Memo 2

Simulating realistic lightcurves for radio sources based on refractive scintillation

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July 21 2010

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1 Introduction

We are interested in a means of introducing the refractive intensity variations of scintillating quasars into the simulations of ASKAP data cubes. We adopt a simple thin screen formalism and make the simplifying assumption that the turbulence is isotropic. We start with a description of the autocovariance of the spatial intensity fluctuations across the two-dimensional observer's plane, and manipulate this expression until it is sufficiently simple to be implemented in a simulation, given a source size and flux density.

2 The power spectrum of refractive scintillation

For a thin screen geometry, the spatial autocovariance of the intensity fluctuations associated with a source undergoing refractive scintillation is

$$C_I(\mathbf{s}) = 8\pi r_e^2 \lambda^2 \int_{-\infty}^{\infty} d^2\mathbf{q} \left| V\left(\frac{\mathbf{q}z}{k}\right)\right|^2 \Phi_{N_e}(\mathbf{q}) \sin^2\left(\frac{q^2 z}{2k}\right) \exp\left[-D_\phi(qz/k) + i\mathbf{q}\cdot\mathbf{s}\right], \tag{1}$$

where $V(\mathbf{r})$ is the source visibility function, $r_e = 2.82 \times 10^{-15}$ m is the classical electron radius, $k = 2\pi/\lambda$ is the wavenumber, z is the distance of the scattering screen from Earth, and $\Phi_{N_e}(\mathbf{q})$ is the electron density power spectrum, which we model to be isotropic and follow a power law,

$$\Phi_{N_e}(q) = \mathrm{SM}\, q^{-\alpha-2} \exp\left[-\left(\frac{ql_0}{2}\right)^2\right], \tag{2}$$

where l_0 is the inner scale of the turbulence power spectrum and we choose $\alpha = 5/3$ corresponding to Kolmogorov turbulence. For this choice the phase structure function is

$$D_\phi(\mathbf{s}) = \frac{8\pi^2}{\alpha 2^\alpha} \mathrm{SM}\, r_e^2 \lambda^2 \frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)} s^\alpha. \tag{3}$$

Here, $\mathrm{SM} = \int dz C_N^2$ is the integrated scattering measure, and we have assumed that $r \gg l_0$. It is useful to recast the phase structure function in the form,

$$D_\phi(s) = \left(\frac{s}{s_0}\right)^\alpha, \tag{4}$$

$$s_0 = \left[\frac{\alpha 2^\alpha}{8\pi^2 r_e^2 \lambda^2 \mathrm{SM}} \frac{\Gamma(1+\alpha/2)}{\Gamma(1-\alpha/2)}\right]^{1/\alpha}. \tag{5}$$

Let us model the source to have a flux density I_0 and a gaussian brightness distribution,

$$I(\boldsymbol{\theta}) = I_0 \exp\left(-\frac{\theta^2}{2\theta_0^2}\right) \tag{6}$$

for which the corresponding visibility is

$$V(\mathbf{r}) = \int d^2\boldsymbol{\theta}\, I(\boldsymbol{\theta}) \exp\left[-ik\boldsymbol{\theta}\cdot\mathbf{r}\right] = 2\pi I_0\theta_0^2 \exp\left(-\frac{k^2r^2\theta_0^2}{2}\right) \equiv V_0 \exp\left(-\frac{k^2\theta_0^2}{2}r^2\right). \tag{7}$$

Putting this all together, we have

$$C_I(\mathbf{s}) = 8\pi r_e^2\lambda^2 V_0^2 \mathrm{SM} \int_{-\infty}^{\infty} d^2\mathbf{q}\, q^{-\alpha-2} \sin^2\left(\frac{q^2 z}{2k}\right) \exp\left[-\left(qz\theta_0\right)^2 - \left(\frac{qz}{2ks_0}\right)^{\alpha} - \left(\frac{ql_0}{2}\right)^2 + i\mathbf{q}\cdot\mathbf{s}\right]. \tag{8}$$

Now, in the regime of refractive scintillation, the third term inside the exponential is not relevant, and the second term inside the exponential cuts the integral off before the $\sin^2(q^2z/2k)$ term begins to oscillate. The intensity autocovariance thus simplifies to

$$C_I(\mathbf{s}) = 8\pi r_e^2\lambda^2 V_0^2 \mathrm{SM} \frac{z^2}{4k^2} \int_{-\infty}^{\infty} d^2\mathbf{q}\, q^{2-\alpha} \exp\left[-\left(qz\theta_0\right)^2 - \left(\frac{qz}{2ks_0}\right)^{\alpha} + i\mathbf{q}\cdot\mathbf{s}\right]. \tag{9}$$

Whether the first or second term cuts the integral off depends on the size of the source relative to the scintillation pattern. The first term inside the exponential will cut the integral off if the angular size of the source exceeds the angular scale of the refractive scintillation pattern, $r_{\mathrm{ref}}/z \equiv 1/ks_0$ (i.e. if $\theta_0 \gtrsim 1/ks_0$).

In the regime of refractive interstellar scintillation at 0.7-1.4 GHz, some sources will be sufficiently compact that the second term will dominate the cut-off of the power spectrum, while others will be large enough that their visibility function cuts the power spectrum off.

3 Temporal Fluctuations

In practice, one does not observe the power spectrum of intensity fluctuations across the observer's plane. One instead observes temporal fluctuations in the intensity due to movement of the scintillation pattern transverse to the line of sight. This scintillation velocity, $\mathbf{v}_{\mathrm{ISS}}$ typically has a speed 30-50 $\mathrm{km\,s^{-1}}$. We obtain the autocovariance of the temporal fluctuations by substituting $\mathbf{s} = \mathbf{v}_{\mathrm{ISS}}\tau$:

$$C_t(\tau) = 2\pi r_e^2\lambda^2 V_0^2 \mathrm{SM} \frac{z^2}{k^2} \int_{-\infty}^{\infty} d^2\mathbf{q}\, q^{2-\alpha} \exp\left[-\left(qz\theta_0\right)^2 - \left(\frac{qz}{2kr_0}\right)^{\alpha} + i\tau\mathbf{q}\cdot\mathbf{v}_{\mathrm{ISS}}\right]. \tag{10}$$

We make simplifying assumptions about the integral depending on whether the scintillation power spectrum is first cut off by the source or by the refractive scintillation filter. In the case where the source cuts off the power spectrum, $\theta_0 > 1/ks_0$, we perform the integral for $\alpha = 5/3$, and find

$$C_t(\tau) = 2\pi^2 r_e^2\lambda^2 V_0 \mathrm{SM}\, \Gamma\left(\frac{7}{6}\right) \frac{z^2}{k^2} \frac{L_{-7/6}\left(-\frac{v_{\mathrm{ISS}}^2\tau^2}{4z^2\theta_0^2}\right)}{(z^2\theta_0^2)^{7/6}}, \tag{11}$$

where $L_n(x)$ is the nth Laguerre polynomial.

An exact expression for the autocovariance is more difficult to obtain when the second term in eq.(10) cuts the integral off. However, if we make replace α by 2 where it appears inside the exponential *only*, then we obtain the approximate expression

$$C_t(\tau) \approx 2^{10/3}\pi^2 r_e^2\lambda^2 V_0 \mathrm{SM}\, \Gamma\left(\frac{7}{6}\right) r_0^2 \left(\frac{k^2r_0^2}{z^2}\right)^{1/6} L_{-7/6}\left(-\frac{k^2 s_0^2 v_{\mathrm{ISS}}^2\tau^2}{z^2}\right). \tag{12}$$

parameter	value
$v_{\rm ISS}$	$3.0 \times 10^4\,{\rm m\,s^{-1}}$
SM	$9.8 \times 10^{15}\,{\rm m^{-17/3}}$
z	$1.2 \times 10^{19}\,{\rm m}$
s_0	$6.4 \times 10^7\,{\rm m}$ (for $\lambda = 0.21\,{\rm m}$)

Table 1: A set of fiducial numbers useful for similations of refractive interstellar scintillation at Galactic mid-latitudes.

4 Fiducial numbers for simulations

Typical screen distances for scattering of extragalactic sources observed off the Galactic plane are in the range $100 - 1000\,{\rm pc}$, and depend on the properties of the individual line of sight. A useful number to adopt for the purposes of a simulation is $z = 400\,{\rm pc} = 1.236 \times 10^{19}\,{\rm m}$.

The only remaining number relates to the scattering measure, SM, which is both a coefficient in the autocovariance amplitude and is used in determining the diffractive scale, s_0 using eq.(5). This also depends on the line of sight through the ISM. It is usually quoted in the units of kpc ${\rm m}^{-20/3}$, but for the purposes of using eqs.(11)-(12), it is more convient to convert it into ${\rm m}^{-17/3}$. (Note that 1 kpc ${\rm m}^{-20/3} = 3.09 \times 10^{19}\,{\rm m}^{-17/3}$.) Typical values are in the range $10^{-5} - 10^{-2}\,{\rm kpc\,m^{-20/3}}$. For a specific line of sight, the Cordes & Lazio NE2001 model can be used to predict the scattering measure: see `http://rsd-www.nrl.navy.mil/7213/lazio/ne_model/`. A fiducial number to use for scattering at Galactic mid-latitude is ${\rm SM} = 10^{-3.5}\,{\rm kpc\,m^{-20/3}} = 9.78 \times 10^{15}\,{\rm m}^{-17/3}$.

A useful check here is that given the above assumptions, one should always obtain $C_I(\tau)/V_0^2 < 1$.

Table (4) summarises a set of fiducial numbers useful for the scintillations.

5 Implementation details

5.1 Evaluation of $L_{-7/6}$

The Laguerre polynomial $L_{-7/6}$ can be evaluated by means of a series expansion

$$L_{-7/6}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^m \Gamma(7/6+m)}{\Gamma(7/6)[\Gamma(1+m)]^2} \tag{13}$$

where $\Gamma()$ is the gamma function. The series converges relatively slowly, especially for $x >\sim 10$. Fig 1 shows the form of $L_{-7/6}$. The function undershoots zero, then slowly comes back to zero.

In practise, the gamma function grows very large and exceeds the capability of a double to store it for $x >\sim 30$, hence it is necessary to use the asymptotic behaviour of the function for large x. The first few terms of the series expansion for large x are given by

$$L_{-7/6}(x) = \frac{x^{-7/6} + \frac{49}{36}x^{-13/6} + \frac{8281}{2592}x^{-19/6}}{\Gamma(-1/6)} \tag{14}$$

At $x = 20$, equations (13) and (14) agree to 3 significant figures, so (14) should be used for $x \geq 20$.

5.2 Making a lightcurve

Equations (11) and (12) provide the theoretical autocovariance of a source's lightcurve. The Fourier transform of this function is therefore the power spectrum. Since the autocovariance is sensitive only the variations of a source, the true source flux density is $I(t) = V_0(1 + C * \delta(t))$ where V_0 is the source mean flux density as above, C is the variability scaling factor given by all the constants outside $L_{-7/6}$ in equations (11) and (12) (except V_0,

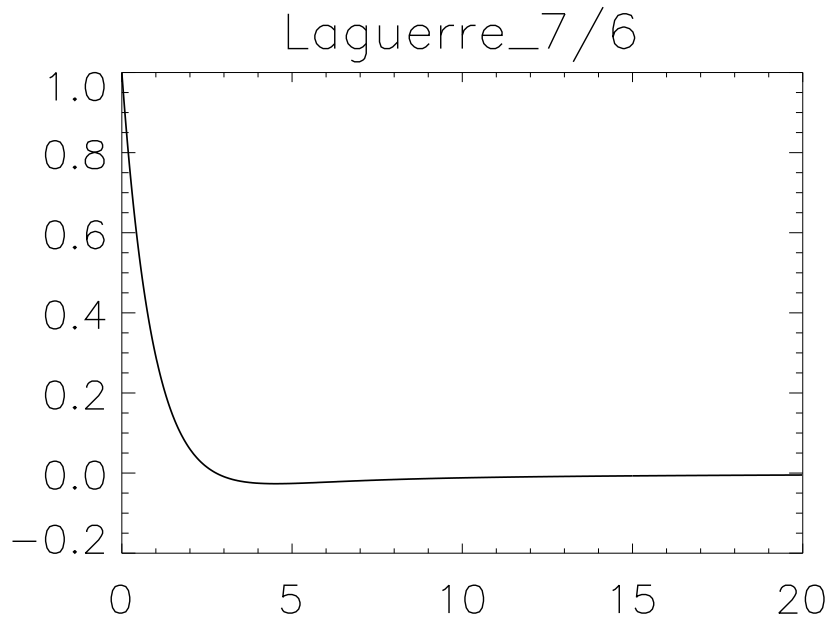


Figure 1: The form of $L_{-7/6}$

of course) and $\delta(t)$ is the random temporal variability due to refractive scattering with unity standard deviation. Hence, to make a random lightcurve we must:

- calculate the square root of the Fourier transform of the autocovariance. Note that the autocovariance is real and symmetric around zero lag, so will have a purely real Fourier transform.
- assign random phase terms to each Fourier component.
- Fourier transform back into time space. This time series is the unscaled variable component $\delta(t)$.

Note that the only variable inside $L_{-7/6}$ in eqs (11) and (12) is τ^2, where τ is in seconds, hence the actual timescale of variability is determined by the various constants listed in Table (4).

Numerical calculation of the lightcurves will provide the most rapid variability when there are many non-zero Fourier components, which in turn means the autocovariance should be narrow in lag space. The number of sample points and timescale should be chosen accordingly.

5.3 Example

The following IDL code calculates a lightcurve given some fiducial numbers that are given in this document.

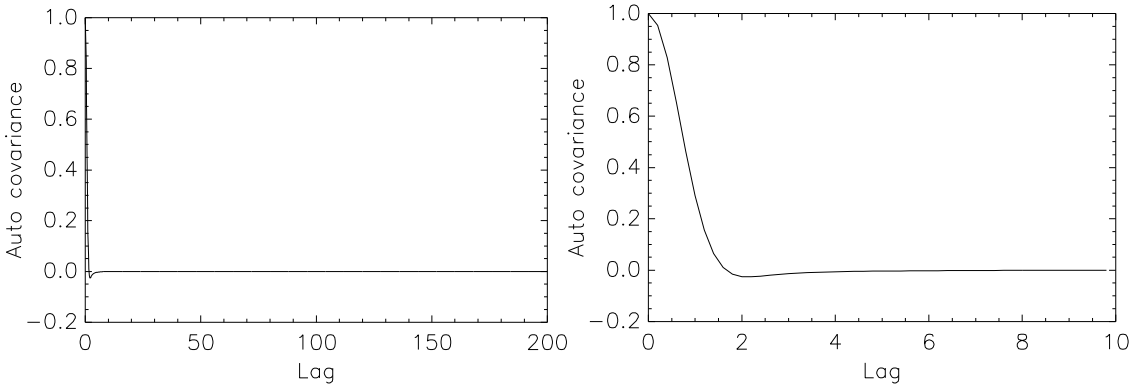
```
.comp laguerre_7on6.pro
viss = 3e4 ; m/s
sm   = 9.8e15   ; m^-17/3
z    = 1.2e19   ; m
s0   = 6.4e7    ; m
lambda = 0.21   ; m
r_e    = 2.82e-15 ; m classical radius of electrron
theta0 = 0.001/3600*!pi/180.0
; amplitude scale factor of variability, not including source intrinsic flux density
```

```
scale = 2.0*(!pi)^2*r_e^2*lambda^2*sm*gamma(7./6)*z^2/(2*!pi/lambda)^2/(z^2*theta0^2)^(7./6)

n_times=1000

; make some time points (no actual dimensions at this point)
x = dindgen(n_times)/n_times*200.
; calc autocovariance - half of what should be a symmetric array around lag = 0
y=laguerre_7on6(x^2)

; create array twice the size and mirror autocovariance in top half of array
y2 = dblarr(n_times*2-1)
y2[0:n_times-1]=y
temp = reverse(y)
y2[n_times:*] = temp[0:n_times-2]

; create some random phases
phases = randomu(seed,n_times)*2*!pi
cphases = complex(cos(phases),sin(phases))

; take sqrt of Fourier transform of autocovariance
fy = fft(y2)
abs_fy = sqrt(abs(fy))

; apply phases including complex conjugates in top half of array
new_y = complexarr(n_times*2-1)
new_y[0:n_times-1] = abs_fy[0:n_times-1]*cphases
temp = reverse(conj(cphases))
new_y[n_times:*] = abs_fy[n_times:*]*temp[0:n_times-2]
;new_y[0] = 0.0

; gets the scaling right with fft by using /inverse
; the inverse fft will be purely real, use float() to make the type real, not complex
t = (1.0 + scale*float(fft(new_y,/inverse)))
```

The autocovariance function generated by this code is shown in Fig 2.

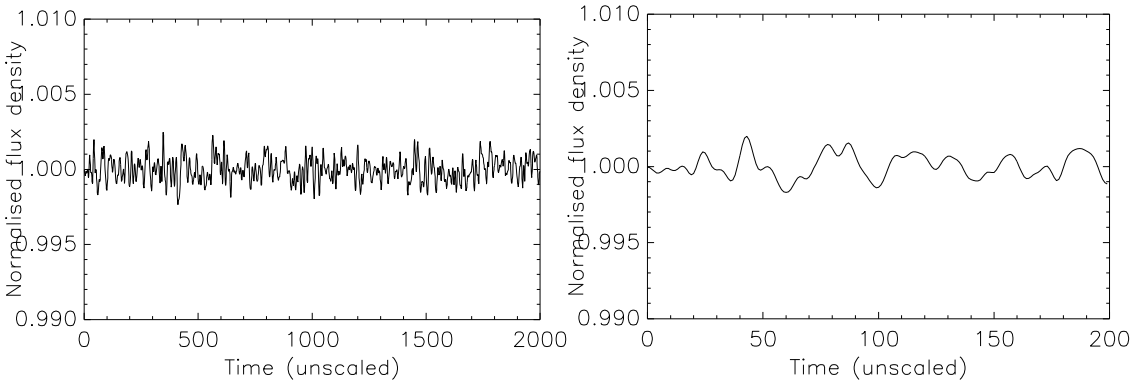
The sample lightcurve, normalised to flux density 1, is shown in Fig 3.

As a verification, we can calculate the autocorrelation of the model lightcurve to compare against the model autocovariance. This is shown in Fig 4.



(a) Full plot (b) First 10 lags

Figure 2: Model autocovariance from example IDL code



(a) Full lightcurve (b) First 200 samples

Figure 3: Model lightcurve from example IDL code

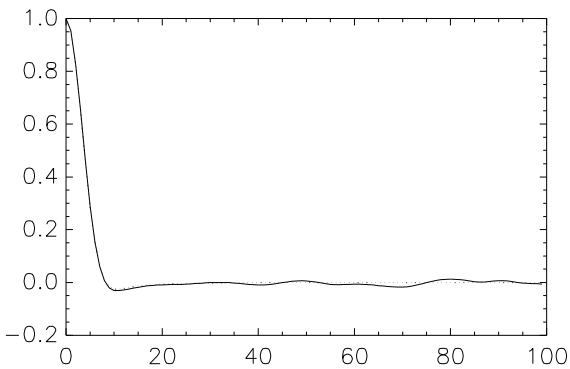


Figure 4: Autocorrelation of the model lightcurve generated by the IDL code (solid) overlaid with the theoretical covariance from the model code (dashed)