1 Introduction

In a sense this section is the magnetic equivalent of Part 1, in which we dealt with dielectrics. There we dealt with charges, here we deal with currents. Moreover, we saw in Part 1 that the way to include a dielectric material to model the atoms as electric dipoles, and we then saw how the dipoles affect the various electric field relations we knew to hold for a vacuum. As part of this effort we introduced the polarization \( \mathbf{P} \) and its associated bound charge, and the displacement field \( \mathbf{D} \), which, to some degree, only depends on the free charges. All of these fields are tied together by a constitutive relation, which defines the type of material that is being dealt with.

We will see that dealing with magnetic materials is somewhat similar: these materials can be modeled as containing magnetic dipoles, and these magnetic dipoles lead to bound current and to a magnetization field \( \mathbf{M} \). In analogy to the electric field \( \mathbf{E} \), the magnetic field \( \mathbf{B} \) is directly related to all currents, free and bound. It is therefore convenient to introduce an auxiliary field, the magnetic intensity \( \mathbf{H} \), which, to some degree, only depends on the free currents. Again, a constitutive relation is needed to tie all these fields together.

A key difference with Part 1 is that magnetic problems are mathematically more difficult to deal with than electric problems. For example, in Part 1 we could use a one-dimensional treatment to get the key results relating the polarization \( \mathbf{P} \) to the density of bound charges. We will see here that a magnetic dipole corresponds to a small current loop, an intrinsically two-dimensional geometry and thus a one-dimensional treatment cannot work. This part is therefore more qualitative than Part 1.

Before studying magnetic materials, it is worth recalling several of the key concepts from last year which apply to magnetic fields in vacuum.

1.1 Ampere’s law

Ampere’s law states that the line integral of magnetic field \( \mathbf{B} \) around a closed loop is proportional to the current through the loop, where the constant of proportionality is \( \mu_0 \), the permeability of free space. In SI units, \( \mu_0 \) has the numerical value \( \mu_0 = 4\pi \times 10^{-7} \text{ N} \cdot \text{A}^{-2} \). The magnetic field can be expressed as an integral equation as follows,

\[
\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i. \tag{1}
\]

Note that this differs from Gauss’ law in that Gauss’ relates the integral over a closed area to the charges contained inside the volume that is bound by the area. In contrast, Ampere’s law relates an integral over a closed loop, to the current through the surface that is bound by it. The subtlety in applying Ampere’s law is the current on the right-hand side. If we were to apply it to a geometry as in Fig. 5 (and we forget the ferromagnetic core for the moment), then the relevant current is not the current through the wire.
Rather, the total current going through the surface bounded by the dashed circle is the current through the wire multiplied by the number of turns. It is for this reason that magnetic devices often have many turns.

As discussed, the current could be in one or more wires passing through the loop, but it could also be extended across a continuous medium such as a plasma. In its most general form we can write the current as an integral of current density over the area of any surface bounded by the loop.

\[ i = \int \mathbf{J} \cdot d\mathbf{A}. \]  

(2)

As for Gauss’s law, Ampere’s law can be used to obtain expressions for magnetic field for simple geometrical configurations. Example include the field due to an infinite wire, an infinite linear solenoid, and a toroidal solenoid.

1.2 Faraday’s law

Faraday’s law is in a sense the electric equivalent of Ampere’s law. Whereas the latter involves a line integral over the magnetic field, we will see that the former involves such an integral over the magnetic field. In first year Faraday’s law was written as

\[ \mathcal{E} = -\frac{d\Phi_B}{dt} \equiv -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A}, \]  

(3)

where \( \Phi_B \) is the magnetic flux, the definition of which leads to the second equality. Recall that \( \mathcal{E} \) is the emf (electromotive force), which in essence comes down to a voltage when we go through the circuit. Loosely speaking, Faraday’s law thus describes the generation of electricity by a time-varying magnetic field, the way almost all electricity in the world is generated.

1.3 Biot-Savart law

This is the magnetic analog of the electric field due to a point charge in electrostatics, i.e. Coulomb’s law. In principle the magnetic field of any current configuration could be determined by integrating over current elements. The field \( d\mathbf{B} \) at a distance \( r \) from a current element \( idl \) is given by

\[ d\mathbf{B} = \frac{\mu_0 i d\mathbf{l} \times \mathbf{r}}{4\pi r^3}, \]  

(4)

which can again be generalized to deal with cases in which the current is extended over a continuous medium.

Exercise

Test your integration skills by deriving the magnetic field at a distance \( r \) from an infinitely long wire carrying a current \( i \) directly from the Biot-Savart law. Recall that Ampere’s law readily yields the result \( B = \mu_0 i/(2\pi r) \).

1.4 Gauss’ law for magnetism

Recall that according Gauss’ law for electrostatics, electric field lines start on the positive charges and finish on the negative ones. Therefore if a closed surface contains net positive charges, there must be a positive flux out of the surface. Recall also that although we formulated Gauss’ law originally for free charges, we saw that it also holds when bound charges are included. The magnetic equivalent which we are considering here is similar, except that there are no magnetic charges. In fact, the two poles of a magnet always come in combination. Therefore, in analogy with the electric case we conclude that

\[ \oint \mathbf{B} \cdot d\mathbf{A} = 0, \]  

(5)

that is, magnetic field lines are closed.
2 Differential form of Ampere’s law

We have seen that the differential form of Gauss’s law relates the electrostatic field at a point to the charge density at that point. By analogy, we would expect to find a differential form of Ampere’s law that relates the magnetic field at a point to the current density at that point. The derivation, given in Appendix A, yields an equation involving the vector cross product of the vector differential operator \( \vec{\nabla} \equiv (\partial/\partial x, \partial/\partial y, \partial/\partial z) \) and \( B \) (called curl \( B \))

\[
\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}.
\]  

(6)

Using the determinant representation of the cross product we can write this as

\[
\begin{vmatrix}
  i & j & k \\
  \partial/\partial x & \partial/\partial y & \partial/\partial z \\
  B_x & B_y & B_z \\
\end{vmatrix} = \mu_0 \left( J_x, J_y, J_z \right) 
\]

(7)

or in component form

\[
\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x,
\]

(8)

\[
\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 J_y,
\]

(9)

\[
\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 J_z.
\]

(10)

One way to understand the curl is to imagine the vector field by drawing the strength and direction of the field as a vector at each position. If a small mill put in this field would start rotating then the field has a curl. If the mill does not turn then the curl of the field vanishes. An example is given in Fig. 1, showing a windmill in a vector field. The field tends to turn the mill clockwise, indicating that the curl is nonzero.

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Recall that in Part 1 we saw that Gauss’ law is equivalent to the \( \vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 \), which involved the divergence of the field. We see here that Ampere’s law involves the curl, another type derivative. The fact that the generalization of the derivative becomes more complicated when dealing with vector fields in three dimensional space is perhaps not surprising. What is perhaps surprising is that this generalization only involves divergences and curls and no other types of derivatives.

3 Magnetic dipole moments

A current loop of radius \( a \) produces a magnetic field at position \( r \), where \( r \gg a \) which has the same pattern as the electric field of an electric dipole. A current loop is, therefore, referred to as a magnetic dipole.

To take a simple situation, consider the magnetic field on the axis of the current loop of radius \( a \) at a distance \( r \) from the loop, as shown in Fig. 2. As with the electric dipole, we take \( r \) as the position vector of the field point with respect to the centre of the dipole. The horizontal components of \( \vec{d}\vec{B} \), the largest components of the magnetic field, sum to zero. In contrast the much smaller vertical components add, giving

\[
dB_v = \frac{\mu_0}{4\pi} \frac{idl' r' \cos \theta}{r'^3} = \frac{\mu_0}{4\pi} \frac{idl}{(a^2 + r^2)^{3/2}}.
\]

(11)
Using \( \int dl = 2\pi a \) and taking \( r \gg a \) we obtain

\[
B = \frac{\mu_0}{4\pi} \frac{2i\pi a^2}{r^3} = \frac{\mu_0}{2\pi r^3} iA.
\]  \hspace{1cm} (12)

The essential point is that opposite parts of the current loop have contributions that almost cancel but not quite. Namely, the large horizontal contributions cancel, whereas the much smaller vertical components add up. As the point of observation moves away from the current loop, the horizontal field components becomes increasingly large and the vertical components increasingly small. Therefore, just as in the case of an electric dipole, the field of the loop increases more rapidly than the field of a small segment of the loop. It is therefore perhaps not surprising that the magnetic field has a \( r^{-3} \) dependence.

Note that Eq. (12) has the same form as the electric field due to an electric dipole at a distant point on the line joining the charges, with the dipole moment \( p \) replaced by \( iA \) where \( A = \pi a^2 \) is the area of the current loop. By analogy this quantity is called the magnetic dipole moment. As a vector quantity the magnetic dipole moment is defined as

\[
m = iA,
\]  \hspace{1cm} (13)

and so it points in the direction orthogonal to the loop, given by by the right-hand rule.

Although at large distances from the respective dipoles, the electric and magnetic field lines are geometrically identical, at close distances they are very different: the electric field lines start at the positive charge and terminate at the negative charge whereas the magnetic field lines are continuous loops.

If we adopt the same geometry for both magnetic and electric dipoles, as shown in Fig. 3, we can translate the vector expression for the electric field (at large distances) due to an electric dipole to an equivalent one for the magnetic field (at large distances) due to a magnetic dipole, by making the substitutions

\[
\frac{1}{4\pi \epsilon_0} \rightarrow \frac{\mu_0}{4\pi}, \hspace{1cm} p \rightarrow m,
\]  \hspace{1cm} (14)

yielding

\[
B = \frac{\mu_0}{4\pi} \frac{3(m \cdot r)r - r^2 m}{r^5}.
\]  \hspace{1cm} (15)
3.1 Torque on a magnetic dipole

By analogy with the case of the electric dipole we can see that the torque on a magnetic dipole in a uniform magnetic field is given by

$$\tau = m \times B$$

(16)

If the field is nonuniform, then the magnetic dipole also experiences a force. If the dipole has a component parallel to $B$, the force is directed towards the region of stronger magnetic field; if the dipole has a component antiparallel to $B$ the force is directed towards regions of weaker magnetic field.

**Exercise**

By considering current loops in a diverging or converging magnetic field, show that the forces on a magnetic dipole in a non-uniform magnetic field are as stated above.

3.2 Potential energy of a magnetic dipole

By analogy with the case of the electric dipole we can see that the potential energy of a magnetic dipole due to its orientation with respect to a magnetic field is given by

$$U = -m \cdot B$$

(17)

Thus the potential energy is a maximum when the dipole moment is antiparallel to $B$, and minimum when it is parallel to $B$. The torque on a dipole in a magnetic field acts to align it with the field where the potential energy is a minimum. If there is no friction the dipole oscillates about this position; in the presence of friction it settles down to an equilibrium where $m$ is parallel to $B$.

4 Magnetic properties of materials

We earlier studied electric dipoles since they are the key to understanding the response of dielectric media to an applied electric field—the dielectric medium can be thought of as consisting of dipoles which are induced by the applied field and which leads to the macroscopic polarization $P$. We also saw that dipoles lead to bound charges which contribute to the electric field via Gauss’ law. Similarly, constituent atoms and molecules which have a dipole moment even without an applied field align with the field due to the applied torque. We will see a similar pattern for magnetic materials: A magnetic medium can be thought of as consisting of magnetic dipoles which are either induced by the applied magnetic field or, if they exist without the need for an applied field, they align with the field. This is further developed below.

The magnetization of a material $M$ is defined to be the magnetic dipole moment per unit volume. It is the magnetic analogue of polarization $P$. There are three kinds of magnetic materials: **diamagnetic**, **paramagnetic**, and **ferromagnetic**.
paramagnetic and ferromagnetic. All materials exhibit diamagnetism as it is associated with magnetic dipoles that are induced by the applied magnetic field; materials which contain atoms or molecules with intrinsic magnetic moments exhibit paramagnetism; a few of these materials exhibit ferromagnetism – the spontaneous alignment of magnetic dipoles to create a permanent magnetic moment per unit volume. In Part 1 it was quite obvious that an applied electric field leads to an induced dipole, but in magnetism it is not so clear that an applied magnetic field leads to an induced magnetic dipole. However, we can get somewhere by using Faraday’s law, discussed in Section 1.2. We leave the argument to be qualitative here. Faraday’s law is usually applied to some circuit; in contrast, here we apply it to an atom, in which the induced current comes about from the motion of the electrons around the nucleus. By Faraday’s law a time varying magnetic field leads to an EMF (electromotive force) and a subsequent current. The EMF corresponds to a voltage and thus to an electric field, which exerts a torque on the atom, which in turn changes its angular momentum. Thus the presence of the magnetic field, leads to a change in the angular momentum, and thus in a change of the electron’s motion. This is equivalent to a current, and therefore a change in the atom’s magnetic moment, which leads to the response field $M$.

The argument then closely follows that for dielectric media and is illustrated in Fig. 4. Briefly, and we return to this below, we need to find the macroscopic consequences of the microscopic currents shown in the figure. If these currents are all identical they cancel out everywhere inside the medium, except at the edges where a macroscopic, bound current forms. If there is a nonuniformity in the currents, for example by a nonuniformity in the atomic composition or density, internal macroscopic bound current are possible. Mathematically, bound currents are expressed in close analogy to the dielectric response, as

$$J_s = M \times \hat{n}, \quad J_i = \nabla \times M,$$

where the subscripts stand for “surface” and “internal”, respectively, and, as in Part 1, $\hat{n}$ is the normal pointing outwards from the medium. Since the magnetic moment in Fig. 4 points into the plane of the paper, the surface current is parallel to the interface, as expected. Note that $J_s$ and $J_i$ have different dimensions. $J_s$ is the bound equivalent of the free current density we encountered earlier and has units of $A \cdot m^{-2}$, whereas $J_i$, since it is confined to the surface, has units $A \cdot m^{-1}$. As in the discussion of dielectrics, the second of Eqs (18) is the more general one, with the first describing the particular situation in which the nonuniformity is a boundary.

![Figure 4: Schematic of the origin of the magnetization $M$ in a medium. An applied field induces a small current in each atom. The macroscopic quantity $M$ derives from the boundary of the medium and, within the medium, from variations in the current’s density or magnitude.](image)

We now wish to follow the same route magnetic materials as for dielectrics: magnetic materials lead to a bound current which can be found $M$, which, in calculating the magnetic field $B$ we need to add to the effect of free currents. It is then helpful to introduce an auxiliary field which we associate with the free current only. Thus, to describe the magnetic behaviour of materials we introduce a new field $H$, the magnetic intensity or simply the $H$ field (which is the magnetic equivalent of $D$). That is, the sources of
\( \mathbf{B} \) are both the free and the bound currents, the sources of \( \mathbf{M} \) are the bound currents. Thus \( \mathbf{H} \) is defined such that its sources are the free currents.

Recall that in vacuum, Ampere’s law can be written in differential form as (see Eq. (6))

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_f
\]

(19)

where the subscript \( f \) explicitly refers to the fact that in vacuum all charges are free. Now as discussed earlier, the sources of the magnetic field \( \mathbf{B} \) are all currents, free and bound. It is therefore natural to generalize Eq. (6) to

\[
\nabla \times \mathbf{B} = \mu_0 (\mathbf{J}_f + \nabla \times \mathbf{M}),
\]

(20)

in the presence of magnetic materials. Equation (20) represents the crucial step as all expression below follow naturally from it. To start, we substitute the second of Eqs (18) to get

\[
\nabla \times \mathbf{B} = \mu_0 (\mathbf{J}_f + \mathbf{M}),
\]

(21)

and we can thus write

\[
\nabla \times (\mathbf{B} - \mu_0 \mathbf{M}) = \mu_0 \mathbf{J}_f.
\]

(22)

As discussed in the previous paragraph, we introduce a field \( \mathbf{H} \), the sources of which are the free currents only. Thus suggest that if we define \( \mathbf{H} \) to be

\[
\mathbf{H} = \mathbf{B} - \mu_0 \mathbf{M}, \quad \text{so that} \quad \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}),
\]

(23)

then indeed the sources of \( \mathbf{H} \) are the free charges only, since

\[
\nabla \times \mathbf{H} = \mu_0 \mathbf{J}_f.
\]

(24)

We now know thus how, in principle, we need to deal with magnetic materials. To write the different forms of Ampere’s law in the integral form which we will use in this course we write, using the analogy between Eqs (1) and (6), we can then write in the presence of magnetic materials

\[
\oint \mathbf{H} \cdot d\mathbf{l} = i_{\text{free}}
\]

(25)

which shows that the component of \( \mathbf{H} \) in the direction of \( d\mathbf{l} \) is unaffected by bound currents. This does not imply that the same is true for the other components of \( \mathbf{H} \) though.

We now consider how the magnetic version of Gauss law is affected by the presence of magnetic materials. In fact, the argument is the same as in the electric case, namely that it remains valid provided that both free and bound charges and currents are included. Therefore, in the presence of magnetic materials,

\[
\oint \mathbf{B} \cdot d\mathbf{A} = 0.
\]

(26)

### 4.1 Constitutive relations

While we now know how formally to deal with magnetic materials, in practice we require a constitutive relation to define the properties of the medium. To this end we introduce the magnetic susceptibility \( \chi_m \), which is defined as

\[
\mathbf{M} = \chi_m \mathbf{H}.
\]

(27)

Therefore the relationships between \( \mathbf{B}, \mathbf{H} \) and \( \mathbf{M} \) are

\[
\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 (1 + \chi_m) \mathbf{H} = \mu \mathbf{H}
\]

(28)

where \( \mu = (1 + \chi_m) \mu_0 \) is the permeability of the material. The quantity \( \mu/\mu_0 = 1 + \chi_m \) is the relative permeability. The various kinds of magnetic properties of materials can be characterized by the value of \( \chi_m \). For both diamagnetic and paramagnetic materials \( |\chi_m| \ll 1 \), with negative values for diamagnetic materials; for ferromagnetic materials \( |\chi_m| \gg 1 \). We discuss these cases separately below.
4.1.1 Diamagnetism

Consider initially an atom or molecule which does not have a permanent magnetic moment. When it is placed in an external magnetic field \( \mathbf{B} \), a dipole moment is induced which is in the direction opposite to \( \mathbf{B} \). A material consisting of such atoms or molecules acquires a magnetic moment per unit volume in the direction opposite to \( \mathbf{B} \); this is the phenomenon of diamagnetism. Thus \( \mathbf{M} \) is antiparallel to \( \mathbf{B} \), and

\[
\mathbf{M} = \chi_m \mathbf{H} = \frac{\chi_m \mathbf{B}}{\mu_0 (1 + \chi_m)} \approx \frac{\chi_m \mathbf{B}}{\mu_0} \tag{29}
\]

since \( |\chi_m| \ll 1 \); moreover since \( \mathbf{M} \) and \( \mathbf{B} \) point in opposite directions, \( \chi_m < 0 \). Further, since the induced magnetic moment is antiparallel to the magnetic field, a diamagnetic material in a non-uniform magnetic field experiences a weak force towards the regions of weaker magnetic field.

4.1.2 Paramagnetism

Paramagnetism occurs in materials which contain atoms with a permanent magnetic moment mainly as a result of the spin of unpaired electrons, but there may also be a contribution from orbital angular momentum. In some atoms the total angular momentum of the electrons adds to zero. For atoms with an odd number of electrons, however, there is always an unpaired electron spin. Thus atoms with an odd number of electrons have permanent magnetic dipole moments.

When a magnetic field is applied the dipoles tend to align with the field, and thus for paramagnetic materials \( \chi_m > 0 \). The degree of alignment is determined by competition between the effect of the field and thermal effects – note the similarity here to electric dipoles in an electric field. The magnetisation is therefore also given by the Langevin equation with \( pE \) replaced by \( mB \). Thus the magnetisation of a paramagnetic material is

\[
M = Nm \left( \coth \left( \frac{mB}{kT} \right) - \frac{1}{mB/kT} \right) \tag{30}
\]

where \( m \) is the magnetic dipole moment of an atom and \( n \) is the atom number density. For small degrees of alignment \( (mB/kT \ll 1) \)

\[
M = Nm \frac{B}{3kT} \tag{31}
\]

Recalling that \( B \approx \mu_0 (H + M) \) and \( M = \chi_m \mathbf{H} \), we find that

\[
\chi_m = \frac{M}{H} = \frac{\mu_0 (1 + \chi_m) M}{B} \approx \frac{\mu_0 M}{B} \tag{32}
\]

since \( \chi_m \ll 1 \). Thus when \( mB/kT \ll 1 \), using Eq. (31)

\[
\chi_m \approx \frac{N m^2}{3kT} \tag{33}
\]

The Langevin equation is obtained with the assumption that dipoles can have all possible alignments with respect to the magnetic field. However the alignment of magnetic dipoles with respect to \( \mathbf{B} \) is quantized, since the orientation of angular momentum is quantized (“space quantisation”). More details of the quantum treatment are given in Appendix B. As \( \mathbf{M} \) is parallel to \( \mathbf{B} \) for a paramagnetic material, in the presence of a non-uniform magnetic field the paramagnetic material experiences a weak force towards regions of stronger magnetic field.

The third type of magnetic response is ferromagnetism. It is of important, and qualitatively so different from diamagnetism and paramagnetism that it the entire next section is devoted to it.

5 Ferromagnetism

A ferromagnetic material is one containing atoms with permanent magnetic dipoles which spontaneously align themselves. The alignment occurs naturally in domains, with different domains aligned in different directions, so that a typical sample of ferromagnetic material may exhibit no average magnetisation.
Application of an increasing magnetic field initially causes those domains with magnetisation directions close to the applied field to grow at the expense of others, and finally the direction of magnetisation turns to that of the field. The magnetisation has then reached its maximum possible value, called saturation, and further increase in the applied field produces no increase in magnetisation.

Ferromagnetic materials include the elements iron, cobalt, nickel and gadolinium at room temperature, as well as some alloys containing these elements and some alloys which do not. A necessary condition for a ferromagnetic material is that it contains atoms with a large magnetic dipole moment, but the structural arrangement is also important. For example the alloy MnBi is ferromagnetic at room temperature even though the elements Mn and Bi themselves are not (Bi is diamagnetic, Mn is paramagnetic). Quantum mechanics is required to explain why in some circumstances the presence of an atom with a permanent dipole moment leads to ferromagnetism, while in other circumstances the same atom produces paramagnetism. Beyond the Curie temperature $T_c$, a ferromagnetic material ceases to be ferromagnetic, reverting to being paramagnetic.

To explore the behaviour of ferromagnetic materials in more detail consider a toroidal coil wound on a ring of ferromagnetic material as shown in Fig. 5. The approach is similar to that in Part 1: Since the core is ferromagnetic, it is easiest to work initially with the magnetic intensity $H$, because its sources are free currents, which are unaffected by the magnetic properties of materials. Moreover, as discussed in Section 1.1, the current that goes through the surface is the product of the current through the wire and the number of turns. Once we have found $H$ everywhere, then we can find $B$ via the constitutive relation that characterizes the material.

Let us now apply this strategy to the geometry in Fig. 5. If the coil has $N$ turns and carries a (free) current $i$, the application of Ampere’s law for the magnetic intensity $H$ to the dashed path shown in the figure gives

$$\oint H \cdot dl = HL = Ni \quad \text{where} \quad L = 2\pi r.$$  \hfill (34)

Thus $H = Ni/L$ in the sample can be changed by varying the current in the coil. To find the magnetic field we use the constitutive relation for ferromagnetic media, according to which the magnetic field in the sample is $B = \mu_0(1 + \chi_m)H$. Here we come to another key difference between dielectric and magnetic systems: we saw in Part 1 that in the former $P \propto E \propto D$, which is essentially so because the applied electric field is usually weak compared to the electric field inside the the atom. In contrast, in magnetic systems the effect collective properties of many atoms, leading to ferromagnetism, is strong, and linear constitutive relations no longer apply.

For a ferromagnetic material the relationship between $B$ and $H$ is nonlinear (see Fig. 6) and at any point in the $B$–$H$ curve the effective value of the relative permeability $\mu/\mu_0 = 1 + \chi_m$ is much greater than unity, typically of the order of 1000. Thus the value of $B$ is determined predominantly by $M$, with only a minor contribution from $H$. With further increase of $H$ the value of $B$ saturates, because $M$ saturates with all dipoles now aligned with $B$. The effect of continuing increase in $H$ has a minimal effect.

If $H$ is reduced, the curve follows the path shown in Fig. 7. When $H$ reaches zero, $B$ remains non-zero, its value known as the remanence, $B_r$. If we now increase $H$ in the opposite direction (i.e. $H$ becomes negative), $B$ falls to zero at a value of $H$ called the coercivity, $H_c$. If $H$ is allowed to oscillate backwards and forwards the path followed is a loop, as shown in Fig. 7. This kind of behaviour, where the path followed is different for increasing and decreasing $H$ is known as hysteresis.
Figure 6: Magnetic field ($B$) as a function of magnetic intensity ($H$) for a ferromagnetic material. Note that the relative permeability $\mu/\mu_0 = 1 + \chi_m$ is not constant and is $\gg 1$. (From J.R. Reitz, F.J. Milford and R.W. Christy, *Foundations of Electromagnetic Theory*, Addison-Wesley (Reading MA, 1980))

Figure 7: A hysteresis loop for a ferromagnetic material
Work is done when a magnetic material moves around a hysteresis loop. Consider the toroidal coil in Fig. 5 as it moves around the loop. The current in the coil is related to $H$ by

$$H = \frac{Ni}{L}$$  \hspace{1cm} (35)

An emf is induced in the coil by the changing magnetic flux through the coil, given by

$$\mathcal{E} = -N\frac{d\Phi}{dt} = -NA\frac{dB}{dt}$$ \hspace{1cm} (36)

The instantaneous electric power dissipated is

$$\mathcal{E}_i = NA\frac{dB}{dt} \times \frac{HL}{N} = (AL)H\frac{dB}{dt}$$ \hspace{1cm} (37)

If we integrate instantaneous power over the time interval required to complete one circuit of the hysteresis loop, i.e., calculate the energy dissipated per hysteresis cycle,

$$\oint \mathcal{E} dt = (AL) \oint H\frac{dB}{dt} dt = AL \oint H dB$$ \hspace{1cm} (38)

where $AL$ is the volume of ferromagnetic material. Thus we find that the energy dissipated (as heat) per unit volume of material is the area of the loop, i.e. $\oint H dB$.  

Materials with large $H_c$, hard magnetic materials, are not easily demagnetised and are suitable for permanent magnets. Those with small $H_c$, called soft magnetic materials, are suitable for electromagnets – when the current is turned off little magnetisation remains. (We will see later that the resting place is not at $H = 0, B = B_r$, but in the second or fourth quadrant of the hysteresis loop where $H < 0$.) Soft materials are also suitable for transformers, as the small hysteresis loop area minimises power dissipated as heat in the core. Fig. 8 shows examples of hysteresis curves for magnetically hard and soft materials.

![Figure 8: Examples of hysteresis loops for a magnetically hard material (solid line) and a magnetically soft material (dashed line). (J.R. Reitz et al., as for Fig. 6)](image)

A material with permanent magnetisation can be demagnetised by repeated traversing the hysteresis loop while gradually reducing the amplitude of $H$. This degaussing process, is illustrated in Fig. 9.

$^1$This is a thermodynamic quantity with units of J/m$^3$, and is basically the magnetic equivalent of more common expressions such as $\oint PdV$.
Figure 9: The degaussing process for removing permanent magnetisation from a material. At the end of the process the material consists of many domains, aligned in different directions, such that the net magnetisation is zero.

5.1 Magnetic circuits

Ferromagnetic materials occur as components of magnetic circuits which channel magnetic flux. Consider the electromagnet shown in Fig. 10: The coil has $N$ turns and carries a current $i$. The length of the loop (the magnetic circuit) is $L$, and the length of the gap is $d$. For simplicity we assume that we can represent the magnetic behaviour of the soft magnetic core by a constant $\chi_m (\gg 1)$.

Magnetic flux ($BA$) is conserved from the material to the gap. This can be seen from the magnetic version of Gauss’ law, by taking the integration volume to be a thin box thin pill box that traverse the interface between the magnet and the air gap. Thus if the bulging of the $B$ field in the gap is negligible, we can take the values of $B$ in the material and in the gap to be the same. Thus $H$ in the gap is $H_g = B/\mu_0$, and in the material $H_m = B/(\mu_0(1 + \chi_m))$. We can now write down Ampere’s law in terms of $H$ and the free current $i$.

$$\oint \mathbf{H} \cdot d\mathbf{l} = H_m(L - d) + H_gd = Ni$$ (39)

which can be written in terms of $B$ as

$$\frac{B}{\mu_0(1 + \chi_m)}(L - d) + \frac{B}{\mu_0}d = Ni$$ (40)

from which we can obtain the value for $B$,

$$B = \frac{\mu_0 Ni}{d + (L - d)/(1 + \chi_m)}$$ (41)

Note that as $\chi_m \gg 1$ this can often be approximated as

$$B = \frac{\mu_0 Ni}{d}$$ (42)

showing that, provided the gap is small, $B$ in the gap is inversely related to the length of the gap. Eq. (41) is often written in terms of magnetic flux ($\phi = BA$) as

$$Ni = \phi \left( \frac{d}{\mu_0 A} + \frac{L - d}{\mu_0 (1 + \chi_m) A} \right)$$ (43)

More generally, when the toroid consists of different uniform sections, each of length $l_j$, cross section $A_j$ and magnetic permeability $\mu_j$, this can be written as

$$Ni = \phi \left( \frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2} + \cdots \right),$$ (44)
where each term in the brackets represents a section.

We can make an analogy here with electric circuits, where $Ni$ (amperes-turns) corresponds to emf or potential difference, $\phi$ corresponds to current, and $1/(\mu A)$ corresponds to resistance. The latter term is called the magnetic reluctance of the magnetic circuit. Low reluctance corresponds to high permeability. Thus magnetic flux tends to “flow” through regions of high permeability, as shown in Fig. 11 which shows the effect of placing a ferromagnetic sphere in a region of uniform magnetic field.

![Figure 11: The effect of placing a sphere of magnetic material in a uniform magnetic field. (From H.E. Duckworth, *Electricity and Magnetism*, Holt, Rinehart and Winston (New York, 1960))](image)

The same effect makes it possible to use ferromagnetic materials to magnetically shield a region of space. Fig. 12 shows the effect of a spherical shell placed in a region of uniform magnetic field. The field lines are channelled through the low reluctance shell leaving a magnetic-free region inside.

![Figure 12: A shell of ferromagnetic material acts as a magnetic shield, creating a magnetic field free region inside the shell. (From J.D. Jackson, *Classical Electrodynamics*, Wiley (New York, 1975))](image)
Exercise

Consider an electromagnet with \( N = 5000 \) turns in two windings, \( I = 1 \) A, \( A_m = 100 \) cm\(^2\), \( A_g = 75 \) cm\(^2\), \( \mu_r = 1000 \), \( L_m = 49 \) cm, \( L_g = 1 \) cm. Calculate

(a) The flux;
(b) The magnetic field in the gap;
(c) The self-induction.

5.2 Permanent magnets

The magnetic circuit can also include permanently magnetised components. For example consider Fig. 13, which shows a permanent magnet in the form of a ring, of length \( L \), with a gap of length \( d \).

![Figure 13: Calculating the fields for a permanent magnet.](image)

Assuming that there is negligible bulging of the magnetic field in the gap, \( B \) in the gap is the same as \( B \) in the material. As for the electromagnet it is the magnetic flux (\( \phi = BA \)) that is the same in the gap as in the material. Note that if there were bulging \( A \) in the material and in the gap would be different, leading to different \( B \) values. Applying Ampere’s law in terms of \( H \) we find, as there is no winding carrying free current in this case, that

\[
\oint H \cdot dl = H_m(L - d) + H_g d = 0 \quad (45)
\]

This implies that \( H_m \) and \( H_g \) point in opposite directions; on the other hand we know that \( B_m \) and \( B_g \) point in the same direction (and, in fact they are equal in magnitude as discussed above). We can summarise the situation as follows: in the gap \( B \) and \( H \) are parallel, and \( B = \mu_0 H \). In the ferromagnetic material \( H \) and \( B \) point in opposite directions, but \( M \) and \( B \) are parallel; as \( H \) is acting in a direction to demagnetise the permanent magnet, it is referred to as the depolarising field. Thus the magnetic state of a permanent magnet corresponds to a point on the hysteresis loop which is in the second or fourth quadrant (\( H < 0, B > 0 \)). For soft materials, the depolarising field is usually sufficient to bring the material back to \( B = 0 \).

Figure 14 shows the \( B \) and \( H \) lines for a uniformly magnetised sphere of ferromagnetic material. This figure shows clearly the depolarising \( H \) field inside the sphere. It also shows that \( B \) lines are loops: they have no beginning or end. In contrast \( H \) lines start where \( M \) lines end, and finish where \( M \) start. In other words sinks of \( M \) are sources of \( H \), and vice versa. This can readily be appreciated from the fact that \( \nabla \cdot B = 0 \). Thus

\[
\nabla \cdot (H + M) = 0 \quad (46)
\]

\[
\nabla \cdot H = -\nabla \cdot M \quad (47)
\]
Appendices

A Differential form of Ampere’s law

We apply Ampere’s law to a loop of area $\delta y \delta z$ in the $y-z$ plane of a right-handed cartesian coordinate system, shown in Fig. 15. The current through the loop, due to the $x$ component of current density is $J_x \delta y \delta z$. The line integral around the loop can be written as

$$
\left( B_y + \frac{1}{2} \frac{\partial B_z}{\partial y} \delta y \right) \delta y + \left( B_z + \frac{1}{2} \frac{\partial B_x}{\partial y} \delta y + \frac{1}{2} \frac{\partial B_z}{\partial z} \delta z \right) \delta z
- \left( B_y + \frac{1}{2} \frac{\partial B_x}{\partial z} \delta z + \frac{1}{2} \frac{\partial B_y}{\partial y} \delta y \right) \delta z - \left( B_z + \frac{1}{2} \frac{\partial B_y}{\partial z} \delta z \right) \delta z
= \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \delta y \delta z
$$

(48)

$$
= \mu_0 J_x \delta y \delta z
$$

(49)

It follows that

$$
\left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) = \mu_0 J_x
$$

(50)

with two other similar equations obtained by by cyclic permutation of the independent variables. As shown in Section 2, these three component equations correspond to the vector equation

$$
\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}
$$

B Quantum theory of paramagnetism (Advanced)

We assume that the magnetic dipole moment is due to the spin angular momentum of a single electron. The component of magnetic moment in the direction of the magnetic field is quantized, and has two possible values

$$
m_z = \pm \mu_b,
$$

(51)
where $\mu_b$ is the Bohr magneton. The dipole therefore has only two energy values

$$U = -m_z B = \mp \mu_b B.$$  

(52)

Using the Boltzmann distribution, the populations of the two states ($-\mu_b$ and $\mu_b$ respectively) are

$$\frac{N_1}{N} = \frac{e^{\mu_B B/kT}}{e^{\mu_B B/kT} + e^{-\mu_B B/kT}},$$

(53)

$$\frac{N_2}{N} = \frac{e^{-\mu_B B/kT}}{e^{\mu_B B/kT} + e^{-\mu_B B/kT}}.$$  

(54)

The magnetisation in the material is therefore

$$M = -N_1 \mu_b + N_2 \mu_b = N \frac{e^{\mu_B B/kT} - e^{-\mu_B B/kT}}{e^{\mu_B B/kT} + e^{-\mu_B B/kT}} = N \mu_b \tanh \left( \frac{\mu_B B}{kT} \right),$$

(55)

This equation has the limiting behaviour

$$\mu_b B/kT \to 0, \quad M \to N \mu_b \frac{\mu_B B}{kT},$$

(56)

$$\mu_b B/kT \to \infty, \quad M \to N \mu_b.$$  

(57)

Comparing now the classical and quantum expressions in the limit $\mu_b B/kT \to 0$

classical: \begin{align*}
  M &\to N m^2 B/(3kT), \\
  \text{quantum:} &\quad M \to N \mu_b^2 B/(kT).
\end{align*}

We can reconcile these results by noting that the magnitude of the magnetic moment in quantum mechanics is related to the component value as the ratio of magnitude of spin angular momentum to spin component i.e. $m/\mu_b = \sqrt{1/2(1/2 + 1)} : 1/2 = \sqrt{3} : 1$. Therefore $m^2/3 = \mu_b^2$. Thus we conclude that the classical and quantum versions of the Langevin equation agree in the linear region. They do however not agree in the saturation limit

classical: \begin{align*}
  M &\to N m, \\
  \text{quantum:} &\quad M \to N \mu_b.
\end{align*}

In this limit there is a real difference in the prediction due to the fact that classically the dipole can completely align with the magnetic field, but in the quantum case saturation corresponds to all quantised component values being in the direction of the field.