3.3 THE RECTANGULAR HYPERBOLA

A hyperbola for which the asymptotes are perpendicular is a rectangular hyperbola and is also called an equilateral hyperbola or right hyperbola. This occurs when the semimajor $a$ and semiminor $b$ axes are equal, $a = b$.

- **Equation**
  \[ x^2 - y^2 = a^2 \]
  rectangular hyperbola opening to the left and right

- **Eccentricity**
  \[ c^2 = a^2 + b^2 \]
  \[ a = b \]
  \[ c = a\sqrt{2} \]
  \[ e = \frac{c}{a} = \sqrt{2} \]

- **Directrix**
  \[ x = \pm \frac{a^2}{c} = \pm \frac{a}{\sqrt{2}} \]

- **Asymptotes**
  \[ y = \pm x \]
Example: Verify the information shown in the figure below

\[ a = 5 \quad b = 5 \quad c = 7.07 \]

\[ P(x, y) = (7.5, 5.59) \]
\[ A_1(x, y) = (-5, 0) \quad A_2(x, y) = (5, 0) \]
\[ F_1(x, y) = (-7.07, 0) \quad F_2(x, y) = (7.07, 0) \]
\[ D = (3.54, 5.59) \]

Eccentricity \( e = 1.41 \)

Directrices 1: \( x = -3.54 \) \quad Directrices 2: \( x = 3.54 \)

Slope tangent \( M_1 = 1.34 \) \quad Slope normal \( M_2 = -0.745 \)

Intercept tangent \( B_1 = -4.47 \) \quad Intercept normal \( B_2 = 11.2 \)

Tangent crosses X-axis: \( x_T = 3.33 \) \quad Normal crosses X-axis: \( x_N = 15 \)

Distances: \[ \text{PF}_1 = 15.6 \quad \text{PF}_2 = 5.61 \quad |\text{PF}_1 - \text{PF}_2| = 10 \]

Distances: \[ \text{PF}_2 = 5.61 \quad \text{PD} = 3.96 \quad \text{PF}_2 / \text{PD} = 1.41 \]

\[ \frac{x_p^2}{a^2} - \frac{y_p^2}{b^2} = 1 \quad \text{asymptotes } y = 1 \times \quad \text{asymptotes } y = -1 \times \]
Rectangular hyperbola opening in the first and third quadrants has the Cartesian equation

\[ xy = \frac{a^2}{2} \quad y = \frac{a^2/2}{x} \]
\[ x^2 - y^2 = a^2 \]
\[ a = \sqrt{2} \Rightarrow x^2 - y^2 = 2 \quad y = \pm (x^2 - 2) \]

Rotate graph 45° anticlockwise to give plot on right

\begin{align*}
xy &= \frac{a^2}{2} \\
y &= \frac{a^2}{x} \\
a &= \sqrt{2} \Rightarrow xy = 1 \quad y = \frac{1}{x}
\end{align*}

Rotate graph 45° clockwise to give plot on right
Example: Verify the information shown in the figure below \( x^2 - y^2 = 2 \quad a = b = \sqrt{2} \)

\[ a = 1.41 \quad b = 1.41 \quad c = 2 \]

\[ P(x, y) = (7.5, 7.365) \]

\[ A_1(x, y) = (-1.41, 0) \quad A_2(x, y) = (1.41, 0) \]

\[ F_1(x, y) = (-2, 0) \quad F_2(x, y) = (2, 0) \]

\[ D = (1, 7.365) \]

eccentricity \( e = 1.41 \)

directrices 1: \( x = -1 \) \quad directrices 2: \( x = 1 \)

slope tangent \( M_1 = 1.02 \) \quad slope normal \( M_2 = -0.982 \)

intercept tangent \( B_1 = -0.272 \) \quad intercept normal \( B_2 = 14.7 \)

T tangent cross X-axis: \( x_T = 0.267 \) \quad N normal cross X-axis: \( x_N = 15 \)

distances: \( PF_1 = 12 \quad PF_2 = 9.19 \quad |PF_1 - PF_2| = 2.83 \)

distances: \( PF_2 = 9.19 \quad PD = 6.5 \quad PF_2 / PD = 1.41 \)

\[ x_p^2 / a^2 - y_p^2 / b^2 = 1 \quad \text{asymptotes } y = 1 \quad \text{asymptotes } y = -1 \]
**Example:** Verify the information shown in the figure below \((x, y = 1/y = 1/x \ a = b = \sqrt{2})\)

The vertices \(A_1\) and \(A_2\) can be found from the solution of the equations

\[
y = 1/x \quad \text{and} \quad y = x \quad \Rightarrow \quad x = 1 \quad y = 1 \quad \text{and} \quad x = -1 \quad y = -1
\]

The Cartesian coordinates are \(A_1(-1, -1)\) and \(A_2(1,1)\)

The parameter \(a\) is equal to the distance \(O A_2\)

\[
a = \sqrt{1^2 + 1^2} = \sqrt{2}
\]

For a rectangular hyperbola \(a = b = \sqrt{2}\)

\[
c^2 = a^2 + b^2 \quad \Rightarrow \quad c = 2
\]
The focal length is $c = 2$ (distance $F_1 = F_2 = 2$), therefore, the Cartesian coordinates of $F_1$ and $F_2$ are

$$F_1(-\sqrt{2}, -\sqrt{2}) \text{ and } F_2(\sqrt{2}, \sqrt{2})$$

The eccentricity $e$ is

$$e = \frac{c}{a} = \frac{2}{\sqrt{2}} = \sqrt{2}$$
**ROTATION OF AXES**

The equation for the rectangular hyperbola \( xy = a^2 / 2 \) is the hyperbola \( x^2 - y^2 = a^2 \) referred to an XY coordinate system that has been rotated anticlockwise through an angle of 45°.

Suppose that a set of XY-coordinate axes has been rotated about the origin by an angle \( \theta \), where \( 0 < \theta < \pi/2 \), to form a new set of \( X'Y' \) axes. We would like to determine the coordinates for a point \( P \) in the plane relative to the two coordinate systems.

From the two right angle triangles shown in the figure, we can give the coordinates of the point \( P \) in Cartesian and polar coordinates for both sets of axes.
\[ P(x, y) \]
\[ x = R \cos(\theta + \phi) = R \cos \theta \cos \phi - R \sin \theta \sin \phi \]
\[ y = R \sin(\theta + \phi) = R \sin \theta \cos \phi + R \cos \theta \sin \phi \]

\[ P(x', y') \]
\[ x' = R \cos(\phi) \]
\[ y' = R \sin(\phi) \]

\[ x = x' \cos \theta - y' \sin \theta \]
\[ y = x' \sin \theta + y' \cos \theta \]

\[ x \cos \theta = x' \cos^2 \theta - y' \sin \theta \cos \theta \]
\[ y \sin \theta = x' \sin^2 \theta + y' \sin \theta \cos \theta \]

\[ x' = x \cos \theta + y \sin \theta \]
\[ y' = -x \sin \theta + y \cos \theta \]

**Coordinate Rotation Formulas**  If a rectangular \( XY \) coordinate system is rotated through an angle \( \theta \) to form an \( X'Y' \) coordinate system, then a point \( P(x, y) \) will have coordinates \( P(x', y') \) in the new system, where \( (x, y) \) and \( (x', y') \) are related by

\[ x = x' \cos \theta - y' \sin \theta \quad y = x' \sin \theta + y' \cos \theta \]

\[ x' = x \cos \theta + y \sin \theta \quad y' = -x \sin \theta + y \cos \theta \]
Example
Show that the graph of the equation \( x y = a^2 / 2 \) is a hyperbola by rotating the \( XY \) axes through an angle of \( \pi/4 \) rad (45°).

Solution
Denoting a point in the rotated system by \((x', y')\), we have
\[
x = x'\cos \theta - y'\sin \theta \quad y = x'\sin \theta + y'\cos \theta
\]
\[\theta = \pi / 4 \text{ rad} \quad \sin \theta = 1 / \sqrt{2} \quad \cos \theta = 1 / \sqrt{2}\]

\[
x y = \left( \frac{1}{\sqrt{2}} \right)(x' - y')\left( \frac{1}{\sqrt{2}} \right)(x' + y')
\]
\[x y = \frac{1}{2}(x'^2 - y'^2) = \frac{a^2}{2}\]

\[x'^2 - y'^2 = a^2\]

In the \( X'Y' \) coordinate system, then, we have a standard position hyperbola whose asymptotes are \( y' = \pm x' \).
The constant \( a \) is the distance from the origin \( O(0, 0) \) to one of the vertices \( (A_1 \text{ or } A_2) \) of the hyperbola.

The constant \( c \) is the distance from the origin \( O(0, 0) \) to one of the focal points \( (F_1 \text{ or } F_2) \).

The constant \( d \) is the length of the perpendicular line joining a point \( (D_1 \text{ or } D_2) \) on one of the directrices to the origin \( O(0, 0) \).
The transformation of points and lines between the $X'Y'$ and $XY$ Cartesian coordinate systems is done by using the relationships

\[
\begin{align*}
\theta &= \pi/4 \text{ rad} = 45^\circ \\
 x &= \frac{1}{\sqrt{2}}(x' - y') \\
 y &= \frac{1}{\sqrt{2}}(x' + y') \\
 x' &= \frac{1}{\sqrt{2}}(x + y) \\
 y' &= \frac{1}{\sqrt{2}}(-x + y)
\end{align*}
\]

**Vertex $A_2$**

$X'Y'$ axes \quad $A_2(a, 0)$ \quad $x' = a$ \quad $y' = 0$

$XY$ axes \quad $A_2\left(a/\sqrt{2}, a/\sqrt{2}\right)$ \quad $x = a/\sqrt{2}$ \quad $y = a/\sqrt{2}$

**Focal Point $F_2$** \quad $c = \sqrt{2}a$

$X'Y'$ axes \quad $F_2(\sqrt{2}a, 0)$ \quad $x' = \sqrt{2}a$ \quad $y' = 0$

$XY$ axes \quad $F_2(a, a)$ \quad $x = a$ \quad $y = a$

**Point $D_2$ on directrix** \quad $d = a/\sqrt{2}$

$X'Y'$ axes \quad $D_2\left(a/\sqrt{2}, 0\right)$ \quad $x' = a/\sqrt{2}$ \quad $y' = 0$

$XY$ axes \quad $D_2(a/2, a/2)$ \quad $x = a/2$ \quad $y = a/2$
Asymptotes

$X'Y'$ axes
$y' = x'$  
$y' = -x'$

$XY$ axes
$x = 0$  
$y = 0$

Directrices

$X'Y'$ axes
$x' = -a/2$  
$y' = 0$  
$x' = a/2$  
$y' = 0$

$XY$ axes
$y = -x + a$  
$y = -x - a$
Parametric equation for a rectangular hyperbola

The equation for the rectangular hyperbola is

\[ xy = \frac{a^2}{2} \]

where \( a \) is the distance from the origin to a vertex. This equation can be expressed in parametric coordinates \( (kt, \frac{k}{t}) \) where \( k \) is a constant and \( t \) is a variable parameter. For a point on the hyperbola

\[ xy = (kt) \left( \frac{k}{t} \right) = k^2 \]

Hence \( xy = k^2 = \frac{a^2}{2} \) \( \Rightarrow k = \frac{a}{\sqrt{2}} \) \( a = \sqrt{2}k \)

The focal length \( c \) (distance from the origin to a focal point) is

\[ c = \sqrt{2}a = 2k \quad k = \frac{c}{2} \]

*** In these notes \( c \) is used exclusively to represent the focal length. However, the syllabus and in exam questions, unfortunately in some instances \( c \) is used as the focal length and at other times it is used as an arbitrary constant. The syllabus expresses the equation for the rectangular hyperbola in parametric form as \( \left( ct, \frac{c}{t} \right) \) but \( c \) is just a constant and not the focal length. In my notes, I will use \( k \) for the constant and \( c \) to be the focal length. This is a much better approach.