FLUID FLOW

IDEAL FLUID

BERNOULLI'S PRINCIPLE

How can a plane fly?
How does a perfume spray work?
What is the venturi effect?
Why does a cricket ball swing or a baseball curve?

IDEAL FLUID

Fluid motion is usually very complicated. However, by making a set of assumptions about the fluid, one can still develop useful models of fluid behaviour. An ideal fluid is

- **Incompressible** – the density is constant
- **Irrotational** – the flow is smooth (streamline or laminar), no turbulence
- **Nonviscous** – fluid has no internal friction (\( \eta = 0 \))
- **Steady flow** – the velocity of the fluid at each point is constant in time.

BERNOULLI'S EQUATION (conservation of ENERGY)

An interesting effect is that, for a fluid (e.g. air) flowing through a pipe with a constriction in it, the fluid pressure is lowest at the constriction. In terms of the equation of continuity the fluid pressure falls as the flow speed increases.

The reason is easy to understand. The fluid has different speeds and hence different kinetic energies at different parts of the tube. The changes in energy must result from work being done on the fluid and the only forces in the tube that might do work on the fluid are the driving forces associated with changes in pressure from place to place.
- Conservation of energy
- A pressurized fluid must contain energy by the virtue that work must be done to establish the pressure.
- A fluid that undergoes a pressure change undergoes an energy change.

**Derivation of Bernoulli's equation**

\[ p_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2 \]

\[ p_1 + \frac{1}{2} \rho v_1^2 + \rho g \Delta x_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho g \Delta x_2 \]
Mass element $m$ moves from (1) to (2)

$$m = \rho A_1 \Delta x_1 = \rho A_2 \Delta x_2 = \rho \Delta V$$
where $\Delta V = A_1 \Delta x_1 = A_2 \Delta x_2$

Equation of continuity $A V = \text{constant}$

$$A_1 v_1 = A_2 v_2 \quad A_1 > A_2 \Rightarrow v_1 < v_2$$

Since $v_1 < v_2$ the mass element has been accelerated by the net force

$$F_1 - F_2 = p_1 A_1 - p_2 A_2$$

Mass element has an increase in KE

$$\Delta K = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = \frac{1}{2} \rho \Delta V v_2^2 - \frac{1}{2} \rho \Delta V v_1^2$$

Mass element has an increase in GPE

$$\Delta U = m g y_2 - m g y_1 = \rho \Delta V g y_2 = \rho \Delta V g y_1$$

The increase in KE and GPE comes from the net work done on the mass element by the forces $F_1$ and $F_2$ (the sample of mass $m$, in moving from a region of higher pressure to a region of lower pressure, has positive work done on it by the surrounding fluid)

$$W_{\text{net}} = F_1 \Delta x_1 - F_2 \Delta x_2 = p_1 A_1 \Delta x_1 - p_2 A_2 \Delta x_2$$

$$W_{\text{net}} = p_1 \Delta V - p_2 \Delta V = \Delta K + \Delta U$$

$$p_1 \Delta V - p_2 \Delta V = \frac{1}{2} \rho \Delta V v_2^2 - \frac{1}{2} \rho \Delta V v_1^2 + \rho \Delta V g y_2 - \rho \Delta V g y_1$$

Rearranging

$$p_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2$$

This is Bernoulli's equation.

We can also write, for any point along a flow tube or streamline

$$\rho + \frac{1}{2} \rho v^2 + \rho g y = \text{constant}$$

This is Bernoulli's theorem.
Bernoulli’s theorem only applies to an ideal fluid. Can not use Bernoulli’s Principle when the viscosity is significant. Can be applied to gases provided there are only small changes in pressure.

Dimensions

\[ p \text{ [Pa]} = [N.m^{-2}] = [N.m.m^{-3}] = [J.m^{-3}] \]

\[ \frac{1}{2} \rho v^2 \text{ [kg.m}^{-3}.m^2.s^{-2}] = [kg.m.s^{-2}.m.m^{-3}] = [N.m.m^{-3}] = [J.m^{-3}] \]

\[ \rho g h \text{ [kg.m}^{-3}.m.s^{-2}.m] = [kg.m.s^{-2}.m.m^{-3}] = [N.m.m^{-3}] = [J.m^{-3}] \]

Each term has the dimensions of energy / volume or energy density.

\[ \frac{1}{2} \rho v^2 \text{ KE of bulk motion of fluid} \]

\[ \rho g h \text{ GPE for location of fluid} \]

\[ p \text{ pressure energy density arising from internal forces within moving fluid (similar to energy stored in a spring)} \]

Consequences of Bernoulli’s Theorem

- A portion of a fluid made to move rapidly sustains a lower pressure than does the portion of fluid moving slowly. Using Galilean principle of relatively it is easier to consider the objects to be stationary and the fluid moving

What happens when two ships or trucks pass alongside each other?

Have you noticed this effect in driving across the Sydney Harbour Bridge?
Venturi Effect

A fast jet of air emerging from a small nozzle will have a lower pressure than the surrounding atmospheric pressure. The Venturi effect is used in many types of drainage systems. When you are in the dentist's chair, the dentist uses a device based on the Venturi effect to remove saliva out of your mouth. The device is connected to a tap. Water flows fast past the constriction, causing the pressure to drop inside the long tube. When the other end of this tube is immersed in a pool of saliva, the higher pressure outside forces the
saliva up the tube, and away. In a cotton picker, air flows instead of water to pick up cotton!

The chimney effect  Just the venturi effect being used to pick material up. In most automobiles, petrol is pushed into the carburettor in this way.

Atomizer  This same effect makes atomizers, perfume sprayers, insect sprayers, nebulizers and spray guns work. It is most important that the free surface of the liquid should be open to the atmosphere, else the high pressure outside the container and the low pressure inside will result in the container being crushed (the enclosure must be vented). Fly sprays always have a small air hole. Remember the fluid is caused to move from a container by a pressure difference – the fluid is pushed not sucked

How can you drink from a straw?

- Model for a knee in fluid flow (river going around a bend)
  Where does the flow speed up and slow down?

- How does the pressure change as the liquid goes round the corner?
  It can be observed that the streamlines hug the sharp corner. But far enough before the corner, and far enough after it, they are parallel and equally spaced. Consider the liquid flowing between lines 1 and 2. Its cross-sectional area decreases near the corner, so the liquid
speeds up there. The fluid between lines 3 and 4 has its area increased near the corner, so it slows down.

Remembering that pressure can also be thought of as force per unit area, we see that the fluid must be exerting an extra large force on the tube at the outside corner. Bernoulli’s principle – slower speed higher pressure, the higher pressure is on the outside corner of the bend.

- **Arteriosclerosis and vascular flutter**
Arteriosclerosis arises when plaque builds up on the inner walls of arteries, restricting blood flow. Any such obstruction results in a Venturi pressure drop.

In an advanced stage of the disease, pressure differences across artery causes it to collapse. Backed up blood will produce extra pressure which will force it open, only to collapse again. This gives vascular flutter which can be heard with a stethoscope.

A similar effect occurs in snoring.
• Why does smoke go up a chimney?
  Partly hot air rises (air less dense)
  Partly Bernoulli's principle (wind blows across top of chimney reducing pressure at the top)
• In a serve storm how does a house loose its roof?

Air flow is disturbed by the house. The "streamlines" crowd around the top of the roof ⇒ faster flow above house ⇒ reduced pressure above roof than inside the house ⇒ room lifted off because of pressure difference.

• How can a table tennis ball float in a moving air stream?

The flow of moving air around the ball can be asymmetrical so that there is a higher pressure on one side of the ball than the other. If the ball begins to leave the jet stream of air, the higher pressure outside the jet of air pushed the ball back into the flow.

• Why do rabbits not suffocate in the burrows?

Air must circulate. The burrows must have two entrances. Air flows across the two holes is usually slightly different ⇒ slight pressure difference ⇒ forces flow of air through burrow. One hole is usually higher than the other and the a small mound is built around the holes to increase the pressure difference.

• Why do racing cars wear skirts?

Racing cars are designed so that the air beneath the car moves faster than above it ⇒ large downward force on the car ⇒ increased normal force ⇒ increased frictional force ⇒ better grip and propulsion of the car.

• How can we describe the flow of a liquid from a tank (many applications)
Assume liquid behaves as an ideal fluid and that Bernoulli’s equation can be applied

\[ p_1 + \frac{1}{2} \rho v_1^2 + \rho gy_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho gy_2 \]

A small hole is at level (2) and the water level at (1) drops slowly ⇒ \( v_1 = 0 \)

\[ p_1 = p_{\text{atm}} \quad p_2 = p_{\text{atm}} \]

\[ \therefore \rho gy_1 = \frac{1}{2} \rho v_2^2 + \rho gy_2 \]

\[ v_2^2 = 2g(y_1 - y_2) = 2gh \quad h = (y_1 - y_2) \]

\[ v_2 = \sqrt{(2gh)} \quad \text{Torricelli formula (1608 – 1647)} \]

This is the same velocity as a particle falling freely through a height \( h \)
How do you measure the speed of flow for a fluid?

One method is to use a manometer type venturi meter.

Assume liquid behaves as an ideal fluid and that Bernoulli’s equation can be applied for the flow along a streamline

\[ p_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2 \]

\[ y_1 = y_2 \]

\[ p_1 - p_2 = \frac{1}{2} \rho_F (v_2^2 - v_1^2) \]

\[ p_1 - p_2 = \rho_m g h \]

\[ A_1 v_1 = A_2 v_2 \implies v_2 = v_1 \left( \frac{A_1}{A_2} \right) \]

\[ \rho_m g h = \frac{1}{2} \rho_F \left\{ v_1^2 \left( \frac{A_1}{A_2} \right)^2 - v_1^2 \right\} = \frac{1}{2} \rho_F \ v_1^2 \left\{ (\frac{A_1}{A_2})^2 - 1 \right\} \]

\[ v_1 = \sqrt{\frac{2 g h \rho_m}{\rho_F \left\{ \left( \frac{A_1}{A_2} \right)^2 - 1 \right\}}} \]
How does a siphon work? How fast does the liquid come out?

Assume that the liquid behaves as an ideal fluid and that both the equation of continuity and Bernoulli's equation can be used.

**Heights:** \( y_D = 0 \quad y_B \quad y_A \quad y_C \)

**Pressures:** \( p_A = p_{atm} = p_D \)

Consider a point A on the surface of the liquid in the container and the outlet point D. Apply Bernoulli's principle to these points

\[
p_A + \frac{1}{2} \rho v_A^2 + \rho g y_A = p_D + \frac{1}{2} \rho v_D^2 + \rho g y_D
\]

\[
v_D^2 = 2 (p_A - p_D) / \rho + v_A^2 + 2 g (y_A - y_D)
\]

\[
p_A - p_D = 0 \quad y_D = 0 \quad \text{assume } v_A^2 \ll v_D^2
\]

\[
v_D = \sqrt{2 g y_A}
\]

Now consider the points C and D and apply Bernoulli's principle to these points
\[ p_C + \frac{1}{2} \rho v_C^2 + \rho g y_C = p_D + \frac{1}{2} \rho v_D^2 + \rho g y_D \]

From equation of continuity \( v_C = v_D \)

\[ p_C = p_D + \rho g (y_D - y_C) = p_{atm} + \rho g (y_D - y_C) \]

The pressure at point C can not be negative

\[ p_{atm} + \rho g (y_D - y_C) \geq 0 \]

\[ p_C \geq 0 \text{ and } y_D = 0 \]

\[ p_C = p_{atm} - \rho g y_C \geq 0 \]

\[ y_C \leq p_{atm} / (\rho g) \]

For a water siphon

\[ p_{atm} \sim 10^5 \text{ Pa} \quad g \sim 10 \text{ m.s}^{-1} \quad \rho \sim 10^3 \text{ kg.m}^{-3} \]

\[ y_C \leq 10^5 / \{(10)(10^3)\} \text{ m} \]

\[ y_C \leq 10 \text{ m} \]
For streamline steady flow what is the difference between a nonviscous and viscous fluid flowing through a pipe?

For the viscous fluid there is a gradual decrease in pressure along the pipe.

Viscosity $\Rightarrow$ frictional forces $\Rightarrow$ energy lost by heating fluid $\Rightarrow$ increase in internal energy of fluid $\Rightarrow$ increase in temperature of fluid.

Where does the energy come from?

Equation of continuity still applies $\Rightarrow$ energy can not come from KE of fluid $\Rightarrow$ energy must be provided from the energy of the fluid $\Rightarrow$ pressure drop along tube.

Home activity

Shake a can of soft drink vigorously when outside. Then carefully open the can. Be careful not to wet yourself or someone else.

What happens and why?

Where did the energy come from and go?
Get a table tennis ball. Blow a steam of air from your mouth vertically and balance the ball in the air stream.

Hold a piece of paper and blow across it – the paper will rise.

Hold a piece of paper in each hand – let the papers hang vertically – blow a stream of air between the pages – they should come together.

Daniel Bernoulli (1700 – 1782)
A large artery in a dog has an inner radius of $4.00\times10^{-3}$ m. Blood flows through the artery at the rate of $1.00\times10^{-6}$ m$^3$.s$^{-1}$. The blood has a viscosity of $2.084\times10^{-3}$ Pa.s and a density of $1.06\times10^3$ kg.m$^{-3}$.

Calculate:
(i) The average blood velocity in the artery.
(ii) The pressure drop in a 0.100 m segment of the artery.
(iii) The Reynolds number for the blood flow.

Briefly discuss each of the following:
(iv) The velocity profile across the artery (diagram may be helpful).
(v) The pressure drop along the segment of the artery.
(vi) The significance of the value of the Reynolds number calculated in part (iii).

Solution
radius $R = 4.00\times10^{-3}$ m
volume flow rate $Q = 1.00\times10^{-6}$ m$^3$.s$^{-1}$
viscosity of blood $\eta = 2.084\times10^{-3}$ Pa.s
density of blood $\rho = 1.060\times10^{-3}$ kg.m$^{-3}$

(i) Equation of continuity: $Q = A \nu$

$A = \pi R^2 = \pi (4.00\times10^{-3})^2 = 5.03\times10^{-5}$ m$^2$
$\nu = Q / A = 1.00\times10^{-6} / 5.03\times10^{-5}$ m.s$^{-1} = 1.99\times10^{-2}$ m.s$^{-1}$

(ii) Poiseuille’s Equation

$Q = \Delta P \pi R^4 / (8 \eta L)$
$L = 0.100$ m

$\Delta P = 8 \eta L Q / (\pi R^4)$

$\Delta P = (8)(2.084\times10^{-3})(0.1)(1.00\times10^{-6}) / ((\pi)(4.00\times10^{-3})^4)$ Pa

$\Delta P = 2.07$ Pa

(iii) Reynolds Number

$Re = \rho \nu L / \eta$
where $L = 2 R$ (diameter of artery)

$Re = (1.060\times10^{-3})(1.99\times10^{-2})(2)(4.00\times10^{-3}) / (2.084\times10^{-3})$

$Re = 81$

use diameter not length
(iv) Parabolic velocity profile: velocity of blood zero at sides of artery

Flow of a viscous Newtonian fluid through a pipe
Velocity Profile

Cohesive forces between molecules ⇒ layers of fluid slide past each other generating frictional forces ⇒ energy dissipated (like rubbing hands together)

Adhesive forces between fluid and surface ⇒ fluid stationary at surface

(v) Viscosity ⇒ internal friction ⇒ energy dissipated as thermal energy ⇒ pressure drop along artery

(vi) $Re$ very small ⇒ laminar flow ($Re < 2000$)
A pitot tube can be used to determine the speed of an aeroplane relative to the surrounding air. The device consists of a U shaped tube containing a fluid with a density $\rho$. One end of the tube $A$ opens to the air at the side of the plane, the other end $B$ is open to the air in the direction the plane is flying so that the air stagnant at this location (zero speed). Show that the speed of the plane can be expressed in terms of the difference in the height $h$ of the fluid in the manometer tube as

$$v = \sqrt{2 gh \frac{\rho}{\rho_{\text{air}}}}$$

where $\rho_{\text{air}}$ is the density of the air.

**Solution**

Apply Bernoulli's Principle to the points at the front ($B$) and side ($A$) of the plane. For air the elevations $y_A$ and $y_B$ are essentially the same and the speed $v_B = 0 \text{ m.s}^{-1}$.
\[ p_A + \frac{1}{2} \rho_{\text{air}} v_A^2 + \rho_{\text{air}} g y_A = p_B + \frac{1}{2} v_B^2 + \rho_{\text{air}} g y_B \]

\[ \Delta p = p_B - p_A = \frac{1}{2} \rho_{\text{air}} v_A^2 \]

For the liquid of density \( \rho \) in the manometer

\[ \Delta p = \rho g h \]

Equating the two expressions for \( \Delta p \) \( \Rightarrow \)

\[ \Delta p = \rho g h = \frac{1}{2} \rho_{\text{air}} v_A^2 \]

\[ v = v_A = \sqrt{\frac{2 g h \rho}{\rho_{\text{air}}}} \]