Arguably the most broad-based evolution in the world view of science in the twentieth century will be associated with chaotic dynamics.

S.N. Rasband Chaotic Dynamics of nonlinear Systems.
**Phase plane analysis** is one of the most important techniques for studying the behaviour of nonlinear systems, since there are usually no analytical solutions.

In this document, we will consider the solutions to a pair of coupled first order differential equations with real and constant coefficients for the state variables \((x_1(t), x_2(t))\) of the general form

\[
\begin{align*}
\frac{dx_1}{dt} &= k_{11} x_1 + k_{12} x_2 + k_{13} \\
\frac{dx_2}{dt} &= k_{21} x_1 + k_{22} x_2 + k_{23}
\end{align*}
\]

If \((k_{13} = 0, k_{23} = 0)\) then we have a **homogeneous** system, otherwise an **inhomogeneous** system.

The first step is to find an **equilibrium solution** to the problem when

\[
\frac{dx_1}{dt} = 0 \quad \frac{dx_2}{dt} = 0
\]

An equilibrium solution corresponds to a **fixed point** called a **critical point** or a **stationary point**.
For a linear system, the solutions to find the equilibrium point \((x_{1c}, x_{2c})\) can be found by writing the equations in matrix form

\[
\begin{align*}
  k_{11} x_{1c} + k_{12} x_{2c} &= -k_{13} \\
  k_{21} x_{1c} + k_{22} x_{2c} &= -k_{23}
\end{align*}
\]

\[
K x_c = k \quad K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \quad x_c = \begin{pmatrix} x_{1c} \\ x_{2c} \end{pmatrix} \quad k = \begin{pmatrix} -k_{13} \\ -k_{23} \end{pmatrix}
\]

The values \((x_{1c}, x_{2c})\) are computed with the Matlab statement

\[x_c = K \setminus k\]

We now make the translations

\[z_1 = x_1 - x_{1c} \quad z_2 = x_2 - x_{2c}\]

to give the homogeneous linear system

\[
(2A) \quad \frac{dz_1}{dt} = k_{11} z_1 + k_{12} z_2 \\
\frac{dz_2}{dt} = k_{21} z_1 + k_{22} z_2
\]

or in matrix form

\[
(2B) \quad \frac{d}{dt}(z) = K z \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}
\]

The critical point for the homogeneous linear system is the Origin \((0, 0)\) if \(\det(K) \neq 0\). If \(\det(K) = 0\), then there are infinitely many solutions. We will only consider the case where \(\det(K) \neq 0\). Since \(\det(K) \neq 0\), both eigenvalues of the matrix \(K\) are non-zero.
The two systems (before and after the translations) have the same coefficient matrix $K$. Hence, their respective critical points will also have identical type and stability classification but with the critical point given by

$$x_1 = z_1 + x_{1c} \quad x_2 = z_2 + x_{2c}.$$ 

A solution to equation 2 can be expressed in terms of the two $2 \times 2$ matrices for the eigenfunctions $a$ and eigenvalues $b$ of the matrix $K$. The solutions can be written as

\begin{align*}
(2A) \quad x_1(t) &= C_1 \left( a_{11} e^{b_{11} t} \right) + C_2 \left( a_{12} e^{b_{22} t} \right) \\
 & \quad x_2(t) = C_1 \left( a_{21} e^{b_{11} t} \right) + C_2 \left( a_{22} e^{b_{22} t} \right)
\end{align*}

where $C_1$ and $C_2$ are determined by the initial conditions $(x_1(t = 0), x_2(t = 0))$.

The final solution is expressed as

\begin{align*}
(2B) \quad x_1(t) &= c_{11} e^{b_{11} t} + c_{12} e^{b_{22} t} \\
 & \quad x_2(t) = c_{21} e^{b_{11} t} + c_{22} e^{b_{22} t}
\end{align*}
The eigenfunctions $a$ and eigenvalues $b$ are computed using the function `eig`:

```
[a, b] = eig(K)
```

The matrices for the coefficients $C$ and $c$ are computed by the Matlab statements

```matlab
% Initial conditions
xI = [x1I(c);x2I(c)];

% C coefficients
C = a\xI;

% c coefficients
cc = zeros(2,2);
cc(:,1) = a(:,1)*C(1);
cc(:,2) = a(:,2)*C(2);
```
Example 1

\[ \frac{dx_1}{dt} = -x_1 + x_2 \]
\[ \frac{dx_2}{dt} = -4x_2 \]
\[ t = 0 \quad x_1(t = 0) = 9 \quad x_2(t = 0) = -9 \]

Command Window Output

D.E. coefficients k11 k12 k13 / k21 k23 k23

\[
\begin{array}{ccc}
-1.00 & 1.00 & 0.00 \\
0.00 & -4.00 & 0.00 \\
\end{array}
\]

Eigenvalues b =

\[
\begin{array}{cc}
-1 & 0 \\
0 & -4 \\
\end{array}
\]

Eigenfunction a =

\[
\begin{array}{ccc}
1.0000 & -0.3162 & 0 \\
0 & 0.9487 & 0 \\
\end{array}
\]

Initial conditions: t = 0 corresponds to array index 1.

\[ x_1(1) = 9 \quad x_2(1) = -9 \quad \Rightarrow \]

\[ c = \]

\[
\begin{array}{rr}
6.0000 & 3.0000 \\
0 & -9.0000 \\
\end{array}
\]

Therefore, the solution is

\[ x_1(t) = 6e^{-t} + 3e^{-4t} \]
\[ x_2(t) = -9e^{-4t} \]
The upper diagram shows two trajectories starting from the initial locations \((x_1(1) = 9, x_2(1) = +9)\) and \((x_2(1) = -9)\). The solutions converge to the fixed equilibrium (critical) point at the Origin \((0, 0)\). The lower diagram shows the time evolution of the state variables for the initial condition \(x_1(1) = 0\) and \(x_2(1) = -9\).
Our starting point to look at the dynamics of a system is to set up a phase plane. A phase plane plot for a two-state variable system consists of curves of one state variable versus the other state variable \((x_1(t), x_2(t))\), where each curve called a trajectory is based on a different initial condition. The graphical representation of the solutions is often referred to as a phase portrait. The phase portrait is a graphical tool to visualize how the solutions of a given system of differential equations would behave in the long run.

We can set up a vector field which is constructed by assigning the following vector to each point on the \(x_1-x_2\) plane:

\[
\begin{pmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt}
\end{pmatrix}
\]

The slope of these vectors is

\[
m = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{dx_2}{dx_1}
\]

Thus, the vector field can be computed without knowing the solutions \(x_1\) and \(x_2\). This allows you to visualize the solution of the system for any given initial condition \((x_1(t = 0), x_2(t = 0))\) as the vector field must be tangential to any solutions at all point of the system.
Next we can plot the $x_1$ and $x_2$ nullclines of the phase plane plot, where the nullclines are the straight lines determined by:

- $x_1$-nullcline $\frac{dx_1}{dt} = 0$
- $x_2$-nullcline $\frac{dx_2}{dt} = 0$

These nullclines lines show the points where $x_1$ is independent of time $t$ and the points where $x_2$ is also no longer changing with time. The intersection of the two nullclines represent steady-state values of fixed points of the system.

Fig. 1. Vector field (quiver function) and $x_1$ and $x_2$ nullclines. The arrows point in the direction of increasing time $t$. The critical point is at the intersection of the two nullclines.
The coupled differential equations (equation 1) are specified by the matrix $K$ and the solution for the two state variables depends upon the eigenvalues $b$ and eigenfunctions $a$ of the matrix $K$. The nature of the eigenvalues (real / imaginary) determine the type of equilibrium for the system. If the eigenvalue is greater than zero, then the term increases exponentially with time and if less than zero, the term decreases exponentially with time, since a solution is of the form: 

$$x(t) = c_1 e^{b_1 t} + c_2 e^{b_2 t}$$

where $b_1$ and $b_2$ are the eigenvalues.

$$b > 0 \quad t \to \infty \quad e^{bt} \to \infty$$

$$b < 0 \quad t \to \infty \quad e^{bt} \to 0$$

The larger the eigenvalue, the faster the response and the smaller the value of the eigenvalue, the slower the response. Due to the two-dimensional nature of the parametric curves, we will classify the type of those critical points by the shape formed of the trajectories about the critical point. For distinct real eigenvalues, the trajectories either move away from the critical point to an infinite-distant away (when the eigenvalues are both positive) or move toward from infinite-distant out and eventually converge to the critical point (when eigenvalues are both negative). This type of critical point is called a node. It is asymptotically stable if eigenvalues are both negative, unstable if both are positive values.
Case 1: real eigenvalues of opposite sign
There is a saddle point at the intersection of the two nullclines. The equilibrium point is unstable.

Case 2: real eigenvalues and both negative
The stable fixed-equilibrium point is called a nodal sink.

Case 3: real eigenvalues and both positive
The unstable fixed-equilibrium point is called a nodal source.

Case 4: imaginary eigenvalues and negative real parts
The stable fixed-equilibrium point is called a spiral sink.

Case 5: imaginary eigenvalues and positive real parts
The unstable fixed-equilibrium point is called a spiral source.

Case 6: purely imaginary eigenvalues
This gives a generic equilibrium called a center.

You can investigate the different types of solutions by running the script chaos10.m for each of the following cases.
Case 1: real eigenvalues of opposite sign

The **unstable** equilibrium point called a **saddle**.

D.E. coefficients $k_1$ $k_2$ $k_3$ / $k_2$ $k_3$ $k_3$

$\begin{array}{ccc}
1.00 & 1.00 & 0.00 \\
4.00 & 1.00 & 0.00
\end{array}$

Eigenvalues $b =$

$\begin{array}{ccc}
3.0000 & 0 \\
0 & -1.0000
\end{array}$

Eigenfunction $a =$

$\begin{array}{ccc}
0.4472 & -0.4472 \\
0.8944 & 0.8944
\end{array}$

If the initial condition for $x_2(t = 0) = 0$, then the trajectory reaches the Origin. Otherwise, the solutions will always leave the origin. Hence, the point (0,0) is an **unstable** equilibrium point for the system and is called a **saddle point**.

$$b_{11} = 3 > 0 \quad \Rightarrow \quad t \to \infty \quad x_1 \to \infty \quad x_2 \to \infty$$
Fig. 1.1. The trajectories are always directed away from the Origin (0, 0). The Origin (0, 0) is an unstable equilibrium point called a saddle point.
D.E. coefficients k11 k12 k13 / k21 k23 k23
-1.00  0.00  0.00
0.00  4.00  0.00

Eigenvalues b =
-1   0
0   4

Eigenfunction a =
1   0
0   1

Initial conditions: t = 0   x1(1) = 10   x2(1) = -0.2
cc =
    10.0000   0
0   -0.2000

\[ x_1(t) = x_1(0) \ e^{-t} \quad t \to \infty \Rightarrow x_1 \to 0 \]
\[ x_2(t) = x_2(0) \ e^{4t} \quad t \to \infty \Rightarrow x_2 \to \infty \]
Fig. 1.2. The trajectories are always directed away from the Origin (0, 0). The Origin (0, 0) is an unstable equilibrium point called a saddle point.
D.E. coefficients k11 k12 k13 / k21 k23 k23
2.00  1.00  0.00
2.00 -1.00  0.00

Eigenvalues b =
2.5616  0
0  -1.5616

Eigenfunction a =
0.8719  -0.2703
0.4896   0.9628

Initial conditions: t = 0  x1(1) = 10  x2(1) = -9  cc =
6.4552  3.5448
3.6249 -12.6249

\[ t \to \infty \Rightarrow x_1 \to \infty \quad x_2 \to \infty \]

The trajectories given by the eigenvectors of the negative eigenvalue initially start at infinite-distance away, move toward and eventually converge at the critical point. The trajectories with the eigenvector of the positive eigenvalue move in exactly the opposite way: start at the critical point then diverge to infinite-distance out. Every other trajectory starts at infinite-distance away, moves toward but never converges to the critical point, before changing direction and moves back to infinite-distance away. All the while it would roughly follow the two sets of eigenvectors. This type of critical point is always unstable and is called a saddle point.
Fig. 1.3. The trajectories are always directed away from the Origin (0, 0). The Origin (0, 0) is an unstable equilibrium point called a **saddle point**.
Case 2: real eigenvalues and both negative

The stable fixed-equilibrium point is called a node sink.

D.E. coefficients $k_{11}$ $k_{12}$ $k_{13}$ / $k_{21}$ $k_{23}$ $k_{23}$

\[
\begin{array}{ccc}
-1.00 & 0.00 & 0.00 \\
0.00 & -4.00 & 0.00 \\
\end{array}
\]

Eigenvalues $b =$

\[
\begin{array}{cc}
-4 & 0 \\
0 & -1 \\
\end{array}
\]

Eigenfunction $a =$

\[
\begin{array}{cc}
0 & 1 \\
1 & 0 \\
\end{array}
\]

$t \to \infty \Rightarrow x_1 \to 0 \quad x_2 \to 0$
Fig. 2.1. The solutions converge to the Origin \((0, 0)\) for all initial conditions. The point \((0,0)\) is a stable equilibrium point for the system and is called a \textbf{stable node} or \textbf{nodal sink}.
D.E. coefficients k11 k12 k13 / k21 k23 k23
-2.00  0.00  0.00
1.00  -4.00  0.00

Eigenvalues b =
-4    0
0    -2

Eigenfunction a =
0    0.8944
1.0000  0.4472

\[ t \to \infty \Rightarrow x_1 \to 0 \quad x_2 \to 0 \]
Fig. 2.2. The solutions converge to the Origin (0, 0) for all initial conditions. The point (0,0) is a stable equilibrium point for the system and is called a stable node or nodal sink.
Nonhomogeneous Linear Systems with Constant Coefficients

D.E. coefficients k₁₁ k₁₂ k₁₃ / k₂₁ k₂₂ k₂₃
1.00  -2.00  -1.00
2.00  -3.00  -3.00

Eigenvalues b =
-1.0000  0
0  -1.0000

Eigenfunction a =
0.7071  0.7071
0.7071  0.7071

The critical point is at (3, 1). It has repeated eigenvalues equal to -1. Hence, there is only one linearly independent eigenvector. Therefore, the critical point at (3, 1) is an asymptotically stable improper node.
Fig. 2.3  Asymptotically stable improper node. The critical point is (3,1).
Case 3: real eigenvalues and both positive

The unstable equilibrium point is called a node source.

D.E. coefficients $k_{11} \ k_{12} \ k_{13} / k_{21} \ k_{23} \ k_{23}$

\[
\begin{array}{ccc}
3.00 & 1.00 & 0.00 \\
1.00 & 3.00 & 0.00 \\
\end{array}
\]

Eigenvalues $b =$

\[
\begin{array}{cc}
2 & 0 \\
0 & 4 \\
\end{array}
\]

Eigenfunction $a =$

\[
\begin{array}{cc}
-0.7071 & 0.7071 \\
0.7071 & 0.7071 \\
\end{array}
\]

\[t \to \infty \implies x_1 \to \infty \quad x_2 \to \infty\]
Fig. 3.1. The solutions diverge to the Origin (0, 0) for all initial conditions. The point (0, 0) is an unstable equilibrium point for the system and is called a nodal source.
Case 4: Imaginary eigenvalues with negative Real parts

The stable equilibrium point is called a spiral sink.

D.E. coefficients $k_{11} k_{12} k_{13} / k_{21} k_{23} k_{23}$

\[
\begin{array}{ccc}
-0.20 & 1.00 & 0.00 \\
-1.00 & -0.20 & 0.00 \\
\end{array}
\]

Eigenvalues $b =$

\[
\begin{array}{ccc}
-0.2000 + 1.0000i & 0.0000 + 0.0000i \\
0.0000 + 0.0000i & -0.2000 - 1.0000i \\
\end{array}
\]

Eigenfunction $a =$

\[
\begin{array}{ccc}
0.7071 + 0.0000i & 0.7071 + 0.0000i \\
0.0000 + 0.7071i & 0.0000 - 0.7071i \\
\end{array}
\]
Fig. 4.1. **Spiral sink.** The solutions for the state variables oscillates as they decay towards zero.
D.E. coefficients $k_{11} \ k_{12} \ k_{13} / k_{21} \ k_{23} \ k_{23}$

\[
\begin{align*}
4.00 & \quad -3.00 & \quad 0.00 \\
15.00 & \quad -8.00 & \quad 0.00
\end{align*}
\]

Eigenvalues $b =$

\[
\begin{align*}
-2.0000 + 3.0000i & \quad 0.0000 + 0.0000i \\
0.0000 + 0.0000i & \quad -2.0000 - 3.0000i
\end{align*}
\]

Eigenfunction $a =$

\[
\begin{align*}
0.3651 + 0.1826i & \quad 0.3651 - 0.1826i \\
0.9129 + 0.0000i & \quad 0.9129 + 0.0000i
\end{align*}
\]
Fig. 4.2. **Spiral sink.** The solutions for the state variables oscillates as they decay towards zero.
Case 5: Imaginary eigenvalues with positive Real parts

The unstable equilibrium point is called a spiral source

D.E. coefficients k11 k12 k13 / k21 k23 k23
   2.00   -1.00   0.00
   2.00   0.00   0.00

Eigenvalues b =
   1.0000 + 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 - 1.0000i

Eigenfunction a =
   0.4082 + 0.4082i   0.4082 - 0.4082i
   0.8165 + 0.0000i   0.8165 + 0.0000i
Fig. 5.1. An unstable spiral source.
D.E. coefficients \( k_{11} k_{12} k_{13} / k_{21} k_{23} k_{23} \)

\[
\begin{array}{ccc}
2.00 & -1.00 & 0.00 \\
2.00 & 0.00 & 0.00 \\
\end{array}
\]

Eigenvalues \( b = \)

\[
\begin{array}{cc}
1.0000 + 1.0000i & 0.0000 + 0.0000i \\
0.0000 + 0.0000i & 1.0000 - 1.0000i \\
\end{array}
\]

Eigenfunction \( a = \)

\[
\begin{array}{cc}
0.4082 + 0.4082i & 0.4082 - 0.4082i \\
0.8165 + 0.0000i & 0.8165 + 0.0000i \\
\end{array}
\]
Fig. 5.2. An unstable spiral source.
D.E. coefficients k11 k12 k13 / k21 k23 k23
-2.00  -6.00  8.00
  8.00  4.00  -12.00

Eigenvalues b =
  1.0000 + 6.2450i  0.0000 + 0.0000i
  0.0000 + 0.0000i  1.0000 - 6.2450i

Eigenfunction a =
-0.2835 + 0.5901i  -0.2835 - 0.5901i
  0.7559 + 0.0000i  0.7559 + 0.0000i

The critical point is at (1, 1). It has complex eigenvalues with positive real parts, therefore, the critical point at (1, 1) is an unstable spiral point.
Fig. 5.3. Unstable spiral with the critical point at (1, 1).
Case 6: Imaginary eigenvalues with zero Real parts

The equilibrium point is called a center.

D.E. coefficients $k_{11} k_{12} k_{13} / k_{21} k_{23} k_{23}$

\[
\begin{array}{ccc}
0.00 & -1.00 & 0.00 \\
1.00 & 0.00 & 0.00 \\
\end{array}
\]

Eigenvalues $b =$

\[
\begin{array}{cc}
0.0000 + 1.0000i & 0.0000 + 0.0000i \\
0.0000 + 0.0000i & 0.0000 - 1.0000i \\
\end{array}
\]

Eigenfunction $a =$

\[
\begin{array}{cc}
0.7071 + 0.0000i & 0.7071 + 0.0000i \\
0.0000 - 0.7071i & 0.0000 + 0.7071i \\
\end{array}
\]
Fig. 6.1. System shows center stability.
D.E. coefficients $k_{11} k_{12} k_{13} / k_{21} k_{23} k_{23}$
-1.00  -1.00  0.00
4.00   1.00   0.00

Eigenvalues $b =$
0.0000 + 1.7321i  0.0000 + 0.0000i
0.0000 + 0.0000i  0.0000 - 1.7321i

Eigenfunction $a =$
-0.2236 + 0.3873i  -0.2236 - 0.3873i
0.8944 + 0.0000i  0.8944 + 0.0000i
Fig. 6.2. System shows center stability.
Summary

Asymptotically stable: All trajectories converge to the critical point as $t \to \infty$. Stable critical point: All eigenvalues are all negative or have negative real part for complex eigenvalues.

Unstable critical point: All trajectories (or all but a few, in the case of a saddle point) start out at the critical point at $t \to \infty$, then move away to infinitely distant out as $t \to \infty$. A critical point is unstable if at least one of the K eigenvalues is positive or has positive real part for complex eigenvalues.

Stable (or neutrally stable): Each trajectory moves about the critical point within a finite range of distance and never moves out to infinitely distant, nor (unlike in the case of asymptotically stable) does it ever go to the critical point. A critical point is stable if the K eigenvalues are purely imaginary.

As $t$ increases, if all (or almost all) trajectories

1. Converge to the critical point $\rightarrow$ asymptotically stable.
2. Move away from the critical point to infinitely far away $\rightarrow$ unstable.
3. Stay in a fixed orbit within a finite (i.e., bounded) range of distance away from the critical point $\rightarrow$ stable (or neutrally stable).
An application of phase plane analysis which model the retina uses the mscript chaos10eye.m

A Simple model of the retina: C-cell / H-cell negative feedback interaction: