

FE8**SCALE****OBJECTIVES****Aims**

From this chapter you should learn to appreciate the great power of physical reasoning using very rough approximations to more exact principles. You should become reasonably adept at applying this kind of reasoning and the physical principles that you already know to problems of scaling - comparing the behaviour of things, including animals, which have roughly the same shape and composition.

Minimum learning goals

When you have finished studying this chapter you should be able to do all of the following.

1. Explain, interpret and use the concept of *scale factor*.
2. Explain how the laws of physics can be applied to a wide range of similar problems by concentrating on the similarities and ignoring particular idiosyncrasies.
3. Apply the concept of scale factor to analyse the way the sizes of different animals determine their ability to perform simple mechanical operations such as running and jumping.
4. Describe and explain the idea that bone strengths and muscular forces per cross-sectional area are much the same for all animals.
5. Recall and use the fact that the rate of energy expenditure by an animal is equal to the rate of energy supply, which is controlled by the amount of oxygen available to the cells.
6. State and apply the scaling laws listed in the table of §8-7.
7. Describe some of the pitfalls to be avoided in scaling problems.

PRE-LECTURE**8-1 INTRODUCTION**

The earlier parts of this unit have mainly been concerned with the application of physical concepts to *exact* calculations, using carefully measured data. This technique is used, for example, by engineers in designing bridges and power stations, by physicians in radiation dosimetry and by optometrists in grinding lenses.

There is an equally important, although less well-known, way of using the concepts of physics in non-exact calculations just to see whether a particular idea is valid. The arithmetic used may be rough and ready - of the $3 \times 3 = 10$ variety. Frequent use may be made of statements involving the words 'about equal'. This is the style of approach used, for example, in forecasting trends in world population and consumption of raw materials.

This second type of application of physics is the theme of this chapter. The concepts of forces in equilibrium, work done by forces and energy balances will be used to examine some of the features of the animal world - the strengths of skeletons, running, jumping and diving. The interest is not in the subtler details of the mechanisms of these things, e.g. how a grasshopper bends its legs prior to jumping or how a whale stocks up on oxygen before diving, but in broader considerations, e.g. why grasshoppers, frogs and kangaroos all raise their centres of gravity by roughly the same amount when jumping, or what advantages large mammals have over small mammals in diving.

In the spirit of the ‘equality’ $3 \times 3 = 10$, the physics used will be stripped to its bare essentials. For the purposes of this chapter a force whose average magnitude is F moving through a displacement d does work Fd - we will neglect niceties like including the cosine of the angle between the force and the displacement. All such extraneous considerations will be lumped together in proportionality constants which will have no bearing on the conclusions we draw from the physics.

The primary aim of this chapter is to show how these perfectly respectable procedures can be used to extract information. The results obtained simply illustrate the power of the method; you are *not* expected to memorise these results for examination purposes. You *will* be expected to be able to reason along the same lines.

Q8.1 There are two solid cylinders made of the same material. All the linear dimensions of the larger one are exactly twice those of the smaller.

Compare their heights, diameters, base areas, curved surface areas, volumes and masses.

LECTURE

8-2 SCALE FACTOR

The question, ‘how much bigger is one tree than another?’ has no unique answer unless we know on what basis the comparison is to be made - on their heights, on the cross-sectional area of their trunks or on the volume of wood in their trunks, for example.

If the trees have roughly the same shape, a comparison can be made in terms of a measure based on length - the **scale factor**, L . Then if one tree were L times taller than the other, the radius of its trunk would be L times bigger, the cross-sectional area of its trunk L^2 times bigger and the volume of its trunk L^3 times bigger than the other. The symbol L has been chosen deliberately as a reminder that our *basic* comparison is being made on corresponding lengths. However L is a *ratio* of lengths; it has no unit; it is a *scale factor*, not a *scale length*.

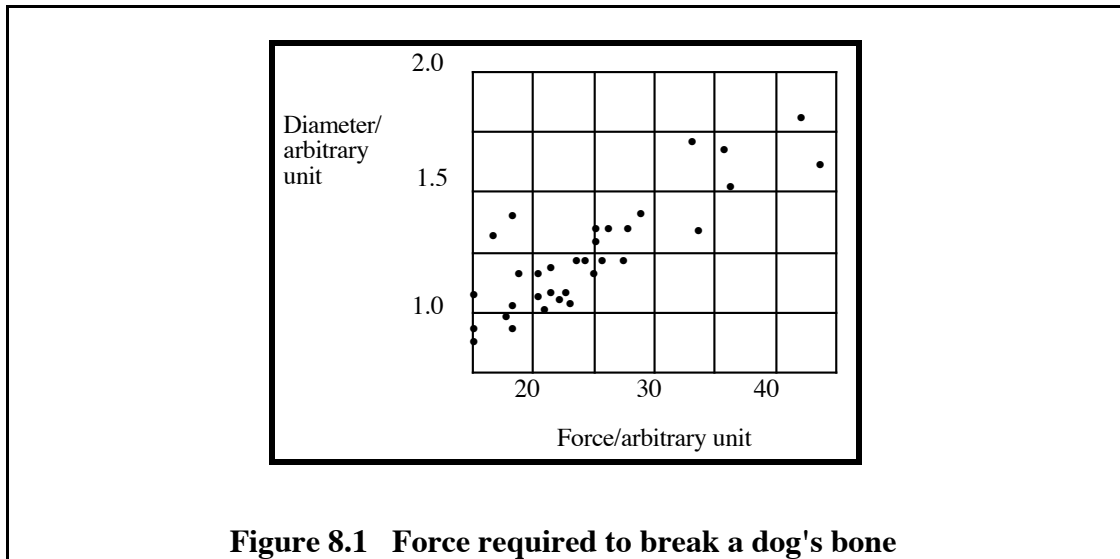
A scale factor can be assigned to any class of objects which have essentially the same shape. In the following we will be comparing the properties of different species of animals which have roughly similar basic shapes, but which differ in size, e.g. dogs, sheep and horses.

8-3 BONE LOADS AND MUSCULAR FORCES

We make the assumptions that the strength of the materials that bones are made of and the way the bones are constructed are much the same for all animals. By strength we mean the maximum magnitude of the force per cross-sectional area that the bone is able to withstand under a compressional or an extensive load (the two values need not be the same). This means that the maximum force that can be applied along the length of a bone without breaking it depends on the cross-sectional area of the bone, i.e. on the square of the bone's diameter.

This assumption is supported indirectly by two sets of experimental results on the breaking of tibia bones of different sizes. (Notice that in this experiment the loads are applied at right angles to the bones - they are not along the bones themselves.)

- In the experiment shown on TV, 37 force units were required to break a dog's tibia of diameter 1.66 length units while 50 force units were required to break a sheep's tibia of diameter 1.84 length units.
- The following results were obtained for the tibia bones of different breeds of dog (figure 8.1).



The same type of assumptions about the maximum forces supplied by muscles or tendons lead to the result that these forces also scale as the square of the diameter.

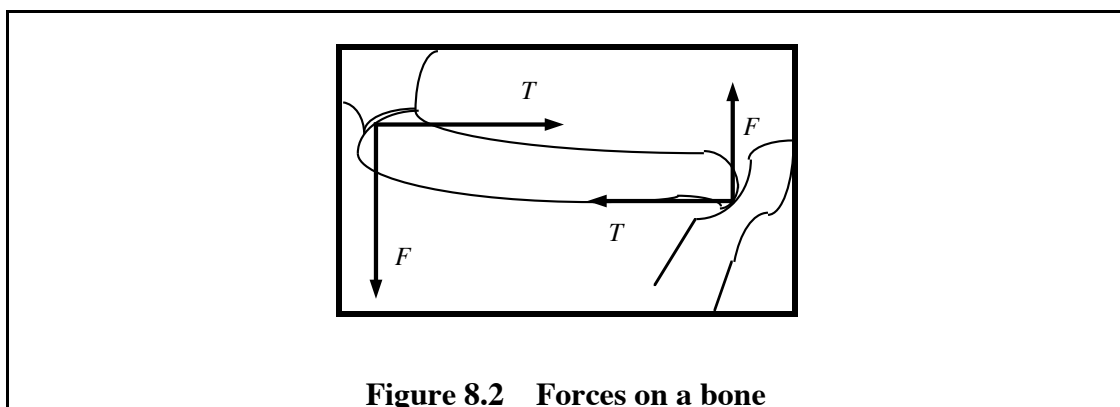
Applications

Animals standing still

The leg bones of a stationary animal must support a compressional load proportional to the animal's weight. The weight of the animal is proportional to its volume and so, between different sized animals, varies as L^3 . We saw above that the maximum compressional force which the bones can support varies as the square of the diameter. The diameter of the leg bones of similar animals must therefore increase faster than the other linear dimensions of these animals. Thus, for example, the leg bones of elephants are comparatively much thicker than those of mice.

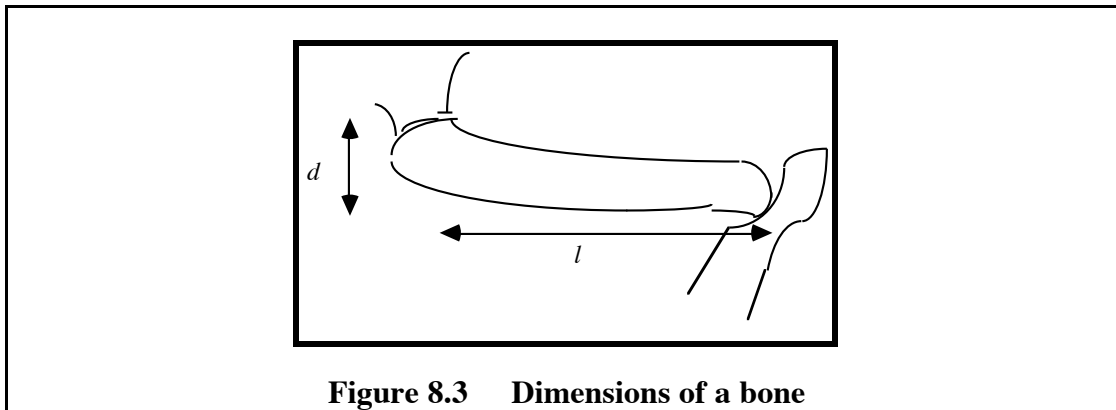
Moving animals

The legs of an animal are also subject to bending forces, particularly when they are moving. Consider for example the forces acting on the humerus bone of a mammal.



The adjacent bones exert vertical forces F proportional to the weight of the mammal (to L^3). To keep the bone from twisting (rotating) there must be a pair of equal and opposite forces T

exerted by the muscles and the tendons along the humerus bone so that there is no net torque: these forces provide a bending (or shearing) load on the bone.



If the length of the humerus bone is l and its diameter is d , since there is no net torque,

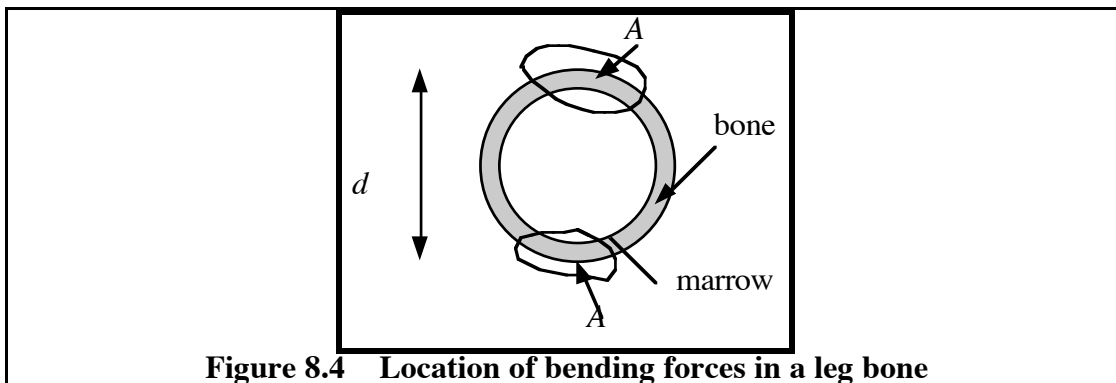
$$Fl - Td = 0$$

so

$$T = \frac{Fl}{d} .$$

Since l and d both vary as L , then T , like F , varies as L^3 .

The bending forces T act over areas A near the surface of the bone as shown in the cross-sectional view below. These areas, and hence the capacity of the bone to withstand the bending, vary as the square of the diameter.



Thus the bending forces in the humerus increase more rapidly with L than the capacity of the bone to withstand them - essentially the same result as in the case of standing still. These bending forces are the reason why giraffes keep their legs straight while running.

8-4 SUPPLY OF CHEMICAL ENERGY IN THE BODY

The maximum continuous rate at which an animal can do work while engaging in various activities is limited by the rate at which the cells in the body can supply energy to the muscles. The latter rate is controlled by the amount of oxygen available to the cells; it is shown in the POST-LECTURE to vary as L^2 .

Thus, for animals of similar shape,

$$\text{the maximum rate of energy supply} = k_1 L^2 , \quad \dots (8.1)$$

where k_1 is a constant independent of the size of the animal and hence of L .

Applications

Running on the flat

Does the size of an animal determine the speed at which it can run on the flat? While the animal is running at a fixed speed, here its maximum speed v , its kinetic energy $\frac{1}{2}mv^2$ is constant. Since it is

running on flat ground its gravitational potential energy is also constant. Hence the energy supplied (by the cells) and the (negative) work being done by the drag force exerted by the air must total zero.

The drag force is proportional to the cross-sectional area of the animal and to the square of the speed. In symbols

$$\text{drag force} = k_2 L^2 v^2,$$

where k_2 is independent of L and v .

In a time interval t the animal travels a distance vt so the work done by the drag force during this time is $-k_2 L^2 v^3 t$.

Equating the magnitude of the rate of doing this work with the maximum rate of energy supply (from equation 1 above),

$$k_2 L^2 v^3 = k_1 L^2,$$

$$\text{or } v^3 = \frac{k_1}{k_2} = \text{constant}.$$

This means that, on the basis of this scaling argument, the maximum speed of an animal running on the flat is expected to be independent of its size. This expectation agrees with the observation that, whilst there are slight individual differences between the maximum speeds of animals, there is no systematic trend with size. For example, dogs and horses can run at about 16 m.s^{-1} while humans, intermediate in size between these, run at about 10 m.s^{-1} .

Running up a steep incline

The dominant force doing work on the animal is the gravitational force. This force (the weight) is proportional to L^3 . As in the previous application, if the animal is travelling at constant speed the energy supplied (by the cells) and the work being done by the gravitational force must total zero.

Suppose that the animal is travelling at its maximum speed v up the incline. In a time interval t the animal travels a distance vt so the work done by the gravitational force during this time is $-k_3 L^3 vt$. (k_3 is also independent of L and v .)

Equating the magnitude of the rate of doing this work to the maximum rate of energy supply,

$$k_3 L^3 v = k_1 L^2$$

$$\text{or } v = \frac{k_1}{k_3} L^{-1}.$$

In other words, the maximum speed of an animal travelling up an incline would be expected to be inversely proportional to its size. For example a greyhound would be expected to travel faster than a horse up a hill.

8-5 JUMPING

An animal standing on the ground exerts a force on the ground equal in magnitude to the weight of the animal. In order to jump, the animal first crouches, lowering its centre of gravity by a small displacement d . It then pushes against the ground with an additional force of average magnitude F . This means that there is an unbalanced force of this average magnitude upwards on the animal as long as it remains in contact with the ground. As the animal leaves the ground it accelerates upwards and its kinetic energy increases as its centre of gravity returns to its original position. This increase in KE actually comes from work done by the leg muscles but its value can be calculated using the product of the net external force (F) and the displacement (d) of the centre of gravity. So the work done on the animal is equal to Fd .

After the animal leaves the ground its centre of gravity rises to a height h above its position when it was crouching.

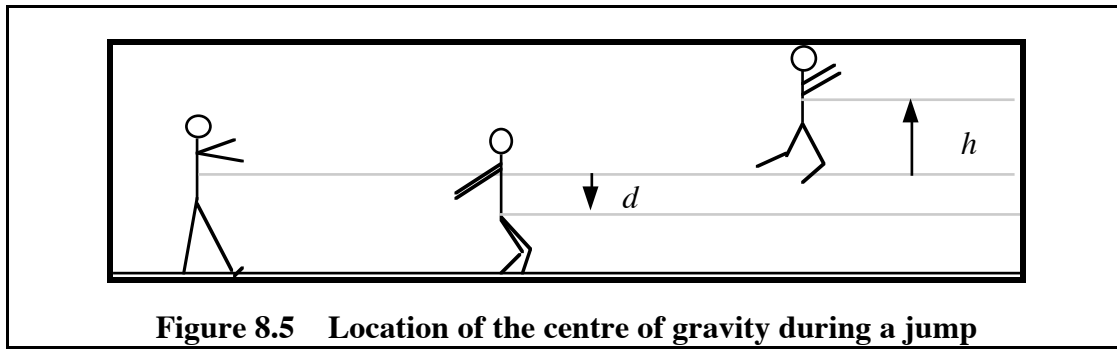


Figure 8.5 Location of the centre of gravity during a jump

The associated increase in gravitational potential energy can be equated to the kinetic energy at take-off or the work done during the launch.

The increase in gravitational potential energy is $k_5 L^3 h$ where k_5 is independent of L and h .

The average force F provided by the muscles is proportional to L^2 (see earlier) and the crouching distance d is proportional to L , so the work done is equal to $k_4 L^3$ where k_4 is independent of L .

Equating these two quantities:

$$k_5 L^3 h = k_4 L^3;$$

hence
$$h = \frac{k_4}{k_5}.$$

This means that the maximum height through which the centre of gravity can be raised is expected to be independent of the animal's size. Observation shows that this is so; for most animals the height involved is a little over 1 metre.

8-6 DIVING

When a mammal dives below the surface of the water, it consumes oxygen at a rate proportional to L^2 (see the section in the post-lecture on the rate of supply of energy); in a time interval t the mammal consumes a quantity $k_6 L^2 t$ of oxygen (k_6 is independent of L and t).

The total oxygen supply depends on the volume of the lungs and hence can be written as $k_7 L^3$. (k_7 is independent of L .) Hence the maximum duration of the dive is given by

$$t = \frac{k_7}{k_6} L,$$

which, since k_6 and k_7 are independent of L , indicates that larger mammals can stay under water for longer periods. Again this is confirmed by observation.

POST-LECTURE

8-7 APPLICATION OF THE CONCEPT OF SCALE FACTOR

Applications involving scale factors fall into two classes.

- The first, rather straight-forward, class is that in which measured or given data lead directly or indirectly to a value of the scale factor L and this value can be used to estimate other data pertinent to the system. In dealing with these problems it is useful to be able to recollect the way certain fundamental quantities are related to the scale factor. Some of these relations are summarised here in tabular form.

L	L^2	L^3
length	surface area	volume
girth	cross-sectional area	mass
height	rate of energy supply	weight
etc.	etc.	etc.

An example of this class of problem is as follows. Consider two oranges, both roughly spherical, one of mass 0.20 kg, the other of mass 0.40 kg and ask the question as to how the surface areas of the two oranges compare. To answer this one uses the mass ratio (2:1) to find the value of L^3 and proceeds to compare surface areas through the derived value of L^2 .

- The second class is where it is required to derive the way in which the variation of some quantity among things of similar shape is related to the scale factor. Such problems usually are more involved and require the use of some linking factor known from everyday life or from the formal study of physics. Many of the illustrations in the lecture material were of this type: running, jumping and diving animals, to name a few.

Q8.2 Suppose that in order to make Australians champions in all sporting events we breed fair dinkum sun-bronzed Aussies who are scaled up $\times 2$ in all linear dimensions. They are made of the same stuff as conventional $\times 1$ people. How would a $\times 2$ person's performance compare with that of a $\times 1$ person in the following events:

- a form of high jumping where, to even things up, the judges measure the increases in heights of the jumpers' centres of gravity,
- weight-lifting,
- sprinting (assume that the forward acceleration from rest at the beginning of the sprint is the dominant influence on the sprinters' performances),
- putting a shot made of the usual material but whose linear dimensions are proportional to the height of the shot putter? (For simplicity assume that the shot is launched horizontally.)

8-8 BREAKING OF DOGS' BONES

The experimental results shown in the lecture for the tibia bones of different breeds of dog are reproduced in graphical form below.

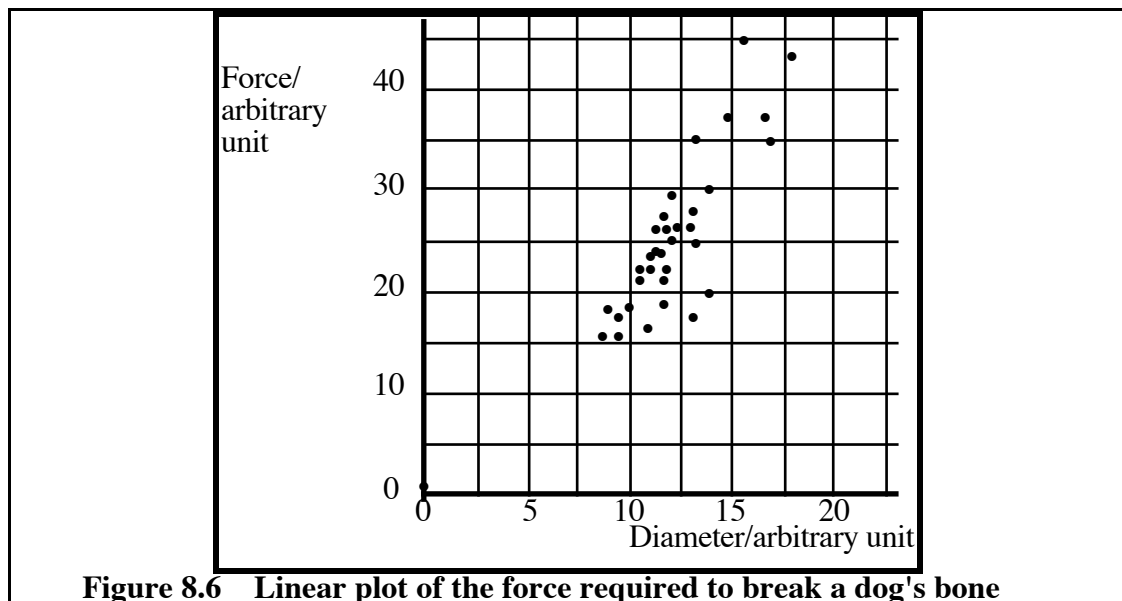


Figure 8.6 Linear plot of the force required to break a dog's bone

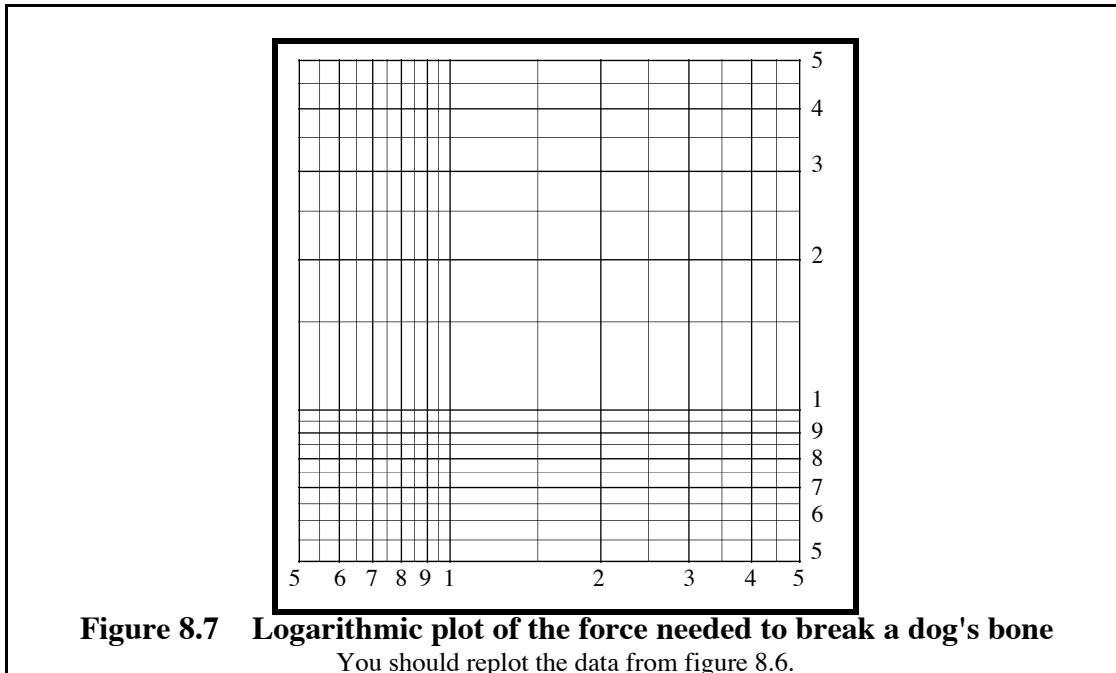
This graph differs from figure 8.1 in that the origin is shown on the graph. This point is a data point on logical grounds rather than experimental evidence. You can now see that the relationship between breaking force and bone diameter is not linear. It may follow some sort of power law. The axes of the graph have been swapped to make the following question simpler.

Q8.3 Suppose the relationship between diameter d and force F is of the form

$$F = k d^n$$

where k and n are constants.

Re-plot the data on a log-log graph to see if the data support the view, held in the lecture, that $n = 2$.

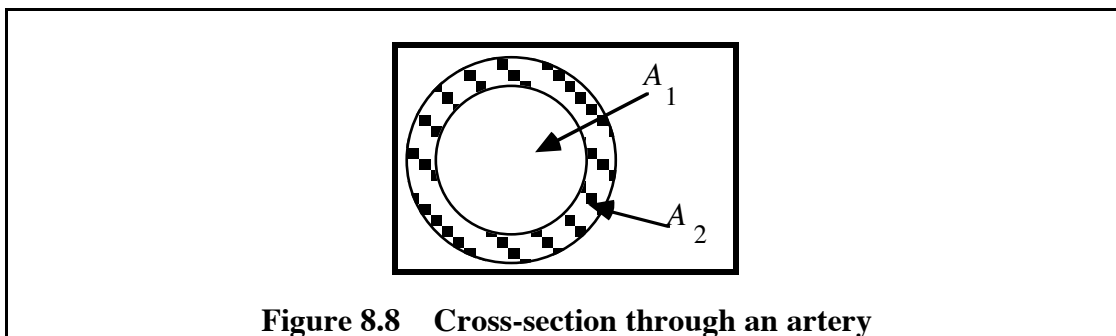


8-8 RATE OF ENERGY SUPPLY AND PULSE RATE

Rate of energy supply in an animal

The rate of supply of energy to muscles depends on the rate of oxygen supply which, in turn, depends on the volume rate of flow of blood passing through the muscle tissues. This volume rate of flow of blood depends on the cross-sectional area of the aorta (proportional to L^2) and the speed of the blood flow.

The first question then is whether the speed of the blood flow depends on size. This question can be answered in a round-about fashion. Since the speed of the blood flow is itself proportional to the blood pressure the question becomes: does the blood pressure depend on size?



The muscular action in the ventricle of the heart keeps the blood moving with more or less constant volume rate of flow so the force exerted by the muscles equals that arising from the pressure. If P is the blood pressure, S the muscular force per cross-sectional area of the muscles, A_1 is the cross-sectional area of the ventricle and A_2 is the cross-sectional area of the muscle,

$$P A_1 = S A_2,$$

that is

$$P = S \frac{A_2}{A_1}.$$

The two areas depend on L^2 while S is independent of L . This means that the pressure, and hence the speed of blood through the aorta, does not depend on size.

Therefore both the rate of flow of blood delivered by the heart and the rate at which energy is supplied are proportional to the cross-sectional area of the aorta and hence depend on L^2 .

Pulse rate

Q8.4 We have seen that the rate of blood supply from the heart varies as L^2 . The volume of the heart will naturally vary as L^3 . What conclusion can you reach regarding the pulse rate of animals of various sizes?

8-9 A CAUTIONARY TALE ABOUT DRUG DOSAGES

Scaling problems can often present traps for the unwary, even for experimenters who should know better. A few years ago some scientists wished to test the reaction of elephants to LSD dosage. They calculated the dose from that already found to put a cat into a rage, (0.1 mg of LSD per kg of body weight) and scaled up by weight to get a dose of 297 mg. On receiving this dose the elephant immediately started trumpeting and running around, then it stopped, swayed and, after five minutes, collapsed, went into convulsions, defecated and died. The scientists concluded that elephants are peculiarly sensitive to LSD.

Was the scaling done correctly? There are several approaches to this problem, leading to a large range of suggested doses.

- Equal concentrations of LSD in body fluids for elephant and cat, i.e. scaling according to weight. Dose: 297 mg.
- Scaling based on metabolic rate, which will control detoxification of the drug and its excretion. Dose: 80 mg.
- Scaling based on an animal which is not as notoriously tolerant to LSD as a cat, for example we could use a human. A dose of 0.2 mg LSD produces severe psychotic symptoms in humans. Using a scaling based on weight, the elephant dose is 8 mg.
- Scaling on metabolic rate, using the human dose as a base. Dose: 3 mg.
- Scaling on brain size, since LSD could be more concentrated there, again using human dose as base (human brain 1.4 kg, elephant brain 3 kg). Dose: 0.4 mg.

8-10 EARLY ATTEMPTS AT SCALING

The idea of scaling in physics is not new. You may be interested in reading the discussion in Galileo's work *Concerning the Two New Sciences*, pp 184-188, vol 28, Britannica Great Books which is available in the Fisher Library Reading Room. Jonathan Swift encountered the problem in *Gulliver's Travels* where the Lilliputian emperor decided that Gulliver needed $(12)^3$, that is 1728, Lilliputian portions of food. Actually the emperor overestimated the situation by scaling according to weight rather than to surface area.

8-11 SCALING APPLIED TO MOTOR VEHICLES

A feature of scaling problems involving animals is that the power available varies as L^2 . The builders of machines are not bound by this constraint and can adjust the scaling factors to best suit their requirements. For example, as the size of vehicles is increased from cars to semi-trailers the prime consideration is to increase the carrying capacity, even if the motive power does not increase in the same proportion as in the animal world.

Q8.5 A family car is about 3 m long, has a mass of about 1000 kg and develops a power of about 75 kW. A semitrailer is about 9 m long, has a laden mass of about 30000 kg and develops a power of about 225 kW. How much faster can the car climb a hill than the semitrailer?