

Workshop Tutorials for Physics

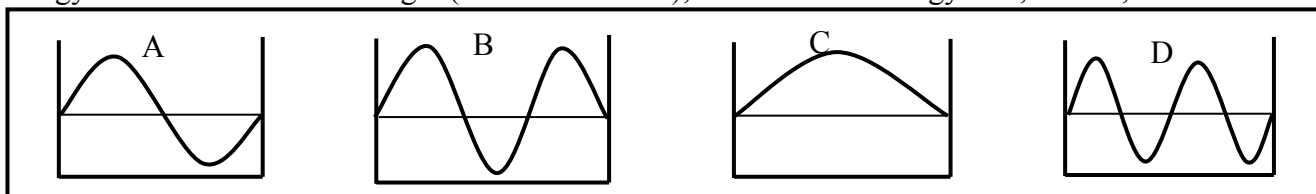
Solutions to QR3: Wave Functions II - Particles in Boxes

A. Qualitative Questions:

1. Shown below are four wave functions of an electron in an infinite potential well.

a. In order of increasing energy of the electron: C, A, B and D.

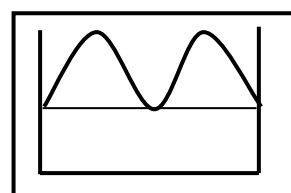
The wave function goes like $\psi = A \sin(n\pi/Lx)$, so C is the ground state with $n = 1$, A has $n = 2$ etc. The energy increases with increasing n (it increases as n^2), so the lowest energy is C, then A, B and D.



b. Diagram A above shows the first excited state, the probability density, shown opposite, is the square of this wave function

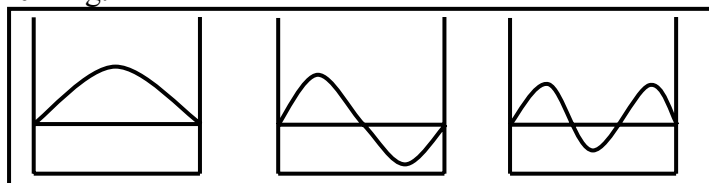
The electron is now replaced with a proton.

c. The energy is given by $E_1 = h^2 / 8mL^2$ where L is the length of the well and m is the mass of the trapped particle.



The proton's zero point energy will be around 2000 times lower than the electron's, because a proton has mass 1.7×10^{-27} kg and an electron has mass 9.1×10^{-31} kg.

d. The wave function will be the same for both, $\psi = A \sin(n\pi x/L)$, which does *not* depend on energy or mass. The wave functions for the first three energy levels are shown.



2. Confinement of particles.

a. When a particle is confined to an infinite well (for example our simple model of an electron trapped in the electric field due to a nucleus) then the probability of finding the particle outside the well must be zero. In our quantum mechanics model a wave is used to represent the probability of finding the particle at a given place. The wave must be zero, i.e. have nodes, at the walls of the well (and be zero outside the well). A standing wave pattern, such as can be observed on a plucked guitar string, fits this model. Since there can only be certain modes of vibration for the standing wave, this results in certain fixed energy levels (discrete rather than continuous) for the trapped particle.

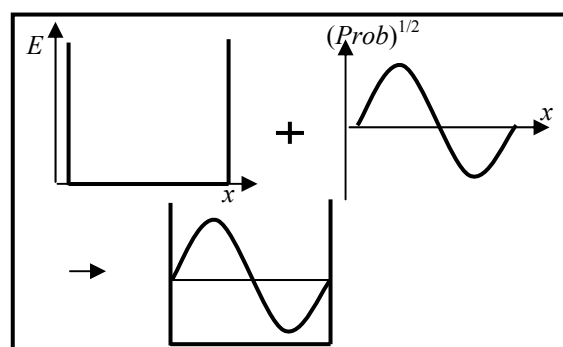
b. The total energy of a confined electron is $E_e = n^2 h^2 / 8mL^2$ and cannot be zero unless n is zero. If n is zero then the wave function, $\psi = A \sin(n\pi x/L) = 0$ would be zero, if the wave function is zero then the probability density (which is the square of the wave function) must also be zero. Hence there is zero probability of finding an electron in a potential well with zero energy!

B. Activity Questions:

1. Potential Wells and Wave functions

a. The horizontal axis represents distance. The vertical axis represents potential energy, overlaid on top of this is the wave function. This is really two diagrams in one, a potential well and a wave function on top of that.

b. The wave function represents a standing wave in the potential well. An electron in a potential well can be modelled as a standing wave, and the square of the wave function at a given point gives the probability of finding the electron at that point.



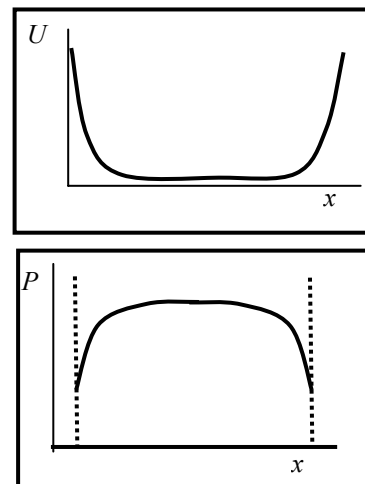
2. Classical particle in an elastic potential energy well

a. The elastic potential energy, U , of a spring-mass system is proportional to $(\Delta l)^2$ where Δl is the compression (or extension) of the spring away from its equilibrium length. The spring is compressed by the glider, and is otherwise at equilibrium. The potential energy as a function of glider position, x , is therefore zero except where the glider compresses the springs at either end of the track, at which positions it is proportional to $(\Delta l)^2$.

b. For a given interval in space, the glider spends more time near the ends of the track as it must be slowed down and change direction, then be accelerated away again at the track ends. The surface is approximately frictionless so the glider travels at constant speed once it is no longer in contact with the end springs.

c. The probability density, P , is shown opposite, the dotted lines show the ends of the air track.

d. The probability density for an electron in a well is the opposite to this for the electron's ground state, and in general is a minimum (zero) at the edges of the well.



3. Waves on a string

a. When an electron is bound to an atom it behaves as if it is trapped in a potential well. There is a coulomb attraction between the nucleus and the electron, and the electron will have lower potential energy closer to the nucleus. The electron becomes trapped in the well and is bound to the atom. However it is not an infinite potential well because it is possible to give an electron enough energy to free itself from a nucleus (ionisation), thus exiting the potential well.

b. The electrons can be modelled as standing waves in a potential well. The well is curved (Coulombic) rather than square and the size of the well gives the size of the orbitals. This is analogous to being trapped in a gravitation potential well, for example falling into a hole.

c. The electron is both a particle and a wave, and for it to fit into the potential well it must have the right wavelength as a standing wave. This restricts the possible energies of the electrons to discrete values, hence the energy is quantised.

C. Quantitative Questions:

1. A pollen grain of mass 2.0 mg moves back and forth under a microscope between two glass slides.

a. The pollen grain travels 0.05×10^{-3} m in 90 s, hence it has a speed of

$$v = d/t = 0.05 \times 10^{-3} / 90 = 5.6 \times 10^{-7} \text{ m.s}^{-1}$$

It's kinetic energy is therefore $K = \frac{1}{2} mv^2 = \frac{1}{2} \times 2.0 \times 10^{-6} \text{ kg} \times (5.6 \times 10^{-7} \text{ m.s}^{-1})^2 = 3.1 \times 10^{-19} \text{ J}$.

This energy is equal to $E = \frac{h^2 n^2}{8mL^2}$ which we can rearrange to solve for n :

$$n = \sqrt{(E \times 8m)L} / h = \sqrt{(3.1 \times 10^{-19} \text{ J} \times 8 \times 2.0 \times 10^{-6} \text{ kg}) \times 0.05 \times 10^{-3} \text{ m}} / 6.63 \times 10^{-34} \text{ J.s} = 1.7 \times 10^{17}.$$

b. No. The average probability of being able to see the pollen grain with the microscope at any point is NOT zero.

c. The probability is proportional to the wave function squared, and hence must always be positive. The total probability is usually set to 1 (normalised), as the grain must be seen somewhere.

2. A particle which is confined has quantised energy.

a. using $E_n = \frac{h^2 n^2}{8mL^2}$

with $L = 1.4 \times 10^{-14}$ m and $m = m_e$, we get $E_1 = 3.07 \times 10^{-10} \text{ J} = 1920 \text{ MeV}$.

b. Repeat using same L , but $m = m_p$, where $m_p \sim 2000 m_e$, which gives $E_1 \sim 1 \text{ MeV}$.

c. Given that it takes $\sim 2000 \text{ MeV}$ to bind an electron into the nucleus, compared to $\sim 1 \text{ MeV}$ for a proton, we would not expect to find electrons in the nucleus. (Although an electron can *come out* of the nucleus when a neutron transforms into a proton and an electron, this is called β^- decay.)