frac{1}{-2E_c} T_1 P_0$$



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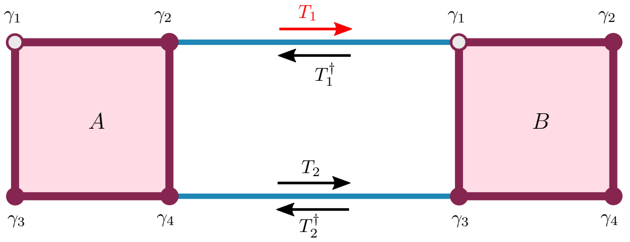

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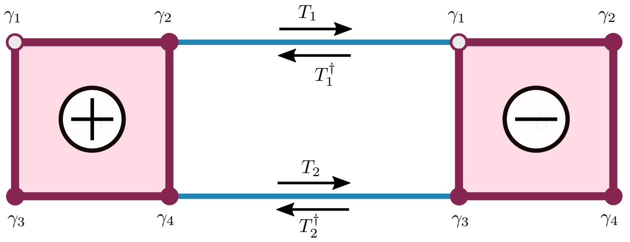

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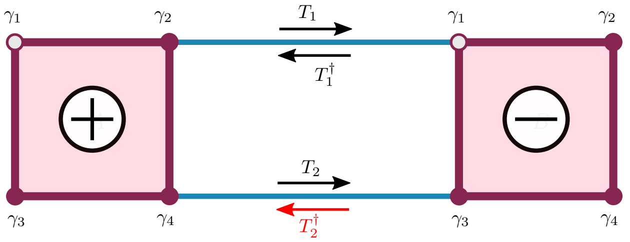

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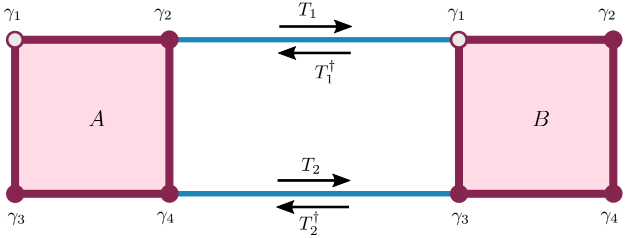

S. Bravyi, D. P. DiVincenzo, and D. Loss, *Ann. Phys.* 326, 2793 (2011)

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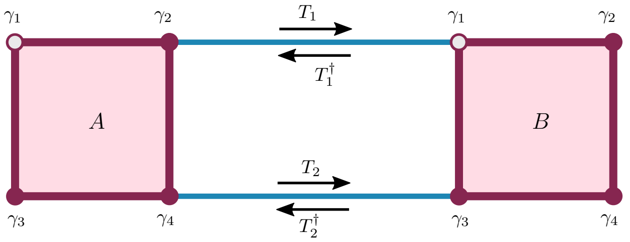
Example $k = 2$

$$O_{12,+} = -\frac{\lambda_1 \lambda_2^*}{2E_C} \gamma_2 \gamma_1 \gamma_3 \gamma_4$$



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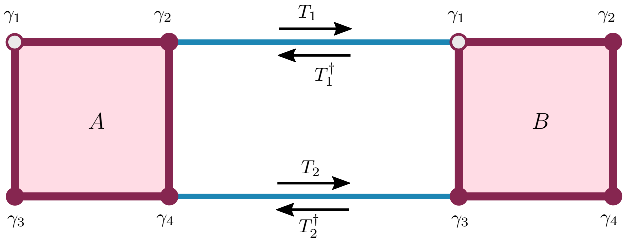
Example $k = 2$

$$O_{12,+} = \frac{\lambda_1 \lambda_2^*}{2E_C} \hat{z}_A \hat{z}_B$$



S. Bravyi, D. P. DiVincenzo, and D. Loss, *Ann. Phys.* 326, 2793 (2011)

Low energy effective theory

Only closed loops contribute to the effective theory.

$$\hat{H}_{\text{eff}} = \sum_{\text{l: closed loops}} \hat{Q}(l) \sum_{\text{d: loop directions}} \sum_{\text{s: sequences}} a_l(s, d)$$

- $\hat{Q}$: Pauli word
- $a_l(s, d) \in \mathbb{C}$
- $a_l(s, d) \propto \frac{1}{E_c}^{|l|}$

Short loops dominate the theory.

S. Bravyi, D. P. DiVincenzo, and D. Loss, *Ann. Phys.* 326, 2793 (2011)

Designing Hamiltonians from MCB networks

Designing $H_{\text{target}} = \hat{x}\hat{x} + \hat{z}\hat{z}$

$$H_{\text{eff}} = a\hat{z}\hat{z}$$

Designing Hamiltonians from MCB networks

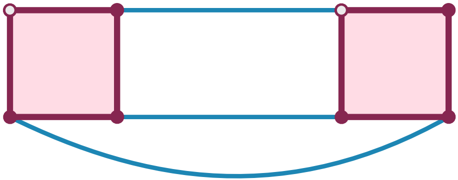
Designing $H_{\text{target}} = \hat{x}\hat{x} + \hat{z}\hat{z}$

$$H_{\text{eff}} = b\hat{x}\hat{x}$$

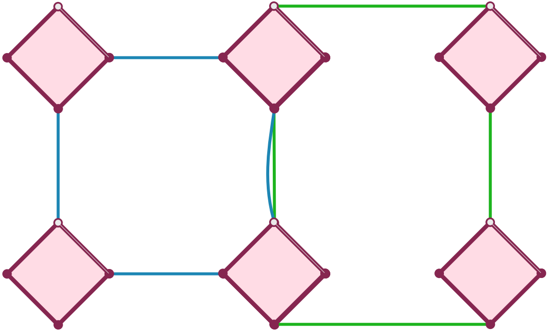
Designing Hamiltonians from MCB networks

Designing $H_{\text{target}} = \hat{x}\hat{x} + \hat{z}\hat{z}$

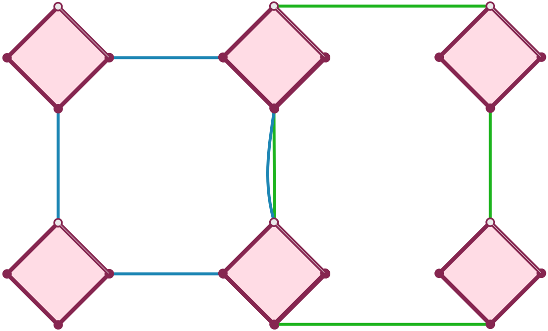
$$H_{\text{eff}} = a\hat{x}\hat{x} + b\hat{z}\hat{z} + c\hat{y}\hat{y}$$



Designing Hamiltonians from MCB networks



Designing Hamiltonians from MCB networks

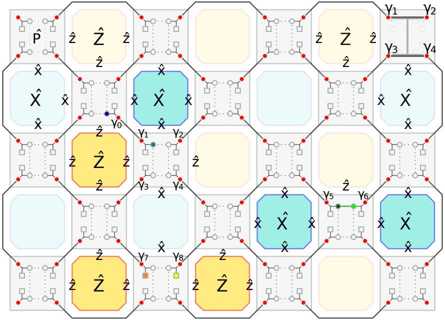


☺Product operators with minimal overlap.

☹Everything else.

Majorana toric code

$$H_{TC} = \sum_p \hat{x} \otimes \hat{x} \otimes \hat{x} \otimes \hat{x} + \sum_v \hat{z} \otimes \hat{z} \otimes \hat{z} \otimes \hat{z}$$



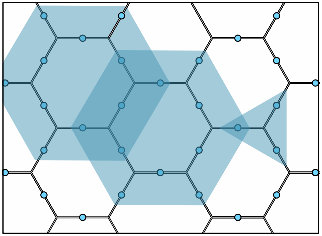
Terhal et al. PRL (2012), Plugge et al. PRB (2016)

Beyond the toric code?

Levin-Wen string-nets

$$\hat{H} = -\sum_{v} \hat{Q}_v - \sum_{p} \hat{Q}_p$$

- Non-chiral topological order
- Commuting projectors
- 12-local interactions
- non-product operators



M. Levin and X. G. Wen, *Phys. Rev. B* 71, 045110 (2005)

Levin-Wen string-nets

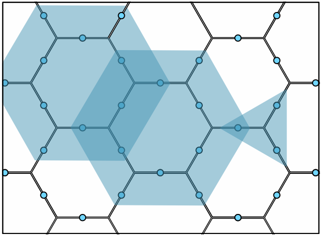
$$\hat{H} = -\sum_{v} \hat{Q}_v - \sum_{p} \hat{Q}_p$$

1) Increase toolbox

   (cancellation mechanisms)

2) Hamiltonian gadgets

3) Tensor networks



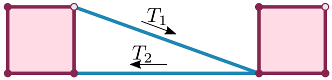
1) Toolbox

Cancellation mechanisms

Overlapping links and symmetries

$$\sum_{s:\text{ sequence}} a(d,s) = 0$$

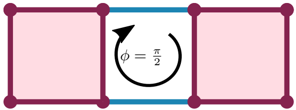
- Anticommuting hopping terms
- symmetry
- $a(\pi[s], d) = -a(s, d)$



Phase cancellation

$$\sum_{d:\text{ directions}} a(d,s) = 0$$

- $\bar{a}(d,s) + a(d,s) = 0$
- $a = |a|e^{i\phi}$
- Loop phase $\phi = \pm\pi/2$

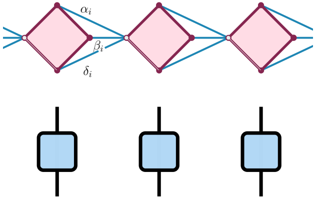


Building blocks

Product operators

$$\hat{O}_+ \propto \hat{q}_1 \hat{q}_2 \cdots \hat{q}_n$$

$$\hat{q}_i = \delta_i \hat{x}_i + \beta_i \hat{y}_i + \alpha_i \hat{z}_i$$

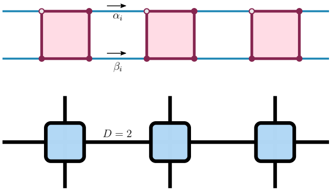


Matrix product operators $D = 2$

$$\hat{O}_+ \propto \mathrm{Tr}\left(\hat{A}^{(1)} \hat{A}^{(2)} \ldots \hat{A}^{(n)}\right)$$

$$\hat{A}^{(i)}_{00} = -\alpha_i \hat{x} \ , \quad \hat{A}^{(i)}_{01} = \beta_i \hat{y} \ ,$$

$$\hat{A}^{(i)}_{10} = \alpha_i \hat{y} \ , \quad \hat{A}^{(i)}_{11} = \beta_i \hat{x}$$

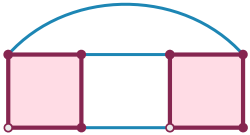


Building blocks

Bell pair 'projection'

$$\hat{H}_B \propto -(\hat{x}\hat{x} + \hat{z}\hat{z} - \hat{y}\hat{y})$$

$$|\Psi\rangle = |0,0\rangle + |1,1\rangle$$

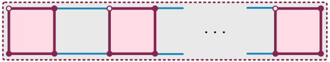


Repetition code qubit

Logical qubit

$|\bar{0}\rangle = |0,\ldots,0\rangle\,, |\bar{1}\rangle = |1,\ldots,1\rangle$

Logical operators $\hat{X}, \hat{Y}, \hat{Z}$



2) Hamiltonian gadgets

Hamiltonian gadgets – locality reduction

$H = H_0 + \epsilon V$ low locality $\rightarrow$ H_{eff} high locality

- 2-local Hamiltonian problem as hard as 3-local Hamiltonian

  J. Kempe, A. Kitaev, and O. Regev , SIAM J. Comp. 35, 1070 (2006)

- $H = H_0 + \epsilon V$: 2-local $\rightarrow$ H_{eff}: k-local

  S. P. Jordan and E. Farhi, Phys. Rev. A 77, 062329 (2008)

- Quantum doubles from 2-local Hamiltonians

  C. G. Brell, S. T. Flammia, S. D. Bartlett, and A. C. Doherty, New. J. Phys 13, 053039 (2011)

- Perturbative parent Hamiltonians

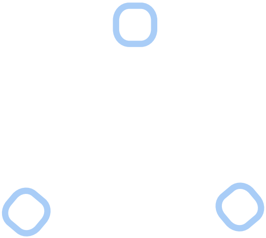
  C. G. Brell, S. D. Bartlett, and A. C. Doherty, New J. Phys. 16, 123056 (2014)

Jordan-Farhi gadgets

$$H_{\text{target}} = \sum h_i \qquad h_i = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$$

Jordan-Farhi gadgets

$$H_{\text{target}} = \sum h_i \qquad h_i = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$$



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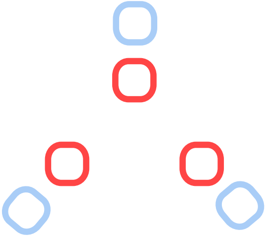
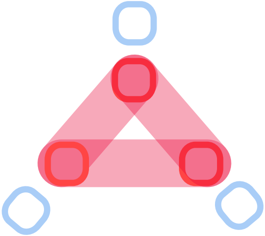
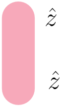


1. Add auxiliary qubits

Jordan-Farhi gadgets

$H_{\text{target}} = \sum h_i \qquad h_i = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$

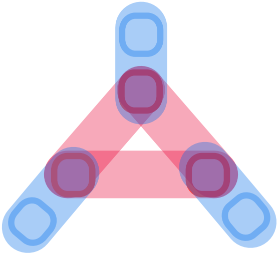


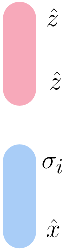


1. Add auxiliary qubits
2. 'polarize' aux. qubits

Jordan-Farhi gadgets

$H_{\text{target}} = \sum h_i$ $\qquad h_i = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$

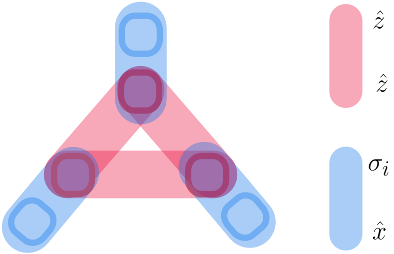




1. Add auxiliary qubits
2. 'polarize' aux. qubits
3. Add target-aux interaction

Jordan-Farhi gadgets

$H_{\text{target}} = \sum h_i$ $\qquad h_i = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$

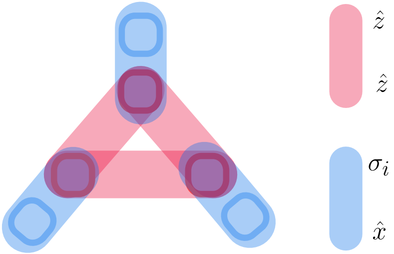


1. Add auxiliary qubits
2. 'polarize' aux. qubits
3. Add target-aux interaction

$$H = \sum \hat{z}_j \hat{z}_{j+1} + \epsilon \sum \sigma_j \hat{x}_j$$

Jordan-Farhi gadgets

$H_{\text{target}} = \sum h_i$ $h_i = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$

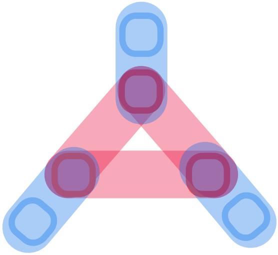


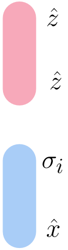
1 Add auxiliary qubits
2 'polarize' aux. qubits
3 Add target-aux interaction

$H = \sum \hat{z}_j \hat{z}_{j+1} + \epsilon \sum \sigma_j \hat{x}_j$ $H_{\text{eff}} \simeq P_0 \hat{x}\hat{x}\hat{x} P_0 \otimes \sigma_1 \sigma_2 \sigma_3$

Jordan-Farhi gadgets

$$H_{\text{target}} = \sum h_i \qquad h_i = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$$



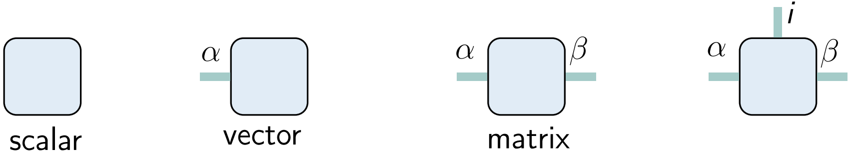


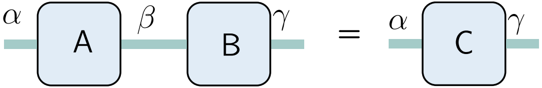
1. Add auxiliary qubits
2. 'polarize' aux. qubits
3. Add target-aux interaction
4. init aux $|+\rangle = |0\ldots0\rangle + |1\ldots1\rangle$

$$H = \sum \hat{z}_j \hat{z}_{j+1} + \epsilon \sum \sigma_j \hat{x}_j \qquad H_{\text{eff}} \simeq P_0 \hat{x}\hat{x}\hat{x} P_0 \otimes \sigma_1\sigma_2\sigma_3$$

3) Tensor network states and again gadgets

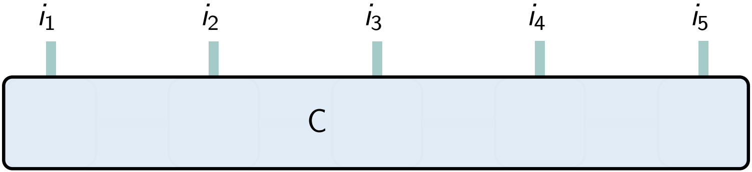
Penrose notation





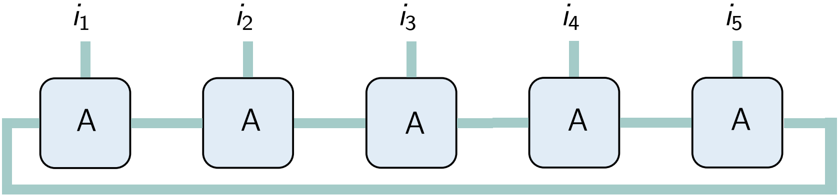
matrix multiplication $\sum_\beta A_{\alpha\beta} B_{\beta\gamma} = C_{\alpha\gamma}$

Matrix Product States – 1D



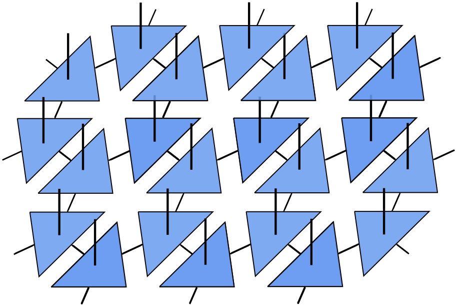
$$|\Psi\rangle = \sum_{i_1, i_2, i_4, i_4, i_5} C_{i_1 i_2 i_3 i_4 i_5} \, |i_1, i_2, i_3, i_4, i_5\rangle$$

Matrix Product States – 1D



$$|\Psi\rangle = \sum_{i_1,i_2,i_4,i_4,i_5} \mathrm{Tr}\left[A^{i_1} A^{i_2} \dots A^{i_5}\right] |i_1, i_2, i_3, i_4, i_5\rangle$$

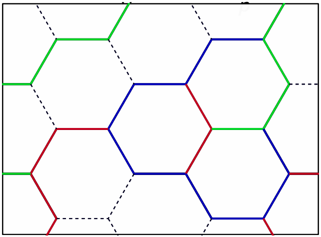
PEPS – 2D



$$|\Psi\rangle = \sum_{i_1,\ldots,i_N} \mathrm{ttr}\left[A^{i_1}\ldots A^{i_N}\right]|i_1,\ldots,i_N\rangle$$

Tensor network ground state of string-net models

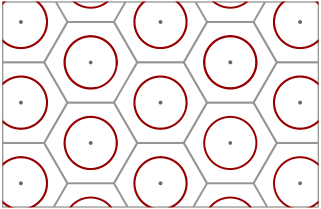
$$\hat{H} = -\sum_v \hat{Q}_v - \sum_p \hat{Q}_p$$



Tensor network ground state of string-net models

$$\hat{H} = -\sum_v \hat{Q}_v - \sum_p \hat{Q}_p$$

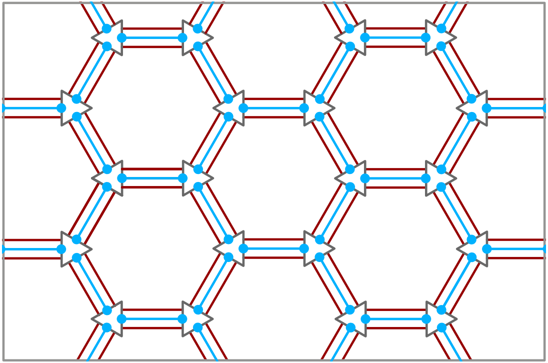
$$|GS\rangle = \prod_p \hat{Q}_p |0\rangle$$



Z.-C. Gu, M. Levin, B. Swingle, and X.-G. Wen, Phys. Rev. B 79, 085118 (2009)

O. Buerschaper, M. Aguado, and G. Vidal, *Phys. Rev.* B 79, 085119 (2009)

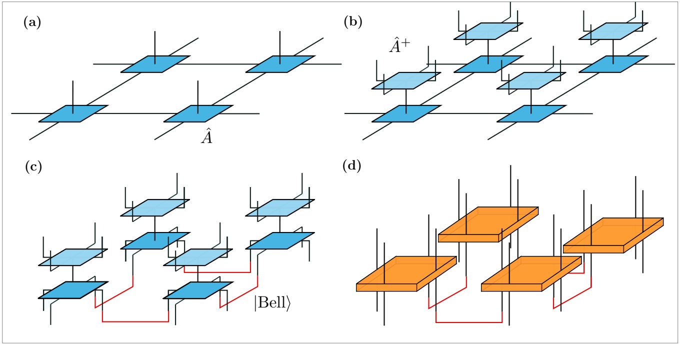
Tensor network ground state of string-net models

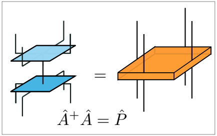


$$|GS\rangle = \sum_{i_1,j_1,k_1,\ldots} \sum_{s,t,u,\ldots} A_{stu}^{i_1 j_1 k_1} \ldots A_{pqr}^{i_n j_n k_n} |i_1, j_1, k_1, \ldots, i_n, j_n, k_n\rangle$$

Perturbative parent Hamiltonian

Virtual encoding $|\Psi'\rangle = A^\dagger \otimes \ldots \otimes A^\dagger |\Psi\rangle = U |\Psi\rangle$





C. G. Brell, S. D. Bartlett, A. C. Doherty, *New J. Phys.* 16, 123056 (2014)

Perturbative parent Hamiltonian

$$\hat{H} = \sum_{v} (1 - A^{\dagger} A) + \varepsilon \sum_{e} -\hat{P}_{\mathrm{Bell}}$$

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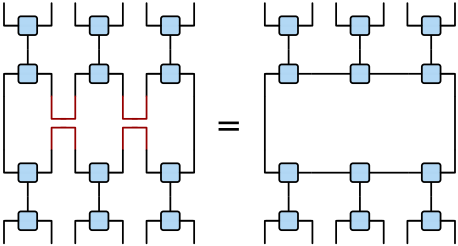
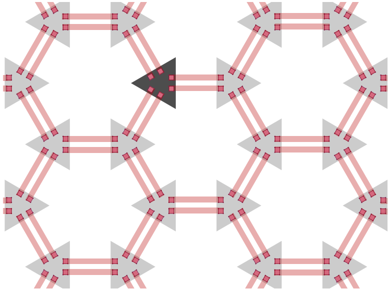


$P_0 = \prod A^\dagger A$ $\quad H^{(2)} \simeq P_0 P_{\mathrm{Bell}} \otimes P_{\mathrm{Bell}} P_0 = A'_3 A'^\dagger_3 = U A_3 A_3^\dagger U^\dagger$

Perturbative parent Hamiltonian

$$\hat{H} = \sum_v (1 - A^\dagger A) + \varepsilon \sum_e -\hat{P}_{\mathrm{Bell}}$$

- 0^{th} order: $\prod_{|\mathcal{R}|=1} A'_{\mathcal{R}} A'^{\dagger}_{\mathcal{R}}$



Perturbative parent Hamiltonian

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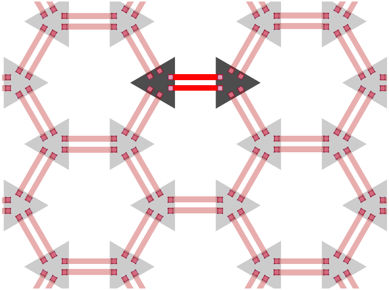
- 0^{th} order: $\prod_{|\mathcal{R}|=1} A'_{\mathcal{R}} A'^{\dagger}_{\mathcal{R}}$
- 2^{nd} order: $\prod_{|\mathcal{R}|=2} A'_{\mathcal{R}} A'^{\dagger}_{\mathcal{R}}$



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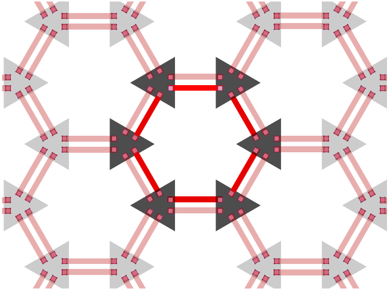
- 2^{nd} order: $\prod_{|\mathcal{R}|=2} A'_{\mathcal{R}} A'^{\dagger}_{\mathcal{R}}$
- 6^{th} order: $\prod_{|\mathcal{R}|=6} A'^{\dagger}_{\mathcal{R}} A'_{\mathcal{R}}$



Perturbative parent Hamiltonian

$$\hat{H} = \sum_v (1 - A^\dagger A) + \varepsilon \sum_e -\hat{P}_{\mathrm{Bell}}$$

- 0^{th} order: $\prod_{|\mathcal{R}|=1} A'_{\mathcal{R}} A'^{\dagger}_{\mathcal{R}}$
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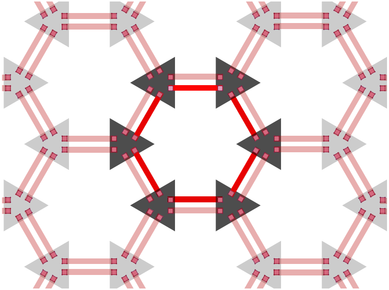
- 6^{th} order: $\prod_{|\mathcal{R}|=6} A'^{\dagger}_{\mathcal{R}} A'_{\mathcal{R}}$

$H_{\text{eff}} \simeq H' \simeq H$



MPO-isometric PEPS

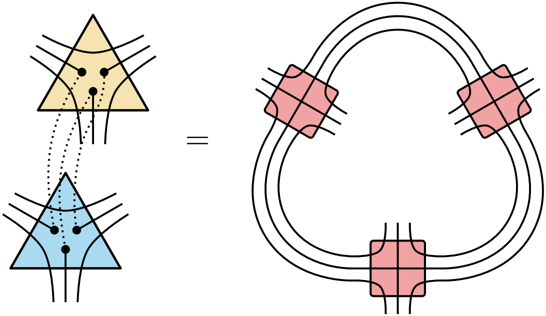
Topologically ordered PEPS

- controlled growth of entanglement
- $A^\dagger_{\mathcal{R}} A_{\mathcal{R}} = P_{|\partial\mathcal{R}|}$
- $P_{|\partial\mathcal{R}|}$: translation invariant Hermitian MPO projector

N. Bultinck, M. Marien, D. Williamson, M. B. Sahinoglu, J. Haegeman, and F. Verstraete
Ann. Phys. *378*, 183 (2017)

MPO-isometry of string-net ground states

$$A^{\dagger}A = P$$

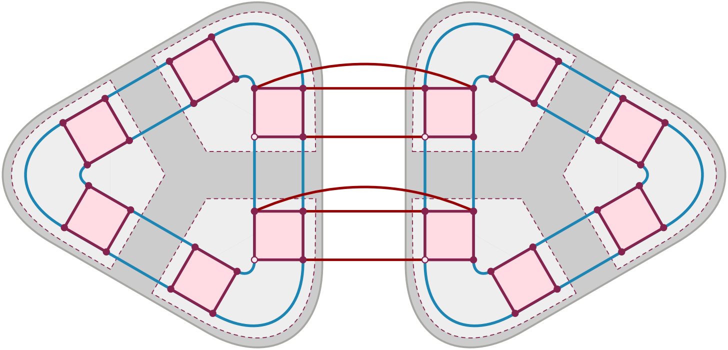


MPO-isometric PEPS

- $1 - A^\dagger A$: MPO projector
- Bond dimension $D \simeq$ complexity
  - $D = 1$ Toric code – Abelian
  - **D = 2 Double semion** – Abelian
  - $D = 5$ Double Fibonacci – non Abelian, universal

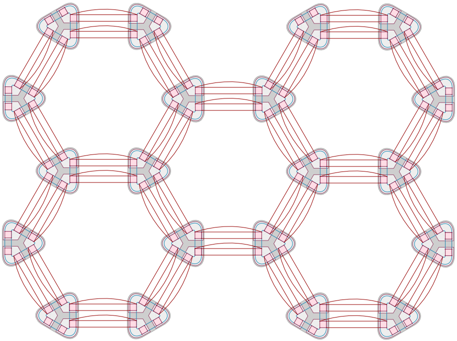
N. Bultinck, M. Marien, D. J. Williamson, M. B. Sahinoglu, J. Haegeman, F. Verstraete *Ann. Phys.* 378 (2017)

Double Semion Blueprint



$$H = \sum_v (1 - \hat{P}_{D=2}) + \varepsilon \sum_e (1 - \hat{P}_{\mathrm{Bell}})$$

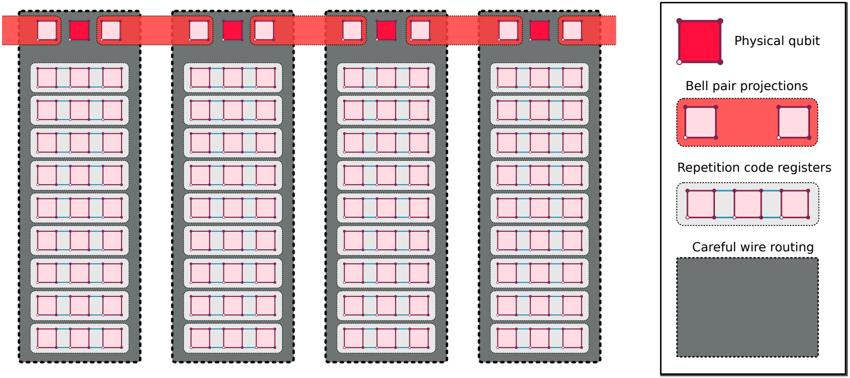
Double Semion Blueprint



$$H = \sum_v (1 - \hat{P}_{D=2}) + \varepsilon \sum_e (1 - \hat{P}_{\mathrm{Bell}})$$

Outlook – more general Hamiltonians?

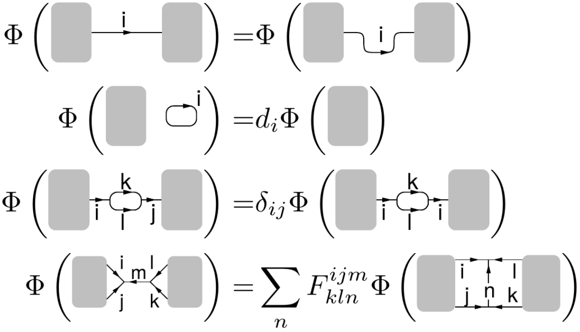
MPOs with $D > 2$: combining gadget ideas



Summary

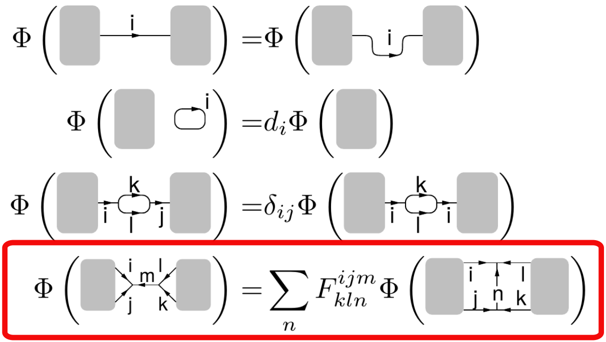
- Majorana Cooper box networks
  - Low energy theory
  - Cancellation mechanisms
  - Building block Hamiltonians
- Topological tensor networks
  - String-net ground states
  - Perturbative parent Hamiltonians
- Blueprint for the double semion model

String deformation rules (RG)



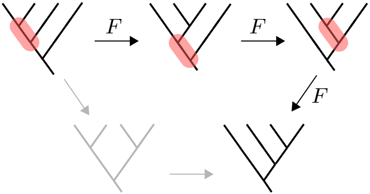
M. Levin and X. G. Wen, *Phys. Rev. B* 71, 045110 (2005)

String deformation rules (RG)

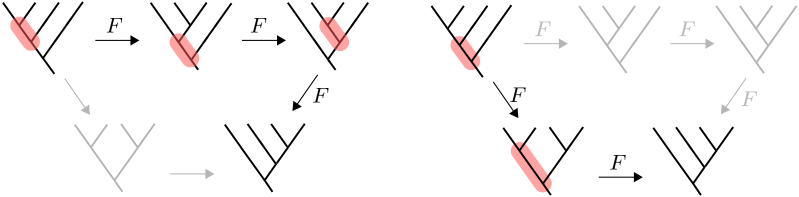


M. Levin and X. G. Wen, *Phys. Rev. B* 71, 045110 (2005)

Self-consistency condition

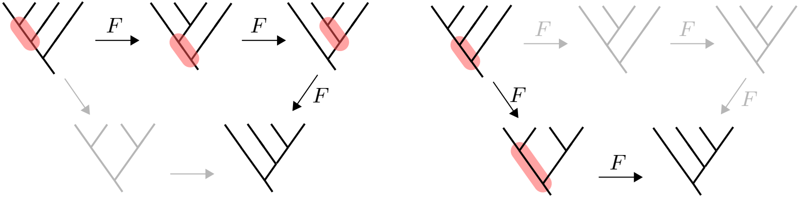


Self-consistency condition



Self-consistency condition

$$\Phi\left(\ \cdot\ \right) = \sum_n F^{ijm}_{kln} \Phi\left(\ \cdot\ \right)$$



Pentagon equation

$$\sum_{n=0}^{N} F^{mlq}_{kp^*n} F^{jip}_{mns^*} F^{js^*n}_{lkr^*} = F^{jip}_{q^*kr^*} F^{riq^*}_{mls^*}$$