

FE7

OSCILLATIONS

OBJECTIVES

Aims

By studying this chapter you can expect to understand the nature and causes of oscillations. You will need to learn a fair number of new terms, but some care and effort in doing that will be well rewarded later because the ideas and principles introduced here can be used to understand a wide range of natural phenomena. In particular, the concepts and language of oscillations form the basis for understanding waves and optics, which come later in this course.

Minimum learning goals

When you have finished studying this chapter you should be able to do all of the following.

1. Explain, interpret and use the terms *oscillation, simple harmonic motion [SHM], amplitude, period, frequency, angular frequency, phase, initial phase, free oscillation, damped oscillation, forced oscillation, natural frequency, restoring force, force constant [spring constant], driving force, damping force, transient motion, steady state oscillation, resonance, resonance peak, resonant frequency.*
2. Explain free, damped and forced oscillations in terms of forces and energy transfers.
3. State and apply the equations and formulae of one-dimensional SHM for
 - (a) displacement, velocity and acceleration in terms of time;
 - (b) restoring force and acceleration in terms of displacement;
 - (c) frequency and angular frequency in terms of force constant and mass for an object on a spring.
4. Describe the important properties of SHM and explain how SHMs can be combined to give more general oscillations.
5. Describe and explain the phenomenon of resonance and recognise examples of resonance.

PRE-LECTURE

7-1 SIMPLE HARMONIC MOTION (SHM) Oscillations (periodic to and fro motions) of objects occur frequently both in nature and technology. For example, sound waves, water waves, elastic waves in stretched strings etc. are made up of large numbers of objects oscillating in a coordinated manner. This chapter will describe the various types of oscillations in terms of the forces and energy transfers necessary for their existence.

Consider a particle moving in one dimension so that its displacement from a fixed point at time t is given by

$$x = A \cos(\omega t)$$

where A and ω are constants. This equation describes a simple harmonic motion. The following questions are designed to lead you through some important characteristics of SHM.

- Q7.1** i) Where is the particle at $t = 0$?
- ii) Sketch a displacement-time graph of the motion. Note that the graph would look the same if you shifted it either to the right or to the left by a time interval $T = \frac{2\pi}{\omega}$.
- iii) What is the largest value of $|x|$?
At what times is the distance from the centre, i.e. $|x|$, largest?
- iv) Verify that the x-component of velocity, $v_x = \frac{dx}{dt}$, at time t is equal to $-A\omega \sin(\omega t)$.
- v) Hence verify that the speed is smallest when $|x|$ is largest and largest when $|x|$ is smallest. (Do these results agree with those obtained directly from the slopes of the displacement-time graph?)
- vi) Verify that the acceleration component, $a_x = \frac{dv_x}{dt}$, at time t is $-A\omega^2 \cos(\omega t)$.

- The acceleration is proportional to the displacement x and is always directed towards the centre, $x = 0$. The proportionality constant is $-\omega^2$; that is, the acceleration component is given by:

$$a_x = -\omega^2 x \quad \dots(7.1)$$

- vii) Show that the displacement, velocity and acceleration will all have the same values again after a further time $T = \frac{2\pi}{\omega}$.

- The motion is cyclic, continually repeating itself.

As you will have seen in (i), at time $t = 0$, the displacement is A ; the particle is at one of the extreme points of the oscillation. It is often useful to start the description of the oscillation at some other part of the cycle. Mathematically this corresponds to writing

$$\begin{aligned} x &= A \cos(\omega t + \phi) \quad \dots (7.2) \\ &= A \cos\left[\omega\left(t + \frac{\phi}{\omega}\right)\right] \end{aligned}$$

or shifting the displacement-time graph a distance $\Delta t = \phi/\omega$ to the left. Since the velocity component at any time t is just the slope of the displacement-time graph, and the acceleration component is the slope of the velocity-time graph, the velocity-time and acceleration-time graphs are also shifted to the left by the same amount, ϕ/ω . The relationships between these three quantities, x , v_x , and a_x are unaltered by the shift. (The same initial phase ϕ appears in each.)

Equation 7.2 is the most general expression for the displacement of a particle undergoing simple harmonic motion. As the name suggests, it is the simplest form of oscillatory motion; its importance lies in the fact that more complex oscillations can be studied in terms of it.

Terminology

Symbol	Name	SI unit
A	amplitude	m
ω	angular frequency	s^{-1} (or $\text{rad}\cdot\text{s}^{-1}$)
$\omega t + \phi$	phase	no unit (or rad)
ϕ	initial phase (i.e. phase at $t = 0$)	no unit (or rad)
T	period	s
f	frequency	Hz

Frequency, angular frequency and period for any kind of oscillation are related as follows:

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \dots (7.3).$$

The SI unit for frequency is the **hertz** (Hz); 1 Hz = 1 cycle per second. Phase is a dimensionless quantity, so it needs no unit. However, since it is often helpful to visualise phases as angles, they are often given the unit radian. Note that there are 2π radians in each cycle, so a frequency of 1 Hz corresponds to an angular frequency of $2\pi \text{ s}^{-1}$ or $2\pi \text{ rad.s}^{-1}$.

Two oscillations of the same frequency are said to be **in phase** at any given time if they have the same phase at that time. Otherwise they are **out of phase** by the difference of their phases ($\phi_2 - \phi_1$).

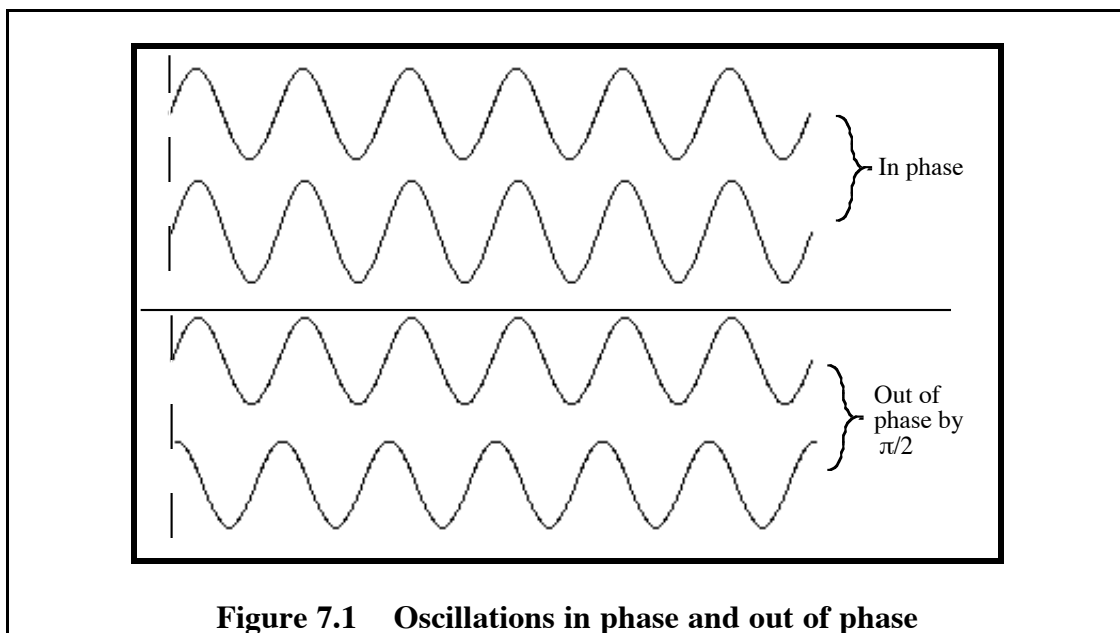


Figure 7.1 Oscillations in phase and out of phase

LECTURE

7-2 FREE OSCILLATIONS

If the oscillating system is isolated (i.e. if no energy is being added to or taken away from the system) the oscillations are called **free oscillations**. Oscillations (free or otherwise) can occur whenever a restoring force capable of transforming potential energy (PE) to kinetic energy (KE) and vice versa is present. In a free oscillation, since the sum of the PE and KE cannot increase, the PE must be largest at the extreme points of the oscillation where the KE is zero.

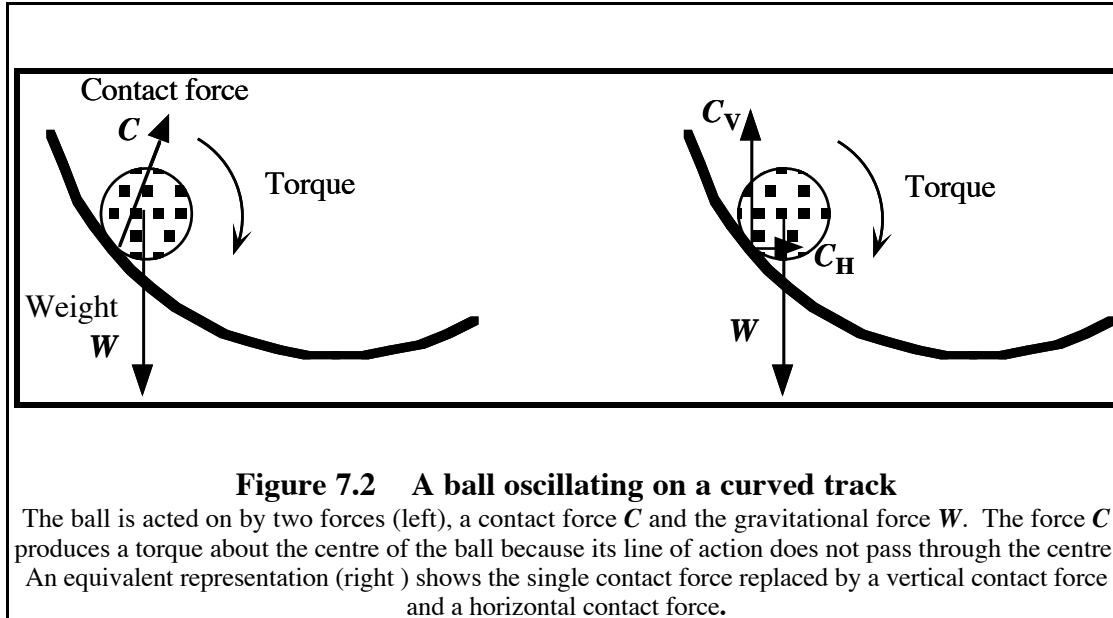
Examples

- Liquid sloshing mode - the restoring forces are due to gravity.
- A vibrating metal plate - elastic restoring forces.
- Stretched string - the restoring force is provided by tension in the string.

In each of these three examples all the oscillating particles together formed a **standing wave** pattern.

Example of a single oscillating object

A ball rolls in a curved track. When viewed from above, the ball is executing a one-dimensional oscillation. The restoring force for the horizontal oscillation is the horizontal component of the force exerted on the ball by the track. (The ball also oscillates vertically.)



Since the track can have any shape (provided that every point in the middle is lower than the ends) there are infinitely many possible restoring force functions and therefore there are infinitely many different one-dimensional oscillations.

SHM - the simplest free oscillation

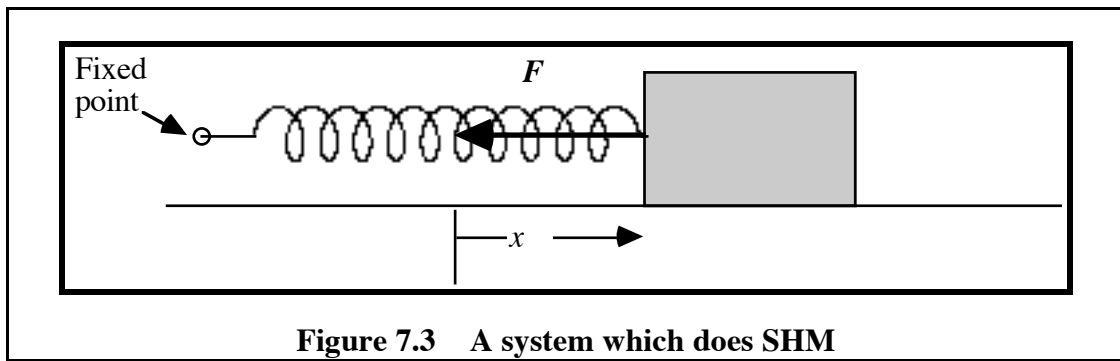
Any oscillation can be represented as a combination of SHMs for the following reasons.

- Any one-dimensional oscillation (free or otherwise) can be represented mathematically as a combination of SHMs with appropriate frequencies, amplitudes and initial phases. An example is the synthesis of the displacement-time graph of a sawtooth oscillation from SHMs.
- Any two- or three-dimensional oscillation can be treated as two or three one-dimensional oscillations at right angles to one another, since the motions in those directions are independent. (Some examples, Lissajous figures in two dimensions, are shown in §7-7, figure 7.10.)

When an object is undergoing SHM, its acceleration and the total restoring force at any time are both proportional to the displacement from the centre and they are both directed towards the centre.

Example: Body on a spring

Consider a body on the end of a spring, moving on a frictionless surface.



For an ideal spring, the x -component F of the **restoring force** is equal to $-kx$, where k is a constant; i.e. F is proportional to displacement and is directed towards the equilibrium point where $x = 0$.

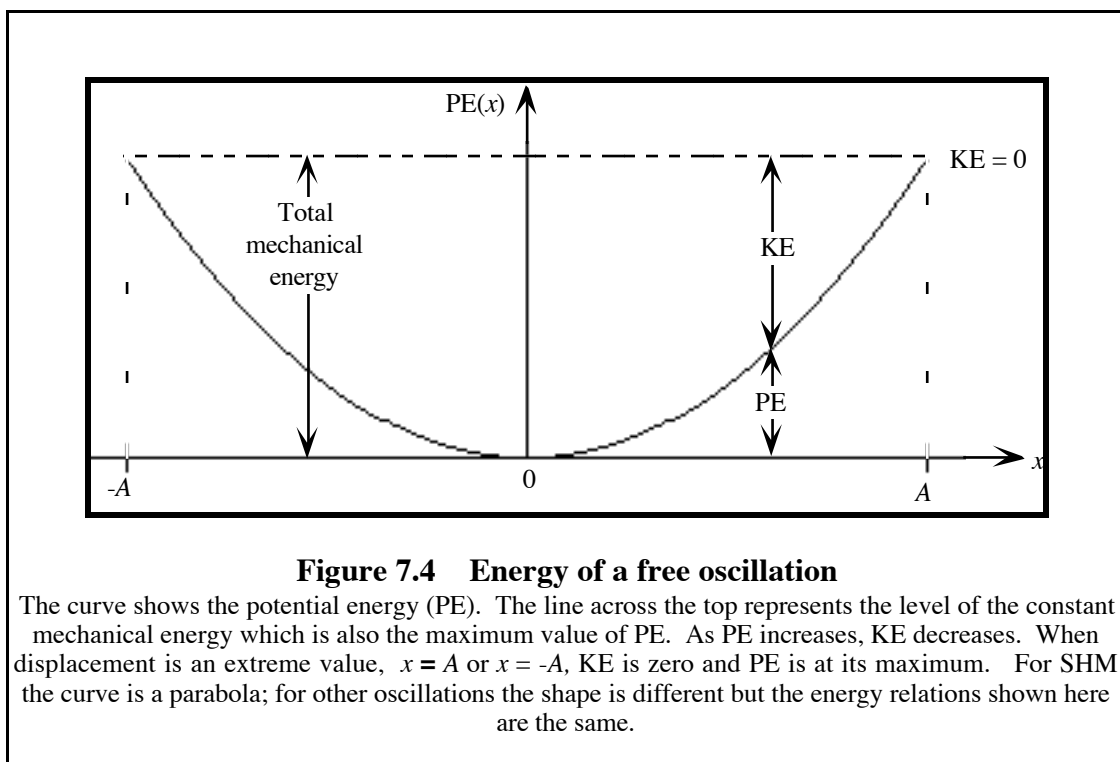
The equation of motion is $ma_x = -kx$ or $a_x = -\frac{k}{m}x$.

Comparison with equation 7.1 gives the **natural frequency** of oscillation:

$$\omega_N = \sqrt{\frac{k}{m}} \quad \text{or} \quad f_N = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \dots (7.4).$$

Energy

In a free oscillation, since the only force doing work is conservative; the total mechanical energy (KE + PE) of the system is a constant.



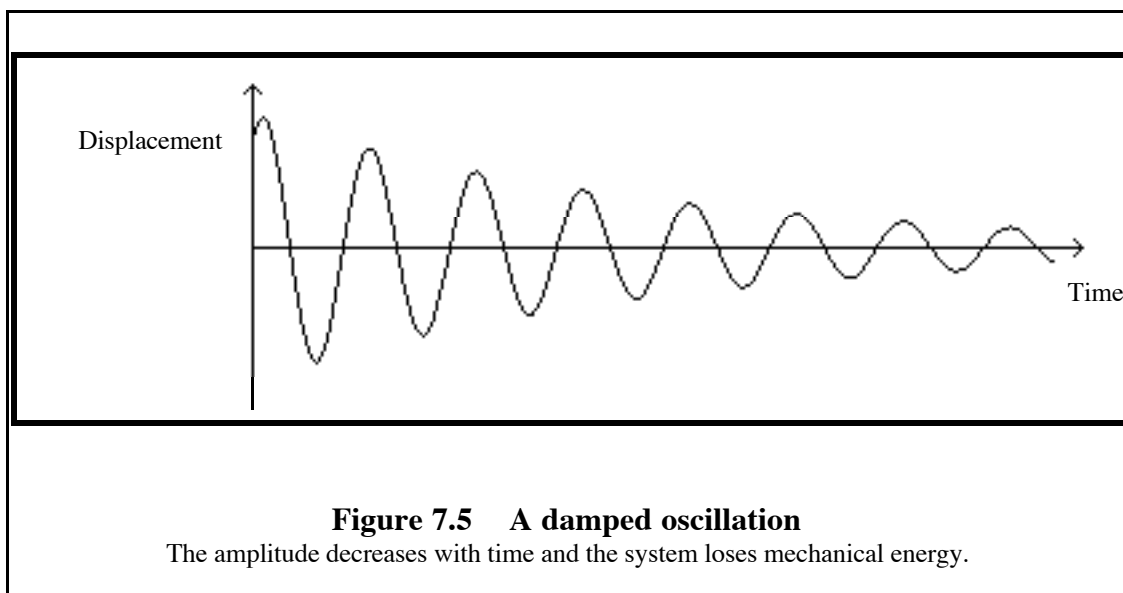
At the extreme points of the oscillation, $|x| = A$, the KE is zero and the total mechanical energy is equal to the PE. In the special case of SHM the restoring force is proportional to the displacement so the force-displacement curve is a straight line (see §5-6, figure 5.6). From the area under the curve, a triangle, the PE when $x = A$ is:

$$\text{maximum } PE = \frac{1}{2} A (kA) = \frac{1}{2} kA^2 = \frac{1}{2} m\omega_N^2 A^2.$$

The energy of the oscillation depends on the square of the amplitude A . This result is true for all SHMs.

7-3 DAMPED OSCILLATIONS

If the total mechanical energy of an oscillating system were conserved, the system would oscillate indefinitely with the same amplitude. In any real situation, however, there are always non-conservative forces such as friction or drag forces present. These dissipative forces decrease the system's total mechanical energy and can be either internal or external to the system. The amplitude will gradually decrease and the oscillations will die out. An oscillation which dies away is an example of a **transient motion**. Examples include pendulums with damping forces.



The frequency, f_D , of a damped system is always less than f_N , the natural frequency that the system would have if the damping forces could be removed, since the damping forces always act to retard the motion.

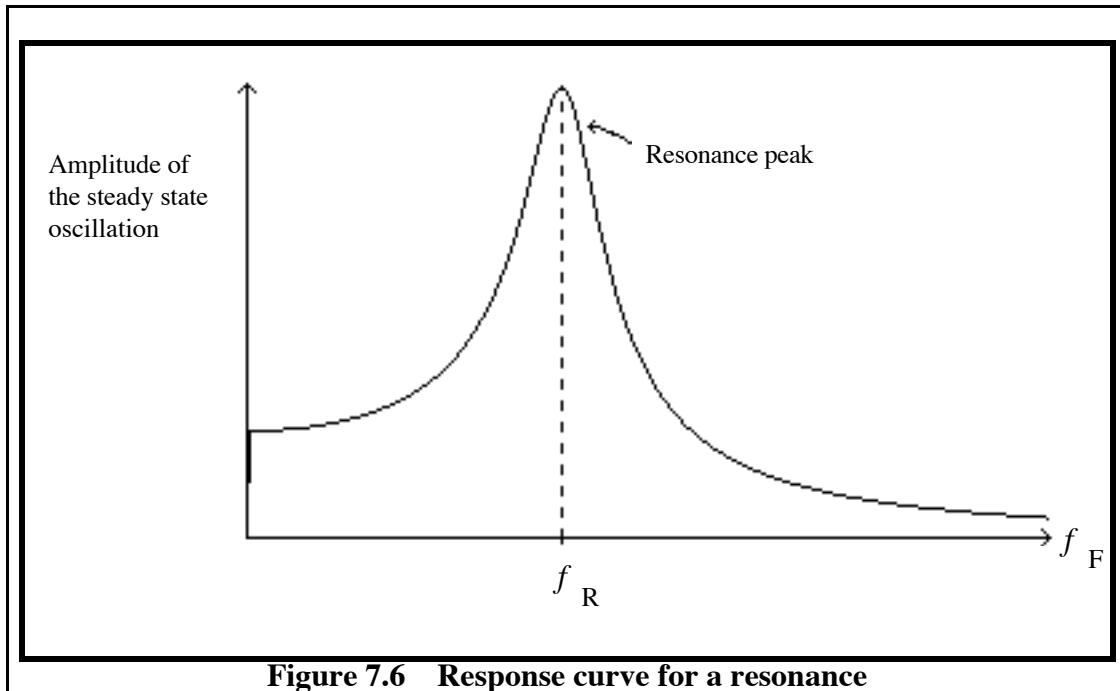
7-4 FORCED OSCILLATIONS

When non-conservative forces are present, an oscillation can die out unless energy is continually supplied to the system by a **driving force**. An oscillation which is kept going by a periodic driving force is called a **forced oscillation**. An example is a car engine.

If the driving force is sinusoidal (i.e. if a graph of the driving force against time looks like a sine curve, perhaps shifted along the time axis), there will be a **steady state oscillation** set up *at the frequency, f_F , of the driving force*. In general, f_F will be quite different from f_D .

For a given amplitude of driving force, the amplitude of the steady state oscillation depends on the driving frequency f_F . An example is a vibrating metal rule driven by an electromagnet.

Each time the frequency of the driving force is changed, the new oscillation takes a while to become stable, while the transient oscillations die away. As the driving frequency is increased from zero, the amplitude of the steady state oscillation gradually increases until it reaches a peak and then dies away again (figure 7.6). This phenomenon is called **resonance**. The frequency (f_F) of the driving force at which the amplitude of the oscillation is a maximum is called the **resonance frequency, f_r** , and the region of the graph near f_r is called the **resonance peak**.



For lightly damped systems, f_N , f_D and f_R are all approximately equal. At resonance the driving force adds to the restoring force in the damped system, producing large accelerations and oscillations.

Examples of forced oscillations

- Shivering to heat the body in response to cold.
- Expansion of the chest in breathing.
- Beating of the heart and dilation and contraction of the arteries.

In each case the driving force is provided by the muscles. When these systems relax from high to low potential energy states, their mechanical energy is decreased by the work done by non-conservative forces. The systems are highly damped and require the driving force to cause the motion.

POST-LECTURE

7-5 QUESTIONS

- Q7.2** List the forces that are acting on the following oscillating objects and in each case state whether the forces are restoring, damping or driving forces:
- i) a tree swaying in the breeze,
 - ii) the earth vibrating after an earthquake,
 - iii) a child on a swing,
 - iv) a vibrating eardrum,
 - v) a marker buoy bobbing in the water,
 - vi) the atoms in a solid when a sound wave passes through.

Simple harmonic motion

In general one can obtain an order of magnitude estimate for the square of the natural angular frequency (ω) of an oscillation by dividing an order of magnitude estimate of the restoring force per

displacement (k) by the mass (m) of the object in motion. This relation is exact for SHM (see equation 7.4) but it also gives results which are roughly OK for other systems.

Q7.3 When a spring is stretched by a pair of 2.4 N forces its length increases by 62 mm. Predict the period of oscillation when an object of mass 0.32 kg is hung from the spring and set vibrating. What will happen to the period if the object is replaced by one with 10 times the mass?

Q7.4

A wooden rectangular pontoon of density $\rho = 0.50 \times 10^3 \text{ kg.m}^{-3}$ floats in water (density $\rho_W = 1.00 \times 10^3 \text{ kg.m}^{-3}$) so that two faces, each of area $A = 20 \text{ m}^2$, are horizontal. The vertical sides each have a length $L = 1.0 \text{ m}$.

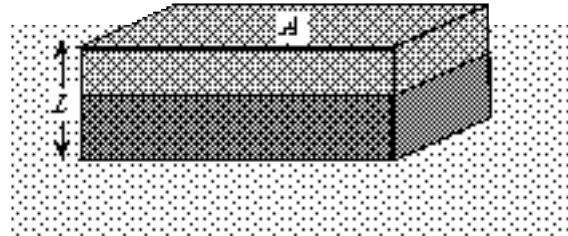


Figure 7.7

- How far is the bottom face below the water at equilibrium? (You may neglect the weight of the displaced air in this problem.)
- What is the total force acting on the pontoon when it is a small distance x below the equilibrium position with the same two faces horizontal?
- What would this force be if the pontoon were a distance x above the equilibrium position?
- Hence show that if the pontoon were released from a position other than its equilibrium position it would execute SHM in the vertical direction if no non-conservative forces were present. (What non-conservative forces would you expect to be present and what effect would they have?)
- What is the natural frequency of the above SHM?

Q7.5 Use energy considerations to find the maximum speed of an object undergoing SHM in terms of the amplitude and natural (angular) frequency of the motion.

Q7.6

A somewhat bizarre proposal for a rapid transit system between Boston and Washington (a distance of 630 km) made use of the Earth's gravitational attraction. (*Scientific American*, August 1965, pp 30 - 40.) Stripped of its technical complications, the proposal had a capsule travelling through an evacuated tunnel drilled in a straight line between those cities.

For the first half of its journey, the capsule would gradually be getting closer to the centre of the earth (i.e. sliding down an incline); for the second half it would be travelling away from the Earth's centre.

In the idealised case where non-conservative forces are neglected, the potential energies at the beginning and the end of the journey would be equal. (In the actual proposal, the capsule was helped along by compressed air to overcome the non-conservative forces.) In this idealised case, the acceleration of the capsule would be

$$a = -Cx$$

where x is the displacement from the midpoint of the journey and C is a constant equal to $1.55 \times 10^{-6} \text{ s}^{-2}$. The capsule would therefore be undergoing part of a SHM.

- How long would the one way trip from Boston to Washington take?
- What would be the maximum speed of the capsule on the journey?

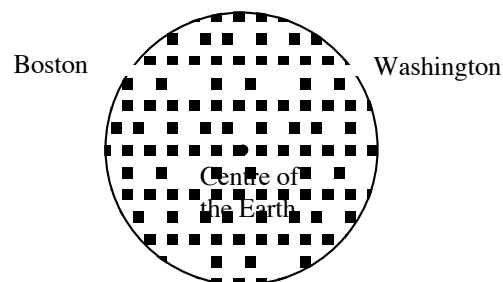
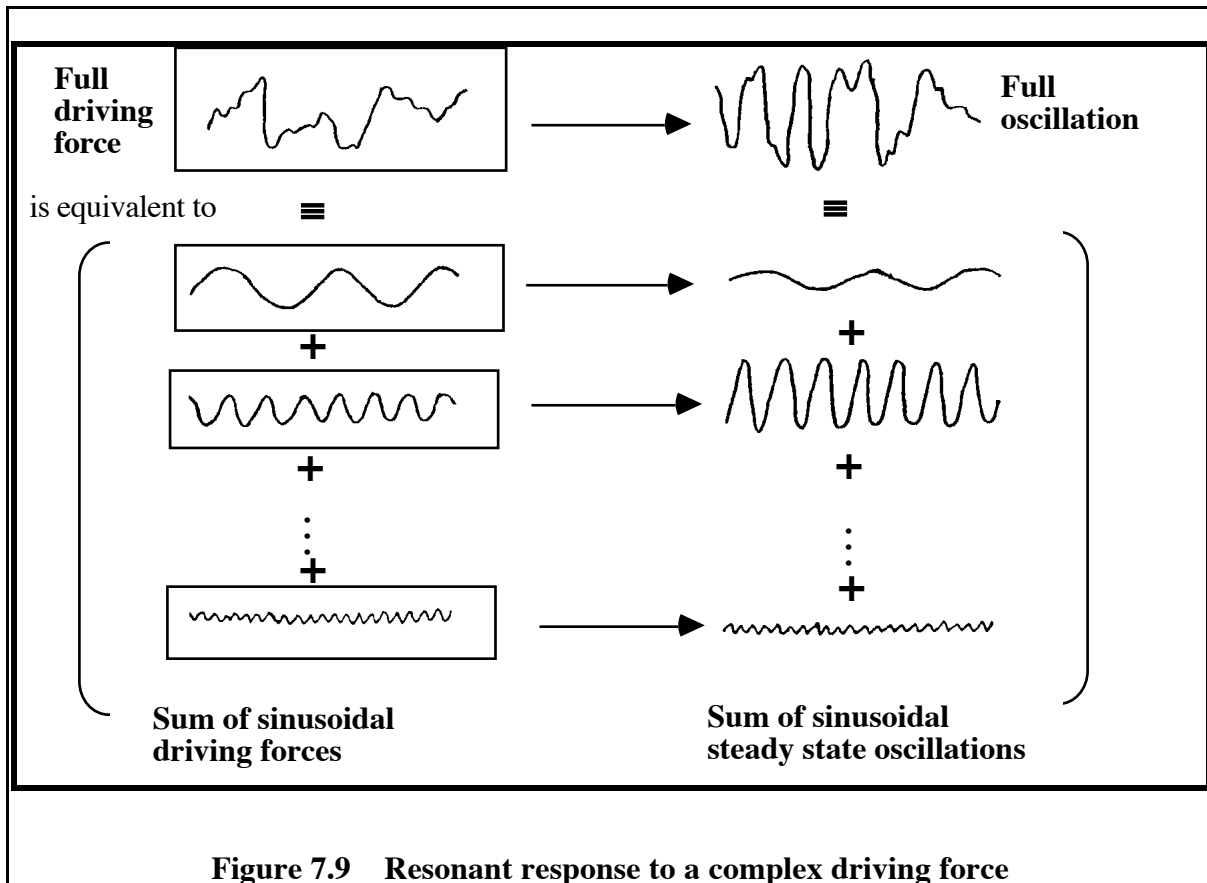


Figure 7.8

7-6 FURTHER DISCUSSION OF RESONANCE

In the video lecture it was stated that the Tacoma Narrows Bridge collapse was caused by resonance. The driving force in that case, however, was *not* sinusoidal nor did it oscillate at the frequency of the steady state oscillations of the bridge. In fact the driving force was provided by a wind blowing *in one direction* for a time long compared to the period of the bridge's oscillation. How then does resonance occur?

Any driving force whatever may be considered mathematically to be a combination of sinusoidal driving forces with appropriate frequencies, amplitudes and initial phases. The procedure for splitting up a force in this way is called **Fourier analysis** (after the French mathematical physicist who developed it to study the flow of heat through metals). Each of these sinusoidal forces can be thought of as acting on the damped oscillating system independently of the other sinusoidal components setting up a sinusoidal steady state oscillation at its own frequency. These oscillations add up to give the actual oscillation caused by the full original driving force. Schematically, this procedure can be summarised as shown in figure 7.9.



The ratio of the amplitude of a steady state oscillation to that of the sinusoidal driving force that caused it is largest when the driving force has a frequency equal to the resonance frequency. Quite often therefore, the combined oscillation is dominated by the steady state oscillations with frequency approximately equal to f_R , particularly if the resonance curve is very sharply peaked.

These ideas are the basis of radio and TV receivers, for example. The driving force is the complicated electrical signal (the sum of all the signals sent out by the various stations plus interference from fluorescent lights, spark plugs etc.) from the aerial or antenna. The tuning circuit of a radio or TV set is an oscillating system whose resonance frequency can be varied to select the steady state oscillations at the frequency of the transmitting station. The resonance peak for these tuning circuits has to be sufficiently narrow that only one station is transmitting at a frequency within the peak at any given setting.

In the bridge collapse, the driving force provided by the wind had a large sinusoidal component at one of the natural frequencies of oscillation of the bridge. This component caused the bridge to resonate and ultimately break apart.

Q7.7 Can you think of any other situations in which resonance plays an important role ?

7-7 APPENDIX: LISSAJOUS FIGURES

When two SHMs at right angles are combined, the path traced out will be a Lissajous figure if the ratio of the two frequencies is equal to a ratio of two integers. The shape is also affected by the phase difference between the oscillations.

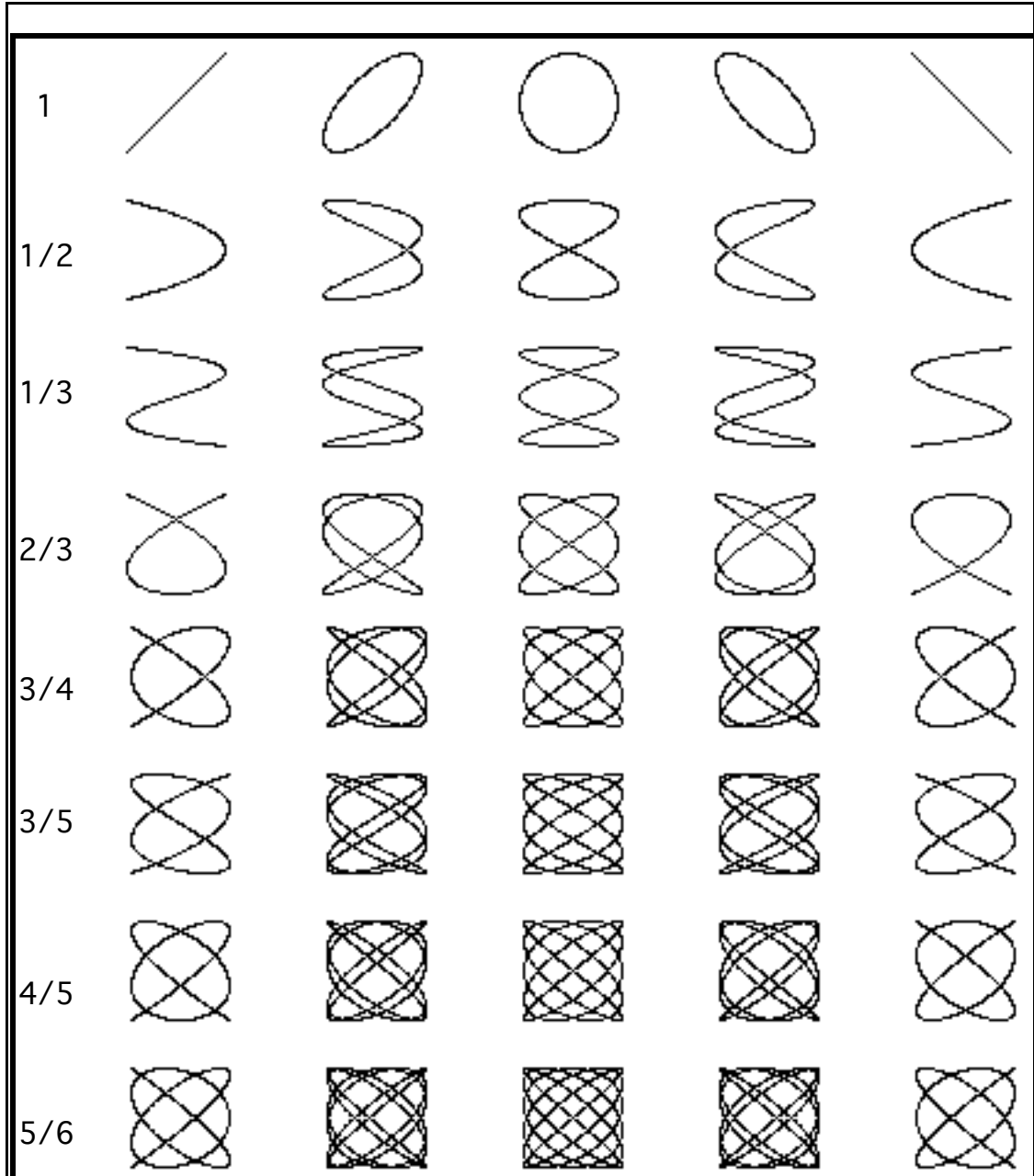


Figure 7.10 Some Lissajous figures

The number at the left of each row is the ratio of the frequency of the vertical oscillation to that of the horizontal oscillation for the figures in that row. The figures in each row correspond to different initial phases of these two oscillations.