DOING PHYSICS WITH MATLAB

MATHEMATICAL ROUTINES

COMPUTATION OF TWO-DIMENSIONAL INTEGRALS: DOUBLE or SURFACE INTEGRALS

$$I = \iint_A f(x,y)\,dA \qquad I = \int_{a_y}^{b_y} \int_{a_x}^{b_x} f(x,y)\,dx\,dy$$

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math_integration_2D.m
Demonstration mscript evaluating the integral of functions of the form $f(x,y)$ using a two-dimensional form of Simpson's 1/3 rule. The code can be changed to integrate functions between the specified lower and upper bounds.

simpson2d.m
Function to give the integral of a function $f(x,y)$ using a two-dimensional form of Simpson's 1/3 rule. The format to call the function is

```
Ixy = simpson2d(f,ax,bx,ay,by)
```

NUMERICAL INTEGRATION:
COMPUTATION OF TWO-DIMENSIONAL INTEGRALS (DOUBLE OR SURFACE INTEGRALS)

The function **simpson2d.m** is a very versatile , accurate and easy to implement function that can be used to evaluate a definite integral of a function $f(x,y)$ between lower bounds and an upper bounds .

We want to compute a number expressing the definite integral of the function $f(x,y)$ between two specific limits (a_x, b_x) and (a_y, b_y)

$$I = \int_{a_y}^{b_y} \int_{a_x}^{b_x} f(x,y)\, dx\, dy$$

The evaluation of such integrals is often called ***quadrature***.

We can estimate the value a double integral by a **two-dimensional version of Simpson's 1/3 rule**.

Simpson's 1/3 rule

This rule is based on using a quadratic polynomial approximation to the function $f(x)$ over a pair of partitions. N-1 is the number of partitions where N must be **odd** and $\Delta x \equiv h = (b - a) / (N-1)$. The integral is expressed below and is known as the ***composite Simpson's 1/3 rule***.

$$I = \frac{h}{3}\left\{\left(f_1 + f_N + 4(f_2 + f_4 + ... + f_{N-2}) + 2(f_3 + f_5 + ... + f_{N-1}\right)\right\}$$

Simpson's rule can be written vector form as

$$I = \frac{h}{3}\mathbf{c}\mathbf{f}^{\mathrm{T}}$$

where $\mathbf{c} = [1\;4\;2\;4\;...\;2\;4\;1]$ and $\mathbf{f} = [\,f_1\;f_2\;...\;f_N\,]$.

c and **f** are row vectors and $\mathbf{f}^{\mathbf{T}}$ is a column vector.

Simpson's [2D] method

The double integral

$$I = \int_{a_y}^{b_y} \int_{a_x}^{b_x} f(x,y)\,dx\,dy$$

can be approximated by applying Simpson's 1/3 rule twice – once for the *x* integration and once for the *y* integration with *N* partitions for both the *x* and *y* values.

x-values: $x_1\ x_2\ x_3 \ldots x_c\ \ldots x_N$

y-values: $y_1\ y_2\ y_3 \ldots y_c\ \ldots y_N$

The lower and upper bounds determine the size of the partitions

$$dx \equiv h_x = \frac{b_x - a_x}{N-1} \qquad dy \equiv h_y = \frac{b_y - a_y}{N-1}$$

The *N* *x*-values and *N* *y*-values form a two-dimensional grid of *N* x *N* points. The function *f*(*x*,*y*) and the two-dimensional Simpson's coefficients are calculated at each grid point. Hence, the function *f*(*x*,*y*) and the two-dimensional Simpson's coefficients can be represented by *N* x *N* matrices **F** and **S** respectively.

The Simpson matrix **S** for *N* = 5 is

1 x 1 = 1	4 x 1 = 4	2x1 = 2	4x1 = 4	1x1 = 1
1 x 4 = 4	4 x 4 = 16	2x4 = 4	4x4 = 16	1x4 = 1
1 x 2 = 8	4 x 2 = 8	2x2 = 4	4x2= 8	1x2 = 1
1 x 4 = 4	4 x 4 = 16	2x4 = 4	4x4 = 16	1x4 = 1
1 x 1 = 1	4 x 1 = 4	2x1 = 2	4x1 = 4	1x1 = 1

Therefore, the **two-dimensional Simpson's rule** which is used to estimate the value of the surface integral can be expressed as

$$I = \left(\frac{h_x\, h_y}{9}\right) \sum_{m=1}^{N} \sum_{n=1}^{N} \left(\mathrm{S}_{mn}\ \mathrm{F}_{mn}\right)$$

The two-dimensional Simpson's coefficient matrix **S** for *N* = 9 is

1	4	2	4	2	4	2	4	1
4	16	8	16	8	16	8	16	4
2	8	4	8	4	8	4	8	2
4	16	8	16	8	16	8	16	4
2	8	4	8	4	8	4	8	2
4	16	8	16	8	16	8	16	4
2	8	4	8	4	8	4	8	2
4	16	8	16	8	16	8	16	4
1	4	2	4	2	4	2	4	1

We will consider a number of examples which demonstrates how to apply the two-dimensional Simpson's rule using the mscript **math_integration_2D.m**.

Example 1 integrate $f(x,y) = x^2\, y^3$ x: 0 → 2 and y: 1 → 5

(1) $$I_{xy1} = \int_{a_y}^{b_y} \int_{a_x}^{b_x} f(x,y)\,dx\,dy = \int_1^5 \left[\int_0^2 x^2\, y^3\,dx \right] dy = 416$$

The exact value of the integral can be found analytically and its value is **416**. So we can compare the numerical estimate with the known exact value.

Steps in estimating the integral numerically using **math_integration_2D.m** and **simpson2d.m**

- Clear all variables, close any Figure Windows and clear the Command Window:

```
clear all
close all
clc
```

- Enter the number of partitions (must be an **odd** number), and the lower and upper bounds for x and y and calculate the range for the x and y values

```
num = 5;
xMin = 0;
xMax = 2;
yMin = 1;
yMax = 5;

x = linspace(xMin,xMax,num);
y = linspace(yMin,yMax,num);
```

- This is a two-dimensional problem, so we need to specify the values (x,y) at all grid points which are determined from the upper and lower bounds. We can do this using the Matlab command **meshgrid**. We can then calculate the value of the function $f(x,y)$ at each grid point (x,y).

```
[xx yy] = meshgrid(x,y);
f = xx.^2 .* yy.^3;
```

To show how the **meshgrid** functions works, see figure (1) and the outputs of the variables *x*, *y*, *xx*, *yy* and *f* that were displayed in the Command Window and the Simpson's [2D] coefficients calculated with the function **simpson2d.m**.

```
x =   0    0.5000    1.0000    1.5000    2.0000
xx =
      0    0.5000    1.0000    1.5000    2.0000
      0    0.5000    1.0000    1.5000    2.0000
      0    0.5000    1.0000    1.5000    2.0000
      0    0.5000    1.0000    1.5000    2.0000
      0    0.5000    1.0000    1.5000    2.0000

y =   1    2    3    4    5
yy =
     1    1    1    1    1
     2    2    2    2    2
     3    3    3    3    3
     4    4    4    4    4
     5    5    5    5    5

f =
     0    0.2500      1.0000      2.2500      4.0000
     0    2.0000      8.0000     18.0000     32.0000
     0    6.7500     27.0000     60.7500    108.0000
     0   16.0000     64.0000    144.0000    256.0000
     0   31.2500   125.0000    281.2500    500.0000
```

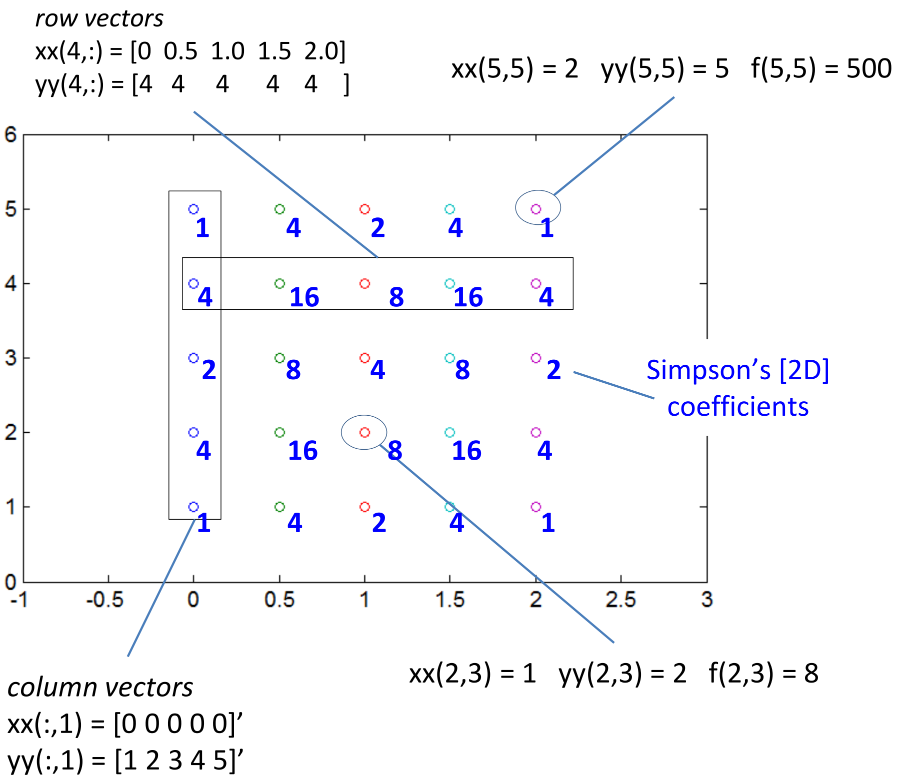


Fig. 1. The grid points for $N = 5$ and how these points relate to the Matlab matrices.

- Calculate the Simpson [2D] coefficients

```
% evaluates two dimension Simpson coefficients ---------------------------
sc = 2*ones(num,1);
sc(2:2:num-1) = 4;
sc(1) = 1;
sc(num) = 1;

scx = meshgrid(sc,sc);
scxy = ones(num,num);
scxy(2:2:num-1,:) = scx(2:2:num-1,:)*sc(2);
scxy(3:2:num-2,:) = scx(3:2:num-2,:)*sc(3);
scxy(1,:) = sc';
scxy(num,:) = sc';
```

- Compute the integral

```
hx = (bx-ax)/(num-1); hy = (by-ay)/(num-1);
h = hx * hy / 9;
integral = h * sum(sum(scxy .* f));
```

The complete mscript to compute the integral is

```
function integral = simpson2d(f,ax,bx,ay,by)

    %num must be odd
    %1 4 2 4 ...2 4 1

    num = length(f);
    hx = (bx-ax)/(num-1); hy = (by-ay)/(num-1);
    h = hx * hy / 9;

    % evaluates two dimension Simpson coefficients ---------------------------
    sc = 2*ones(num,1);
    sc(2:2:num-1) = 4;
    sc(1) = 1;
    sc(num) = 1;
    scx = meshgrid(sc,sc);
    scxy = ones(num,num);
    scxy(2:2:num-1,:) = scx(2:2:num-1,:)*sc(2);
    scxy(3:2:num-2,:) = scx(3:2:num-2,:)*sc(3);
    scxy(1,:) = sc';
    scxy(num,:) = sc';
     % evaluates integral ---------------------------------------------------
    integral = h * sum(sum(scxy .* f));
```

The exact value of the integral is **416** $I_{xy1} = \int_1^5 \left[\int_0^2 x^2 \, y^3 \, dx \right] dy = 416$

With only 5 partitions and 25 (5x5) grid points, the numerical estimate is **416**, the same as the exact value.

Example 2 Double Integrals and Volumes

$$Volume = \iint_A f(x,y)\,dx\,dy$$

To gain an intuitive feel for double integrals, the volume of the region enclosed by the area A is equal to the value of the double integral.

Volume V of a rectangular box

$f(x,y) = k$ height of box $k > 0$

Base of box – the lower bounds (a_x and a_y) and upper bounds (b_x and b_y) determine the area of the rectangular base of the box

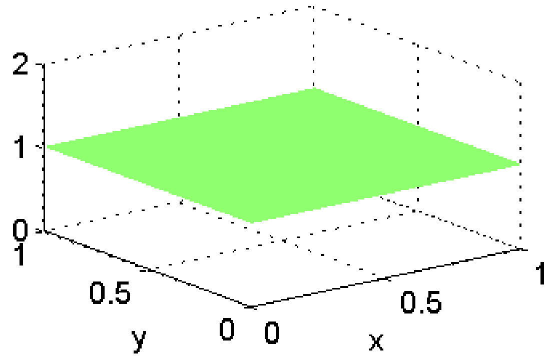
$$Volume\;\; V = \int_{a_y}^{b_y}\int_{a_x}^{b_x} k\,dx\,dy$$

Box

$k = 1$ $a_x = 0$ $b_x = 1$ $a_y = 0$ $b_y = 1$ $N = 299$

Exact volume (analytical) $V = 1.0000$

Simpson's [2D] rule $V = 1.0000$



Volume V of half box

$f(x,y) = 1 - x$

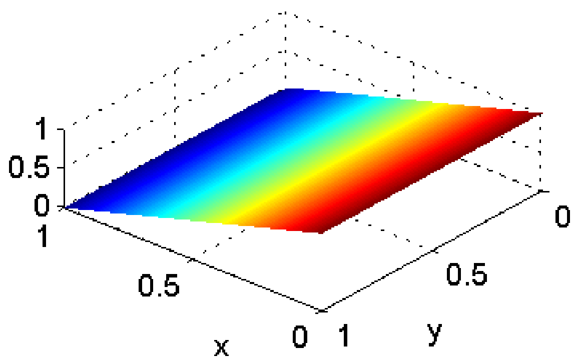
Base of box – the lower bounds (a_x and a_y) and upper bounds (b_x and b_y) determine the area of the rectangular base of the box

$$Volume\quad V = \int_{a_y}^{b_y}\int_{a_x}^{b_x} (1-x)\,dx\,dy$$

$a_x = 0$ $b_x = 1$ $a_y = 0$ $b_y = 1$ $N = 299$

Exact volume (analytical) $V = 0.50000$

Simpson's [2D] rule $V = 0.50000$



Volume *V* of a part-bowl

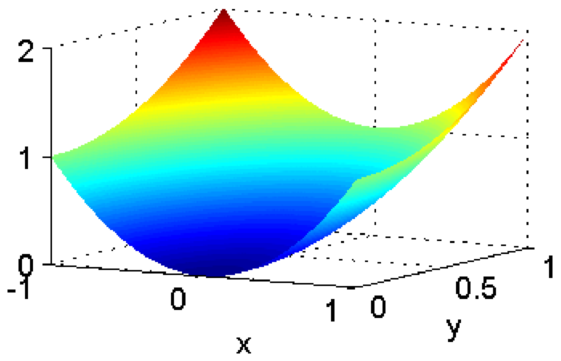
$f(x,y) = x^2 + y^2$

Base of box – the lower bounds (a_x and a_y) and upper bounds (b_x and b_y) determine the area of the rectangular base of the surface

$$Volume \quad V = \int_{a_y}^{b_y}\int_{a_x}^{b_x}\left(x^2+y^2\right)dx\,dy$$

$a_x = -1 \quad b_x = 1 \quad a_y = 0 \quad b_y = 1 \quad N = 299$

Exact volume (analytical) $V = 1.3333$
Simpson's [2D] rule $V = 1.3333$



Volume *V* over a rectangular base

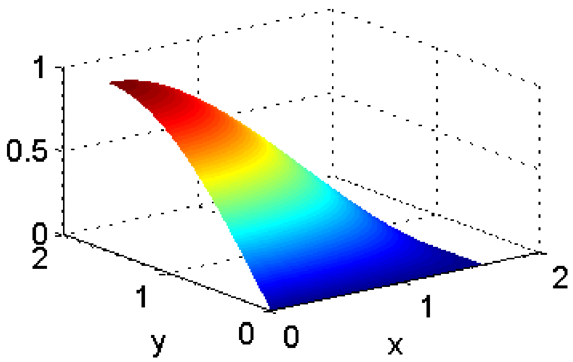
$$f(x,y) = \cos(x)\,\sin(y)$$

Base of box – the lower bounds (a_x and a_y) and upper bounds (b_x and b_y) determine the area of the rectangular base of the surface

$$Volume \quad V = \int_{a_y}^{b_y}\int_{a_x}^{b_x}\left(\cos(x)\,\sin(y)\right)dx\,dy$$

$a_x = 0 \quad b_x = \pi/2 \quad a_y = 0 \quad b_y = \pi/2 \quad N = 299$

Exact volume (analytical)
$V = 1.000$
Simpson's [2D] rule
$V = 1.000000000008578$



Volume *of* a hemisphere using Cartesian coordinates

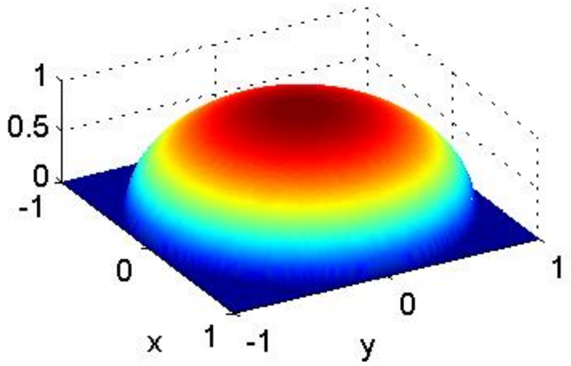
Volume of a hemisphere of radius a $\quad V = \dfrac{2\pi a^3}{3}$

Function

$$x^2 + y^2 \le a^2 \quad f(x,y) = \sqrt{a^2 - x^2 - y^2} \qquad x^2 + y^2 > a^2 \quad f(x,y) = 0$$

$$\text{Volume} \quad V = \int_{a_y}^{b_y}\int_{a_x}^{b_x}\left(\sqrt{a^2 - x^2 - y^2}\right)dx\,dy$$

$a_x = -1 \;\; b_x = 1 \;\; a_y = -1 \;\; b_y = 1$



Exact volume (analytical)

V = 2.094395102393195

Simpson's [2D] rule

N = 99 $\quad V$ = 2.094**417986583109**

N = 999 $\quad$ V = 2.094395**646847362**

A logical Matlab function is used to define the function when $y > 1 - x$. The code to define the function is

```
f = real(sqrt(a^2 - xx.^2 - yy.^2));
f((xx.^2 + yy.^2) > a^2) = 0;
```

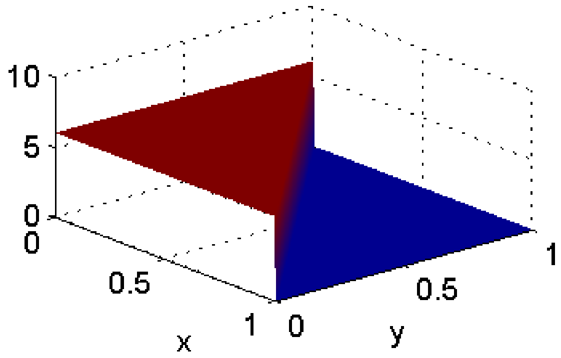
Volume V over a triangular base

$$y \le 1 - x \quad f(x,y) = h \qquad y > 1 - x \quad f(x,y) = 0$$

$$Volume \quad V = \int_{a_y}^{b_y} \int_{a_x}^{b_x} h \, dx \, dy$$

$a_x = 0$ $b_x = 1$ $a_y = 0$ $b_y = 1$ height $h = 6$

Exact volume (analytical)
$V = 3.0000$



Simpson's [2D] rule

$N = 299$ $V = 3.00$**6673873549240**
$N = 999$ $V = 3.00$**1944436635462**
$N = 999$ $V = 3.00$**1944436635462**
$N = 2999$ $V = 3.000$**632027607761**

Even with $N = 2999$ the calculation took less than 1.0 s on a fast Windows computer.

The differences between the exact and computed values is due to the rectangular grid and the condition on the function being zero when $y > 1 - x$
($y = 1 - x$ is a diagonal line and the grid is rectangular).

A logical Matlab function is used to define the function when $y > 1 - x$. The code to define the function is

```
f = h .* ones(num,num);
f(yy > 1 - xx) = 0;
```

Example 3 Polar coordinates

$$V = \int_{a_\phi}^{b_\phi} \int_{a_\rho}^{b_\rho} f(\rho,\phi)\, d\rho\, d\phi$$

where a point (x_P, y_P) has polar coordinates (ρ, ϕ) where

$$x_P = \rho\cos(\phi) \quad y_P = \rho\sin(\phi)$$

The grid pattern for the integration is shown in figure (2) for $N = 9$. At each point the function and Simpson [2D] coefficients are calculated.

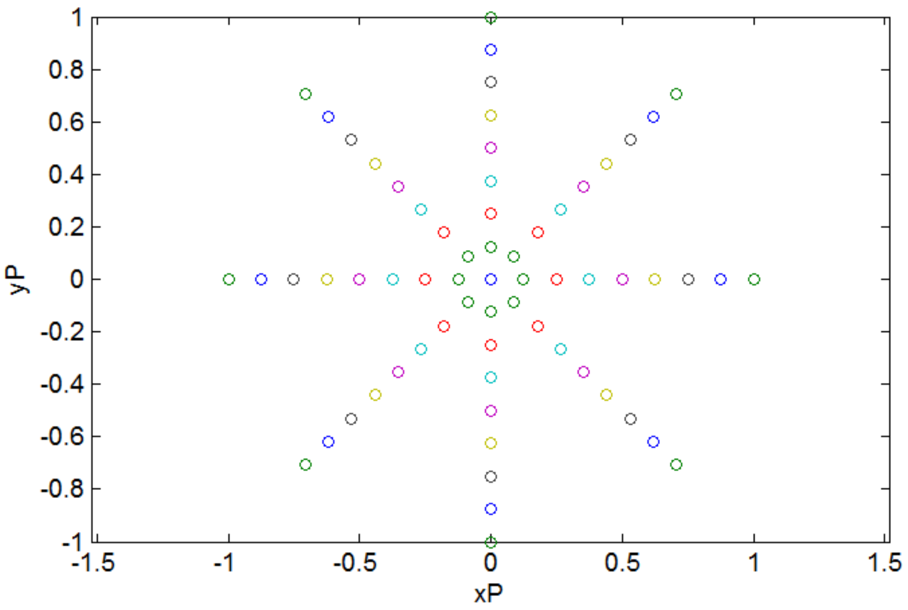


Fig. 2. The grid pattern when using polar coordinates.
Number of partitions $N = 9$ and number of grid points N x $N = 81$.

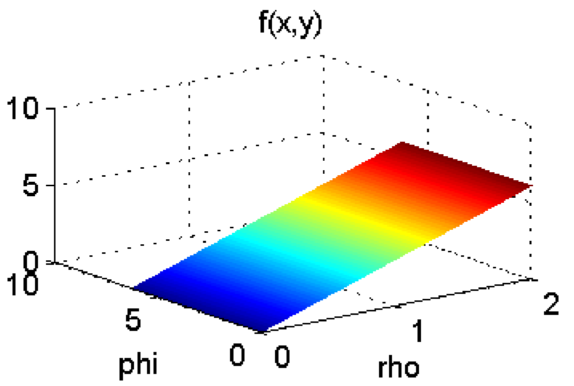
Volume of a cylinder of radius *a* and height *h*

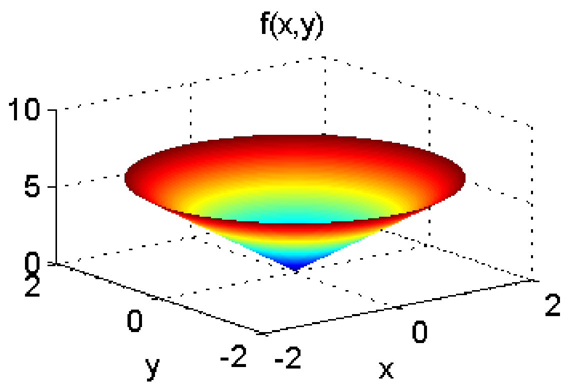
$$f(x,y) = h\rho \qquad V = \int_0^{2\pi}\int_0^a h\,\rho\,d\rho\,d\phi$$

$a_x = 0 \;\; b_x = 2 \;\; a_y = 0 \;\; b_y = 2\pi \quad N = 299$

Exact volume (analytical) $V = 37.699111843077517$

Simpson's [2D] rule $V = 37.699111843077510$





Volume of a hemisphere of radius *a*

$$f(x,y) = \rho^2 \qquad V = \int_0^{2\pi}\int_0^a \rho^2\,d\rho\,d\phi$$

$a_x = 0 \;\; b_x = 2 \;\; a_y = 0 \;\; b_y = 2\pi \;\; N = 299$

Exact volume (analytical) $V = 16.755160819145562$

Simpson's [2D] rule $V = 16.755160819145559$