

PM1

ELASTICITY

*The lion is the king of beasts
And husband of the lioness.
Gazelles and things on which he feasts
Address him as Your Lioness
There are those who admire that roar of his
In the African jungles and veldts.
But I think wherever the lion is
I'd rather be somewhere else.*

OBJECTIVES

Aims

In this chapter you will see that all elastic deformations can be described in terms of linear, shear and bulk changes. You will be introduced to the concepts of stress, strain and material strength. You will apply these ideas to some real-world deformations. You will learn how to do calculations involving simple situations of deformation.

Minimum Learning Goals

When you have finished studying this chapter you should be able to do all of the following.

1. Explain, interpret and use the terms
elastic deformation, plastic flow, permanent set, ductile, brittle, compressive, tensile, linear extension, uniform compression, shear, pressure, fracture, stress, strain, elastic modulus, Young's modulus, shear modulus, bulk modulus, strength, elastic limit.
2. (i) Recall and state Hooke's Law.
(ii) Use the relations between the three elastic moduli and stress and strain in simple numerical problems.
3. Recall that the elastic moduli have dimensions of force per area.
4. Use a model of the microscopic structure of materials to explain elastic behaviour.
5. (i) Describe the experimental measurement of elastic moduli by direct determinations.
(ii) Use mechanical oscillations to measure elastic moduli indirectly.
6. Describe situations involving strength and/or deformation in the human body and in fibrous materials.

PRE-LECTURE

1. In mechanics, forces acting on an extended body are assumed to produce only translational and/or rotational accelerations of the body: the body is assumed to be rigid. However, no body is completely rigid: forces also deform bodies.
2. Remember that pressure is defined as force per area.
3. Refer back to chapter FE3 and/or chapter FE5, where inter-molecular forces are discussed in terms of the distance between molecules.

LECTURE

1-1 STRESS, STRAIN AND THE BASIC DEFORMATIONS.

The study of elasticity is concerned with how bodies deform under the action of pairs of applied forces. In this study there are two basic concepts: **stress** and **strain**.

The pairs of forces act in opposite directions along the same line. Thus there is no resulting acceleration (change of motion) but there is a resulting deformation or change in the size or shape of the body. This is described in terms of strain.

The **strain** is the relative change in dimensions of a body resulting from the external forces.

As a result of the deformation, internal forces are set up and these give rise to stresses. In many simple cases, these stresses are simply related to the external forces, because when these two are in balance the deformation will be maintained without further change. For these simple cases we make the following definition.

The **stress** is the external force divided by the area over which this force is applied.

There are three particular cases we will consider.

Linear Extension

Demonstration

The first type is the linear extension.

An oppositely directed pair of forces along a line extend the body in along that line.

Write the magnitude of these forces as F , the cross-sectional area at right angles to F as A , the original length as L and the extension as e . The stress and strain are then defined as follows:

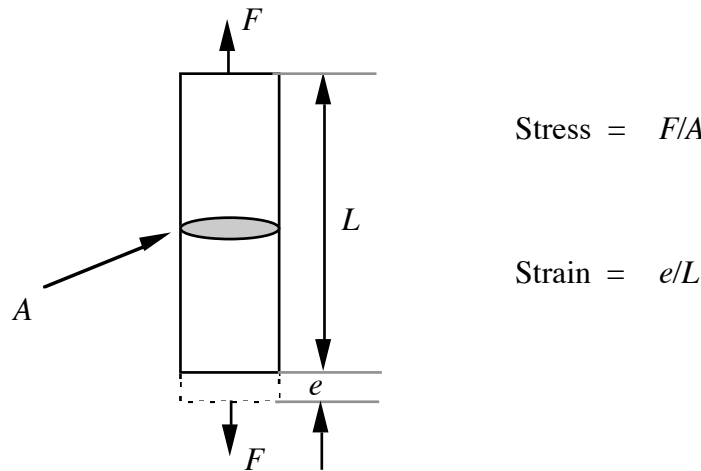


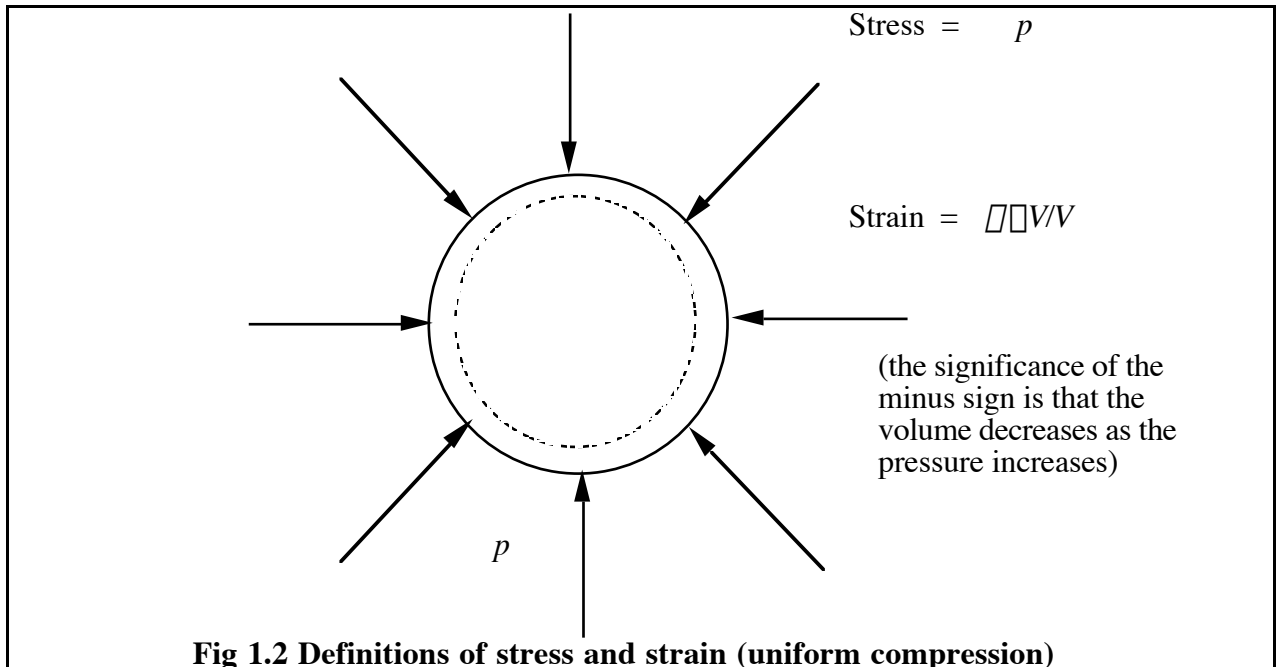
Fig 1.1 Definitions of stress and strain (linear extension)

Uniform Compression

Demonstration

If the forces are applied uniformly in all directions, we have a deformation typified by that produced by a uniform hydrostatic pressure.

Write the pressure as p , the original volume as V and the change in volume as ΔV . The stress and strain are then defined as follows:

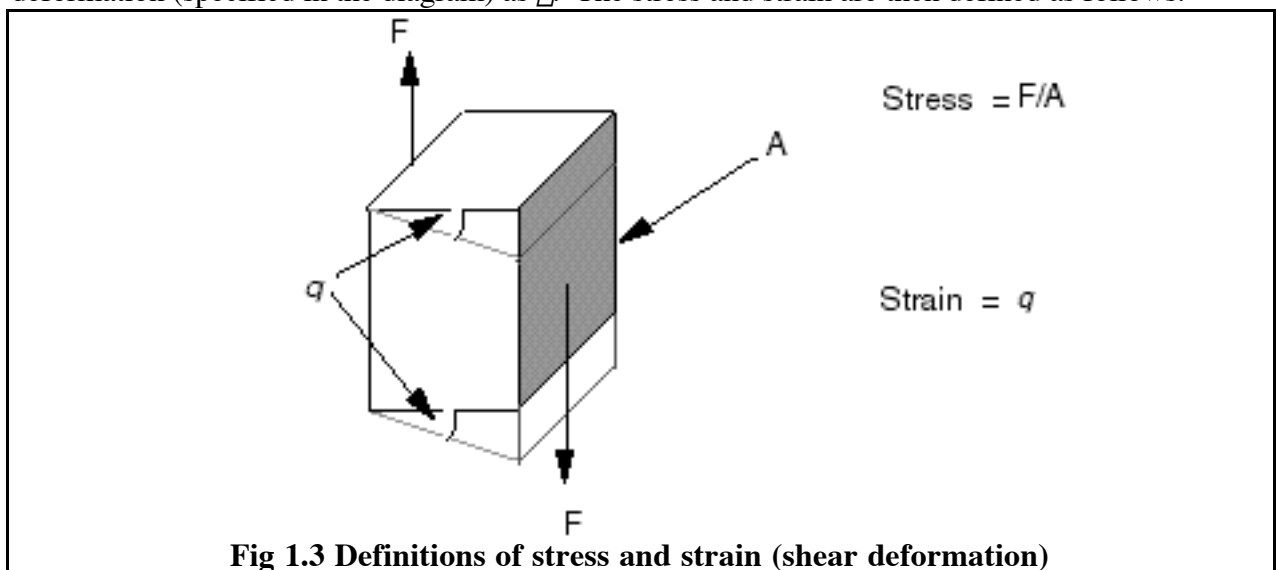


Shear

Demonstration

In the previous two deformations, either the length or volume of a body was changed. In the **shear** deformation, only the shape of a body is changed. Shear occurs for example when oppositely directed tangential forces are applied across opposite faces of a rectangular block of material. These forces deform the rectangular block into a parallelogram.

Write the force as F , the area across which the force is applied as A and the angle of deformation (specified in the diagram) as q . The stress and strain are then defined as follows.

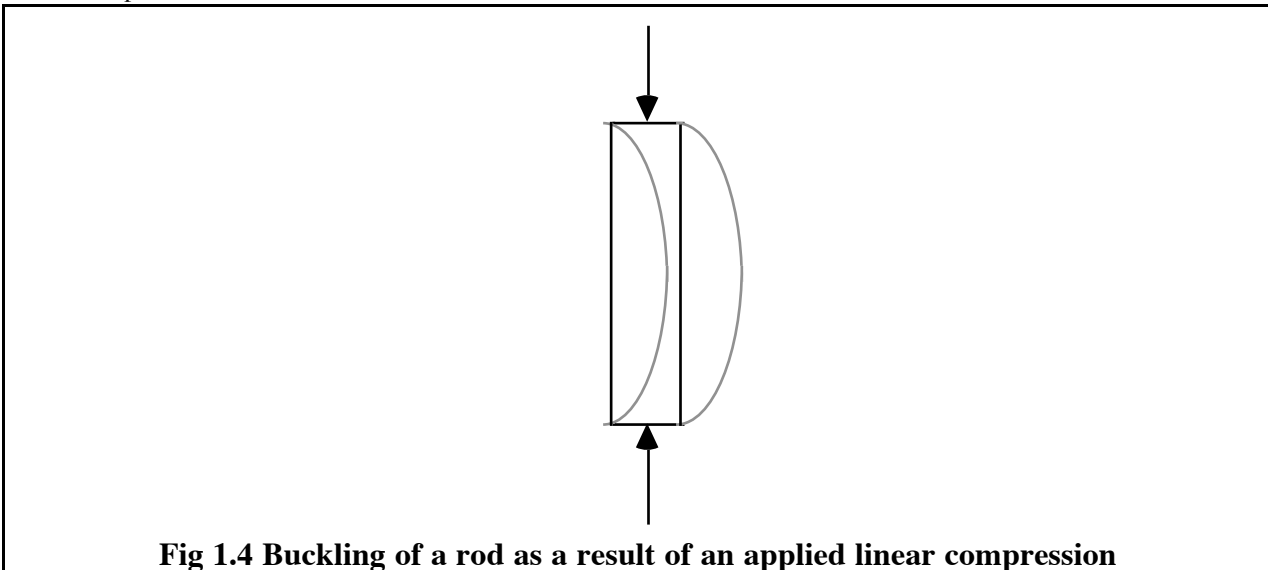


These three deformations are the three basic types.

In general, things are more complicated than this but can be resolved in terms of these basic deformations.

Demonstration

As an example of the more complicated behaviour one can get, consider a rod under the action of a compressive force in the direction of the rod.



If there are no complications, this is merely the opposite of the linear extension. However, if the rod is thin enough, one does not get a linear compression but rather the rod buckles.

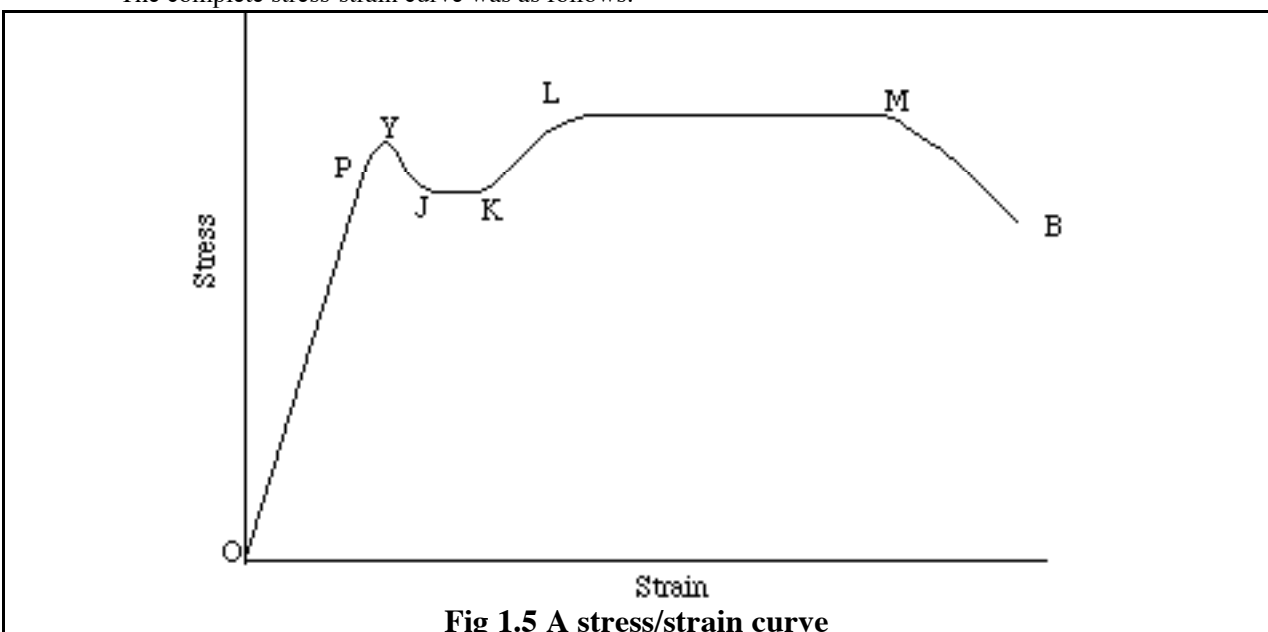
1-2 ELASTIC AND NON-ELASTIC BEHAVIOUR

Let us now consider what happens to a body under the action of one of these types of deforming force as the force is gradually increased from zero.

Demonstration

This was done for the case of linear extension using one of the testing machines in the Civil Engineering Department. A sample of mild steel was tested and the stress as a function of strain was recorded on a chart recorder.

The complete stress-strain curve was as follows:



As the stress was increased it was at first proportional to the strain; if, in this region, the stress were removed, the strain would return to zero i.e. the body would return to its original length. This region OP is known as the **elastic regime** and the point P is called the **elastic limit**.

As the stress was further increased, a point Y, known as the **yield point**, at which the stress rapidly dropped, was reached. From J to K the material flowed like a fluid; such behaviour is called **plastic flow**. After a region K to L of partial elastic behaviour, plastic flow continued from L to M. Eventually (when the point B was reached) the material fractured.

It should be noted that once the material was taken out of the elastic regime (into the non-elastic regime, where plastic flow occurred) the body suffered a permanent deformation or **permanent set**, i.e. removal of the stress did not reduce the strain to zero.

The behaviour described above for mild steel is not typical of all materials. Materials that behave approximately like this, showing elastic behaviour and plastic flow, are called **ductile**.

Demonstration

Other materials, such as concrete, do not flow plastically; such materials are called **brittle**.

1-3 HOOKE'S LAW and ELASTIC MODULI

As we have seen, when a material is stressed there are basically two different regimes: the elastic and the non-elastic. The latter is difficult to describe in a way which is easily applicable but in the former the stress is proportional to the strain.

*This proportionality between stress and strain is known as **Hooke's law***; it applies to all of the three basic deformations. Hence the ratio stress/strain is a constant; this constant is known as the **elastic modulus**. There are three elastic moduli, one for each of the three basic deformations.

Linear Extension

$$\begin{aligned}\frac{\text{Stress}}{\text{Strain}} &= \frac{F/A}{e/L} \\ &= Y \\ &\text{named Young's modulus}\end{aligned}$$

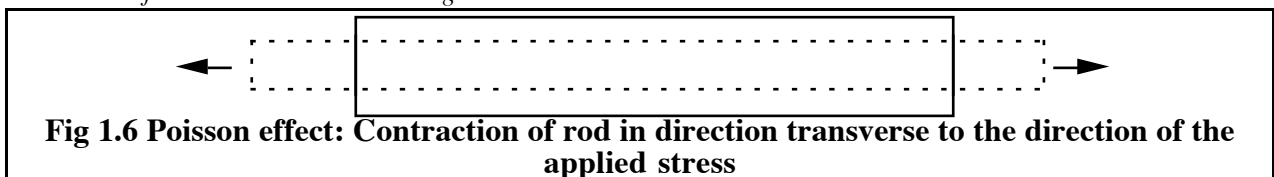
Uniform Compression

$$\begin{aligned}\frac{\text{Stress}}{\text{Strain}} &= \frac{\Delta P}{\Delta V/V} = \Delta \frac{pV}{V} \\ &= k \\ &\text{named bulk modulus}\end{aligned}$$

Shear

$$\begin{aligned}\frac{\text{Stress}}{\text{Strain}} &= \frac{F/A}{\Delta} = \frac{F}{A\Delta} \\ &= n \\ &\text{named shear modulus or modulus of rigidity}\end{aligned}$$

<< There is one other parameter which is necessary to describe the elastic behaviour of materials. This is **Poisson's ratio**. When a body is linearly extended, it contracts in the direction at right angles. *Poisson's ratio, ν , is the ratio of the lateral strain to the longitudinal strain.*



>>

A Microscopic Model

The values of the various parameters we have defined must depend on the microscopic structure of the material. In the unstressed state the atoms or molecules are in equilibrium positions, such that if

they are pulled apart the forces between them are attractive and if they are pushed together the forces are repulsive.

Where these forces as a function of distance between the atoms or molecules are known, one could, in principle, calculate the elastic moduli. Such calculations can and have been made, particularly for crystals, where there is a regular array of atoms. However, the values obtained are always too high, due to the presence, in even the purest crystals, of imperfections such as dislocations and impurity atoms.

1-4 EXPERIMENTAL MEASUREMENT OF ELASTIC MODULI

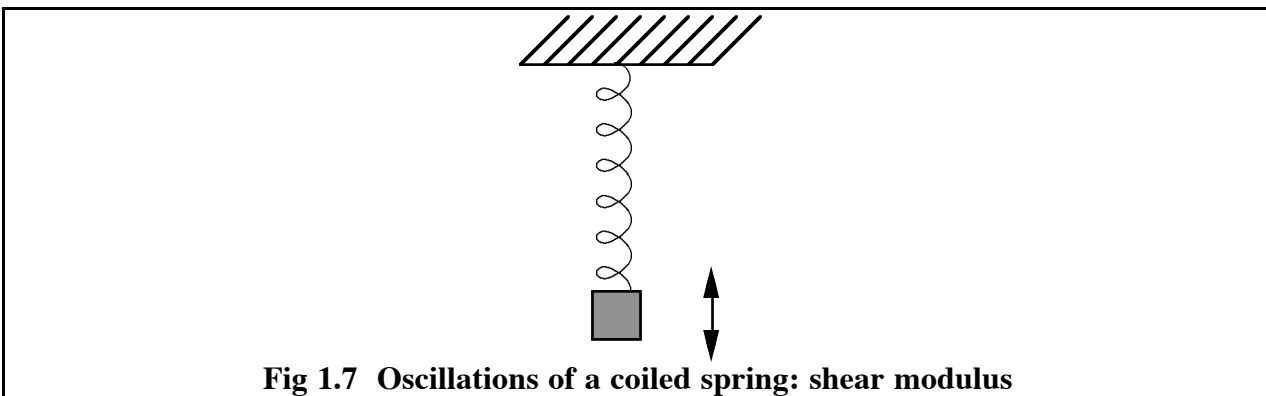
The elastic moduli can be determined in two basically different ways.

The most direct way is to use one of the engineering-type machines you have seen and to measure the strain appropriate to different stresses.

An alternative method is to make use of the fact that the mechanical oscillations of bodies and the characteristics of pressure waves propagating through them depend on the elastic moduli.

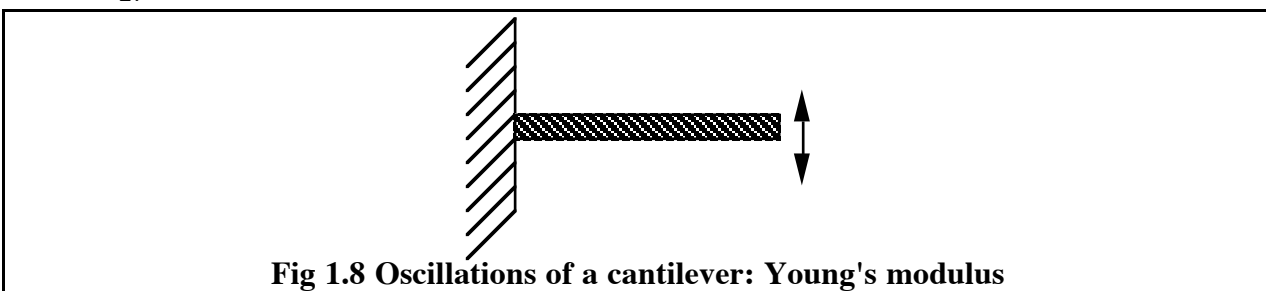
Demonstration

1.



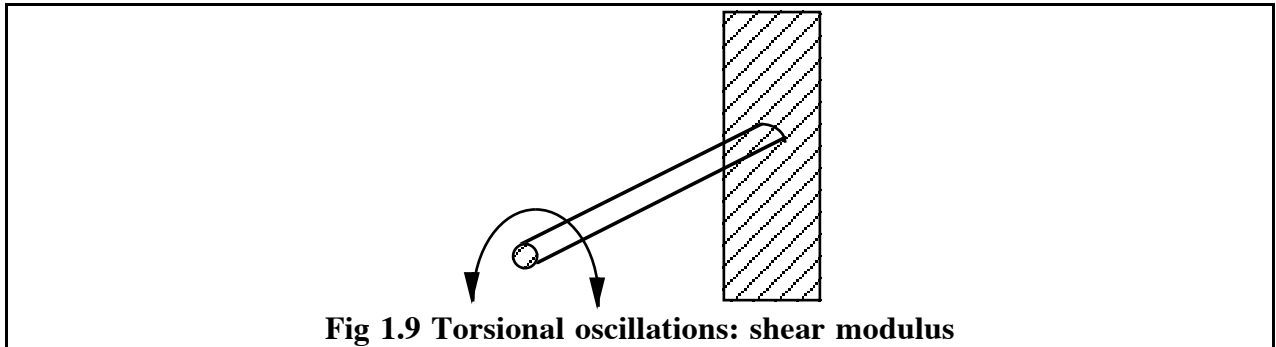
The frequency of oscillation of a coiled spring is determined by the **shear modulus** of the material of which it is made.

2.



The oscillations of a cantilever are determined by its **Young's modulus**.

3.



The torsional oscillations of a rod are determined by its **shear modulus**.

This alternative method can be particularly useful when it is not possible to obtain a sample suitable for the test machines. Investigations of possible changes in the elasticity of bones in the body with age and disease have been made, for example, by setting the bones into oscillation and measuring the oscillation frequencies.

The propagation characteristics of a pressure wave are determined by the **bulk modulus** of the material in which it is propagating. This will be discussed in more detail in the lecture on sound (chapter PM7).

A table of the elastic moduli of various materials is included in the post-lecture material.

1-5 APPLICATIONS

Repair of the Human Body

Many materials are used in the repair of the body. The prime consideration in these applications is that the materials be strong enough. External to the body there are, for example, artificial limbs, and internal to the body there are, for example, plates used for repairing fractures. In these latter applications the materials must also be bio-compatible as well as strong enough.

The strength of a material is defined as that stress which causes the material to break.

For some materials this breaking stress will be different under compression (**compressive strength**) than under tension (**tensile strength**) or under shear (**shear strength**).

Demonstration

An example of a material used for repair of the body is material used for filling teeth. One such material is "composite". This is a polymer mixed with quartz and is quite strong in compression, as it must be, since large compressive stresses are experienced in biting. Its compressive strength is about 2.5×10^8 Pa which compares favourably with that of tooth enamel viz 4×10^8 Pa. Since as well it looks like tooth enamel, it is a very suitable material for anterior fillings.

If the strength of a material is exceeded it will fail. It is interesting that a material can fail at stresses much less than this if the stress is applied and removed a large number of times. This phenomenon is known as **fatigue**.

Demonstration

Dentures for example can fail by fatigue.

Bones etc. as Structural Elements

The basic point in designing any element to withstand stress is to properly assess what the stresses are. The element is then designed so as to withstand these stresses without being unnecessarily big.

Weight bearing structures which occur in nature are of good design. Of particular interest in this regard are trees. These are basically columns and are in a state of compression due to their own weight. One might think that their heights would be limited only by the requirement that the compressive strength be not exceeded; thus no relationship between height and diameter would be expected.

This, however, is not the case. A column fails not by compression but by bending. Failure occurs when the tree's length becomes too great in comparison with its diameter.

Demonstration

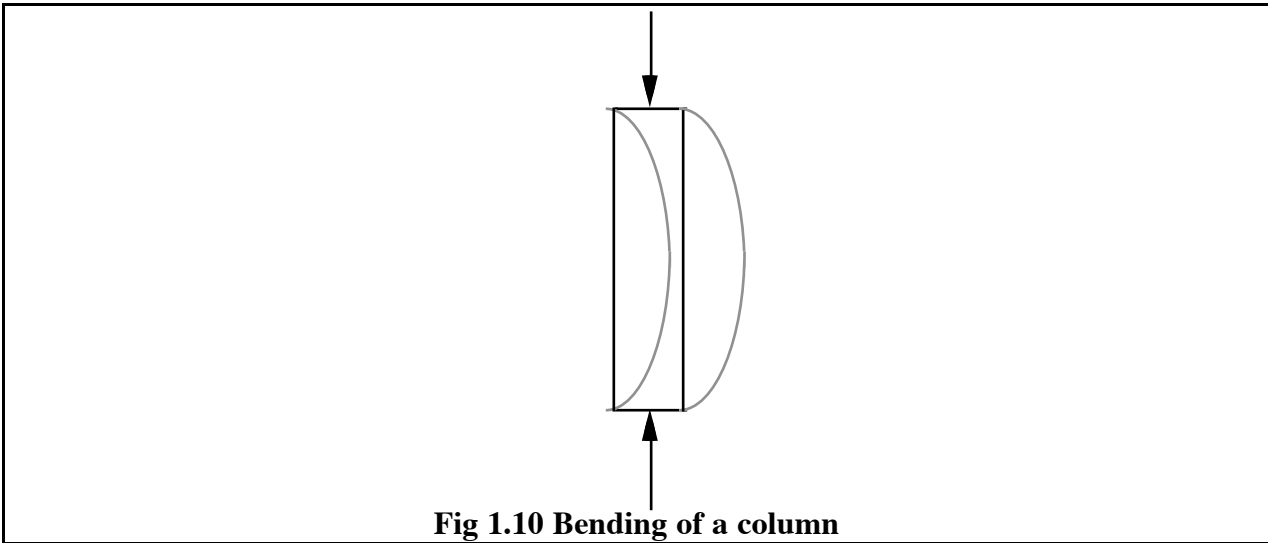


Fig 1.10 Bending of a column

To prevent this failure by bending the diameter should increase as the $3/2$ power of length. This is observed on average for trees.

Demonstration

Scaling, with this same relation, is also observed for bones of animals.

Scaling is not the only good design feature found in bones.

Demonstration

For a given weight/unit length, beams of cross-section such as these

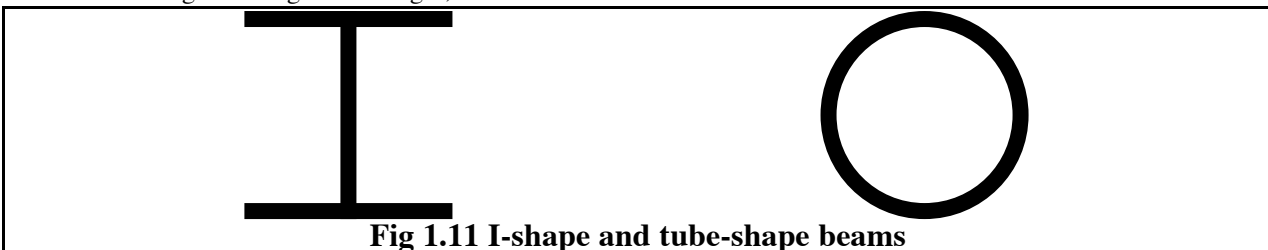


Fig 1.11 I-shape and tube-shape beams

are much stronger against bending than solid beams such as this.

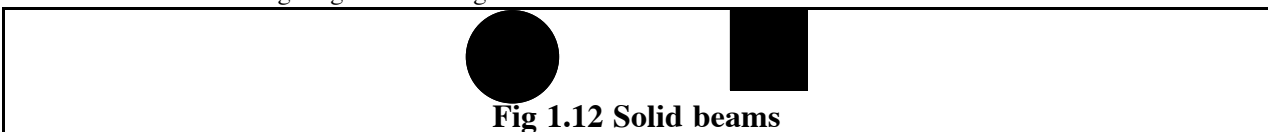


Fig 1.12 Solid beams

Demonstration

Many bones indeed are of tubular shape. In others their good design leads to the bone being arranged differently: it is all a matter of the nature of the stresses. For example, in the top of the femur, the bone is arranged in thin sheets separated by marrow, the sheets being so arranged to give the greatest strength when the bone is experiencing those forces to which it is normally subjected.

Demonstration

No matter how well-designed bones are, they will fracture when the strength of the bone material is exceeded. This is most likely to happen when the bone is stressed in a direction other than usual i.e. when it is stressed in a way for which it was not designed.

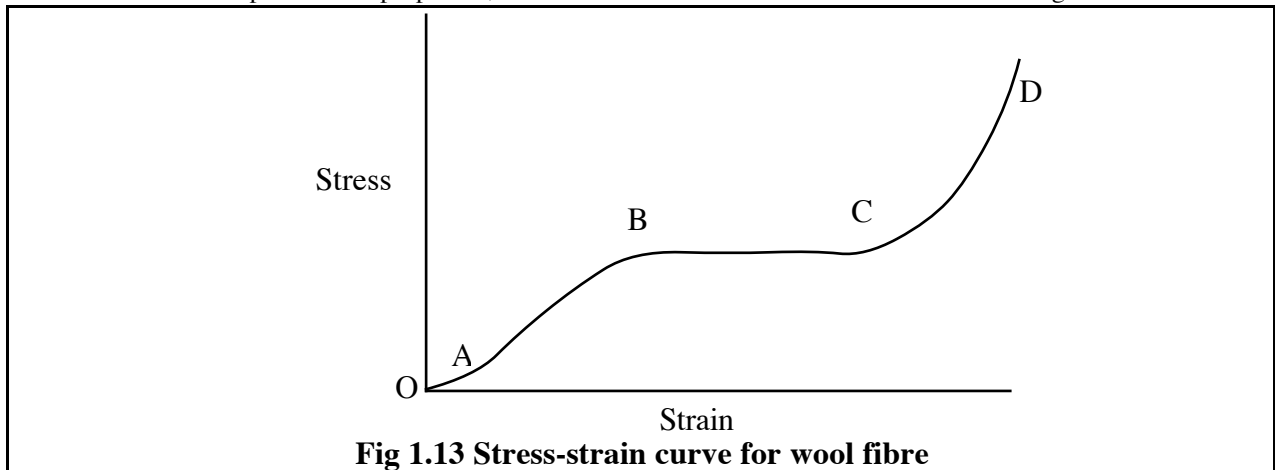
Fibres

The elastic properties of bone and timber are different in different directions. This is so because these materials are fibrous. There are many fibrous materials in nature. One important class of fibres are those used in making textiles: the natural fibres wool and cotton and the various synthetic fibres.

The elastic properties of these fibres is obviously important in that they determine the properties of the textiles made from them.

Demonstration

As an example of these properties, the stress-strain curve for a wool fibre in tension is given.



The narrow region OA corresponds to the "crimp" in the fibre being removed. This region is followed by a linear region AB. As the stress is further increased the curve flattens out into the region BC. If the stress is removed in this region the strain returns to zero. Therefore this region does not correspond to a region of plastic flow, as for steel. It results from the long keratin molecules, of which the fibre is composed, changing from a coiled shape to a more extended one. With further stress, the curve again rises and finally the fibre ruptures at the point D.

Arteries and the Lung

Strong fibrous materials, such as bone, are common in the body. There are other materials in the body where strength is not the important thing but stretchability. The walls of the arteries fall into this category. It is only because they are elastic that the blood flow is smooth.

Demonstration

As the heart pumps, the pressure in the arteries increases and the artery walls stretch. When the aortic valve shuts and the pressure in the arteries drops, the walls relax maintaining the blood flow. The hardening of the artery walls, which occurs with age, inhibits this process.

The elasticity of the lung tissues plays a very significant part in respiration. Muscular effort is required in inspiration to extend the lungs but expiration is mainly due to the relaxing of the stretched tissues.

Demonstration

The stretching can be shown by measuring the pressure to fill the lungs with air. If the lungs are filled with saline solution, a much lower pressure is required. This difference is because forces associated with surface tension play a large part in the operation of the lung; when the lung is filled with saline solution these forces do not act. If the lung is washed out with kerosene and the experiment of inflation with air is repeated, it is found a much higher pressure is required than before. The kerosene washes out a chemical known as "surfactant" which regulates the surface tension. When surfactant is present it decreases the surface tension during inspiration. (This will become clearer after the surface tension lecture - PM2 - when more details will be given.)

Muscle and Skin

Demonstration

Other tissues in the body where stretchability is important are muscle and skin. The skin's elasticity decreases noticeably with age.

POST-LECTURE

1.6 UNITS

You will have noticed that in the television lecture, on occasions, units other than SI. units have been used.

The correct unit, as agreed by the International Conference on Weights and Measures, for stress (or strength, or any of the elastic moduli) is the pascal (Pa). That unit is used exclusively in these notes.

1.7 TABLES

The calculated values are based on microscopic models. The lack of correspondence is the result of dislocations and impurity atoms.

MATERIAL	TENSILE STRENGTH / 10^8 Pa	
	THEORETICAL	OBSERVED
rock salt (NaCl)	2.7	0.004 (bulk material) 1.0 (single crystal)
iron	46	2.0 (bulk) 14.0 (single crystal)
cellulose	11	1.0 (fibre)

For liquids and gases the shear modulus is zero; for liquids the bulk modulus is about the same value as for solids but it is much smaller for gases.

SUBSTANCE	$Y/10^{10}$ Pa	$k/10^{10}$ Pa	$n/10^{10}$ Pa
aluminium	7.05	7.46 2.67	
steel	19 - 21	16.4 - 18.1	7.9 - 8.9
glass (crown)	6.5 - 7.8	4.0 - 5.9	2.6 - 3.2
water		0.2	0
mercury		2.1	0
air (atmospheric pressure)		1.4×10^{-5}	0

1.8 TENSILE AND COMPRESSIVE MODULI

A crystalline solid exhibits the same stress vs. strain relation whether it is under tension or compression. On the other hand, bone and other biological materials show different behaviour under tension and compression.

1.10. PROBLEMS

Q1.1 The effective cross sectional area of a horse's femur (leg bone) is $7.0 \times 10^{-4} \text{ m}^2$ and the Young's modulus of this bone is $8.3 \times 10^9 \text{ Pa}$.

Calculate the strain that occurs in the femur when the horse (mass $\sim 600 \text{ kg}$) puts its full weight on one leg.

Q1.2

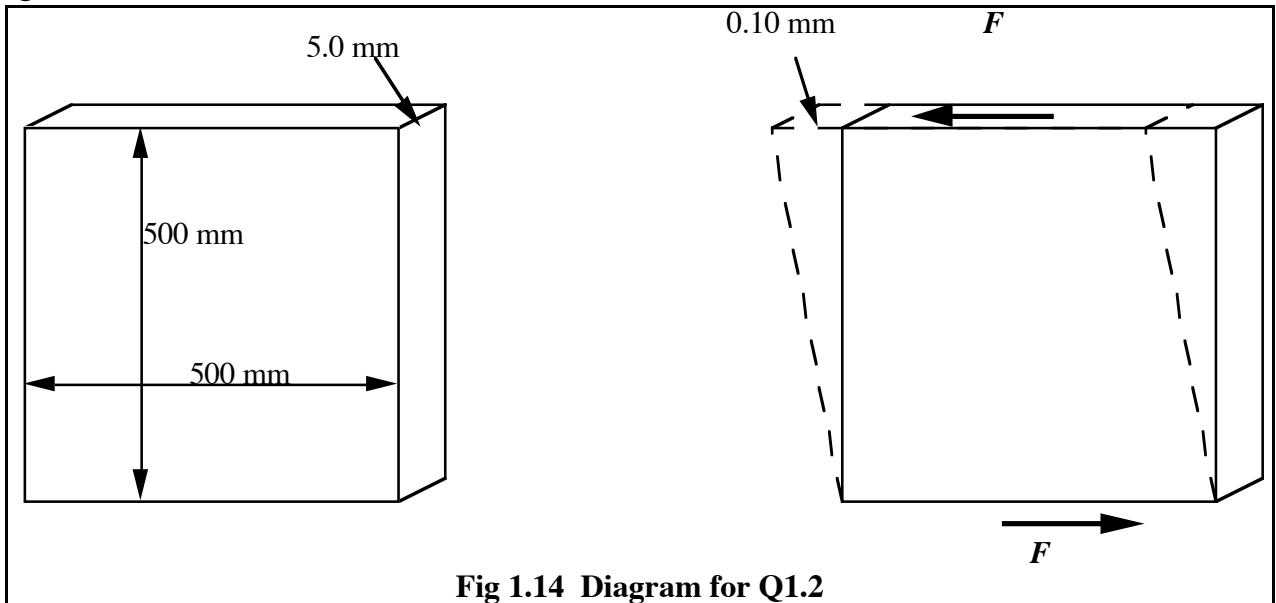


Fig 1.14 Diagram for Q1.2

The square brass plate shown is sheared to the position of the dotted lines by the forces F . The distortion is exaggerated, for clarity, in the diagram. Calculate the magnitude of these forces.

The shear modulus of brass is $3.5 \times 10^{10} \text{ Pa}$.

Q1.3 By what fraction does the density of water at a depth where the pressure is $4 \times 10^5 \text{ Pa}$ increase over the surface density.

The bulk modulus of water is $2 \times 10^9 \text{ Pa}$.