Ian Cooper
School of Physics, University of Sydney
ian.cooper@sydney.edu.au

DOWNLOAD DIRECTORY FOR MATLAB SCRIPTS

op_rs_rxy_01.m
Calculation of the irradiance in a plane perpendicular to the optical axis for a uniformly illuminated rectangular aperture.

simpsonxy_coeff.m
Function to calculate Simpson’s two-dimensional coefficients

fn_distancePQ.m
Function to calculate distance between two points

turningPoints.m
Function to find the zero crossings of a function and its maxima and minima

op_rs_rxy_02.m
Updated version of op_rs_rxy_01.m. Calculation of the irradiance in a plane perpendicular to the optical axis for a rectangular aperture. The irradiance is not scaled but calculated in W.m^{-2}. The energy per second from the aperture to the observation screen is calculated and the energy per second within a circle of radius equal to the position of the first minima in the X direction in the observation screen is also calculated.

simpson2d.m
Function to calculate the value of a two-dimensional integral using the Simpson’s [2D] method.

fn_distancePQ.m
Function to calculate distance between two points
FRAUNHOFER DIFFRACTION – RECTANGULAR APERTURE

We will consider the Fraunhofer diffraction patterns for a single rectangular aperture that is uniformly illuminated. The X and Y widths the rectangular aperture are \(a_x\) and \(a_y\) respectively. The irradiance distribution \(I\) is given by the product of two single slit functions

\[
I = I_o \left( \sin\left(\frac{v_{px}}{v_p}\right) \right)^2 \left( \sin\left(\frac{v_{py}}{v_p}\right) \right)^2
\]

Fraunhofer diffraction

where \(I_o\) is a normalizing constant and the optical coordinates \(v_{px}\) and \(v_{py}\) are

\[
v_{px} = \frac{1}{2} k a_x \sin \theta_x = \left(\frac{\pi}{\lambda}\right) a_x \sin \theta_x \quad \quad v_{py} = \frac{1}{2} k a_y \sin \theta_y = \left(\frac{\pi}{\lambda}\right) a_y \sin \theta_y
\]

and the angles \(\theta_x\) and \(\theta_y\) define the direction of the diffracted ray

\[
\sin \theta_x = \frac{x_p}{\sqrt{x_p^2 + z_p^2}} \quad \sin \theta_y = \frac{y_p}{\sqrt{y_p^2 + z_p^2}}
\]

The geometry for the diffraction pattern from a rectangular aperture is shown in figure (1).

![Geometry for a rectangular aperture with dimensions \(a_x \times a_y\).](image)

The radial coordinates are scaled perpendicular distance from the optical axis.

\[
\sin \theta = \sqrt{\frac{x_p^2 + y_p^2}{x_p^2 + y_p^2 + z_p^2}}
\]
The resulting diffraction pattern for the irradiance has lines of zeros when

\[(3a)\] \[v_{px} = m_{x} \pi \quad m_{x} = \pm 1, \pm 2, \pm 3, \ldots \quad \text{and} \quad v_{py} = m_{y} \pi \quad m_{y} = \pm 1, \pm 2, \pm 3, \ldots \]

or

\[(3b)\] \[m_{x} \lambda = a_{x} \sin \theta_{x} \quad m_{y} \lambda = a_{y} \sin \theta_{y}\]

Fig. 1. Photograph like image for the Fraunhofer diffraction pattern of a rectangular aperture \(a_{y} = 2a_{x}\).

However, a more versatile approach to study the diffraction from an aperture is to evaluate the Rayleigh-Sommerfeld diffraction integral of the first kind. No approximations need to be made and the irradiance can be calculated in the near and far fields in an XY plane and along the Z axis right. There is excellent agreement between the far field predictions of the Fraunhofer theory and the results of evaluating the Rayleigh-Sommerfeld diffraction integral.
RAYLEIGH DIFFRACTION INTEGRAL OF THE FIRST KIND

The Rayleigh-Sommerfeld region includes the entire space to the right of the aperture. It is assumed that the Rayleigh-Sommerfeld diffraction integral of the first kind is valid throughout this space, right down to the aperture. There are no limitations on the maximum size of either the aperture or observation region, relative to the observation distance, because no approximations have been made.

The Rayleigh-Sommerfeld diffraction integral of the first kind (RS1) can be expressed as

\[ E_P = \frac{1}{2\pi} \int_A \int_{S_A} E_Q \frac{e^{jk r_{PQ}}}{r_{PQ}} \left( jk r_{PQ} - 1 \right) dS \]

where \( E_P \) is the electric field at the observation point \( P \), \( E_Q \) is the electric field within the aperture and \( r_{PQ} \) is the distance from an aperture point \( Q \) to the point \( P \). The double integral is over the area of the aperture \( S_A \).

Approach 1 to evaluating The Diffraction Integral

The double integral can be estimate numerically by a two-dimensional form of Simpson’s 1/3 rule. The electric field \( E_P \) at the point \( P \) is computed by

\[ E_P = E_0 \sum_{m=1}^{N} \sum_{n=1}^{N} S_{mn} E_{Qmn} \frac{e^{jk r_{Qmn}}}{r_{Qmn}} \left( jk r_{Qmn} - 1 \right) \]

where \( S_{mn} \) are the Simpson’s two-dimensional coefficients and \( E_0 \) is a normalizing constant. Each term in equation (2) can be expressed as a matrix of size \( N \times N \) and the matrices can be manipulated very easily in Matlab to give the estimate of the integral. The irradiance is proportional to the square of the magnitude of the electric field, hence the irradiance in the space beyond the aperture can be calculated by

\[ I = I_0 |E^*|^2 \]

where \( I_0 \) is a normalizing constant and \( E^* \) is the complex conjugate of \( E \).
**Approach 2 to evaluating The Diffraction Integral**

Instead of using arbitrary units for electric field and irradiance we can perform all calculations in S.I. units.

The irradiance or more generally the term intensity has S.I. units of W.m\(^{-2}\). Another way of thinking about the irradiance is to use the term energy density as an alternative. The use of the letter \(I\) can be misleading, therefore, we will use the symbol \(u\) to represent the irradiance or energy density.

The irradiance or energy density \(u\) of a monochromatic light wave in matter is given in terms of its electric field \(E\) by

\[
(7) \quad u = \frac{cn\varepsilon_0}{2}|E|^2
\]

\(n\) is the refractive index of the medium, \(c\) is the speed of light in vacuum and \(\varepsilon_0\) is the permittivity of free space. This formula assumes that the magnetic susceptibility is negligible, i.e. \(\mu \approx 1\) where \(\mu\) is the magnetic permeability of the light transmitting media. This assumption is typically valid in transparent media in the optical frequency range.

At a point \(Q\) in the aperture plane, the energy density is given by the symbol \(u_Q\) and in the observation plane at a point \(P\) the symbol used is \(u_P\). The energy transferred \(U\) from the aperture plane to observation plane per second is found by integrating the energy density with respect to an area

\[
(8a) \quad U = \int_S u \, dS \quad \text{[W or J/s]}
\]

The energy per second radiated from the aperture is

\[
(8b) \quad U_Q = \int_{S_0} u_Q \, dS \quad \text{integration over the area of the aperture}
\]

The energy per second reaching an area \(S_P\) of the observation plane is

\[
(8c) \quad U_P = \int_{S_r} u_P \, dS \quad \text{integration over the area element of the observation plane}
\]

In the mscript `op_rs_rxy_02.m` for the aperture, the dimensions \(a_x\) and \(a_y\) are specified [m] and the energy density \(u_Q\) [W.m\(^{-2}\)]. Hence, the electric field \(E_Q\) within the aperture is
(9) \[ E_Q = \sqrt{\frac{2u_Q}{cn\varepsilon_0}} \]

The integral to find the electric field \( E_P \) at a point \( P \) in the observation plane is

(4) \[ E_P = \frac{1}{2\pi} \int \int_{S_A} E_Q \frac{e^{jkS_Q}}{r_{PQ}^3} z_p (jkS_Q - 1) \, dS \]

where \( E_Q \) is given by equation (9) and the integral is computed using a two-dimensional form of Simpson’s 1/3 rule.

The integrals given in equations (8) are also computed using the two-dimensional form of Simpson’s 1/3 rule.

Once \( E_P \) is known, the energy density \( u_P \) is calculated from equation (7) and energy \( U_P \) from equation (8c).
**SIMULATIONS**  
**op_rs_rxy_01.m**

**Rectangular aperture (far field)**

Input parameters

\[
\begin{align*}
nP &= 495; \quad \text{% observation points for P must be ODD} \\
nQ &= 147; \quad \text{% aperture points for Q must be ODD} \\
wL &= 632.8 \times 10^{-9}; \quad \text{% wavelength [m]} \\
aQx &= 1 \times 10^{-4}; \quad \text{% defines aperture size [m]} \\
aQy &= 2 \times 10^{-4}; \\
aP &= 30 \times 10^{-3}; \quad \text{% defines size of observation space [m]} \\
zQ &= 0; zP = 1; \quad \text{% z distance between aperture plan and observation plane [m]} \\
\end{align*}
\]

Fig. 2. Irradiance patterns in the along the X and Y axes. The maximum irradiance has been normalized to 1. The bottom curves use a log scale for the irradiance $I_{\text{dB}} = 10\log_{10}(I)$. 
Figures (2) and (3) shows the irradiance for a rectangular aperture with $a_y = 2a_x$. The diffraction pattern is characterized by a strong central maximum and very weak peaks of decreasing magnitude away from the optical axis. The separation of the dark bands along the $Y$ axes is narrower than along the $X$ axis. Figure (3) shows photograph like images of the diffraction pattern. The irradiance is scaled to represent the different exposure times.

![Image](image.png)

Fig. 3. Photograph like images for the diffraction patterns for a rectangular aperture. The image on the right is scaled to represent a longer exposure time.

In Tables 2 and 3 are listed values of the first few maxima and zeros of the diffraction pattern that were calculated using op_rs_rxy_01.m and the function turningPoints.m. The function returns the values of indexMax and indexMin. Then in the Command Window the values of the radial coordinates for the peaks and zeros can be displayed.

View document of Turning points of a function

The theoretical values are for the zero crossing and minima and maxima of the function $y_m^2 = (\sin x / x)^2$ which gives the Fraunhofer diffraction pattern for a single slit.

View document of the sinc function

Table 2. Relative irradiances of the maxima of the diffraction pattern.

<table>
<thead>
<tr>
<th>peaks</th>
<th>central</th>
<th>1$\text{st}$</th>
<th>2$\text{nd}$</th>
<th>3$\text{rd}$</th>
<th>4$\text{th}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>theory</td>
<td>0</td>
<td>± 1.429</td>
<td>± 2.462</td>
<td>± 3.470</td>
<td>± 4.478</td>
</tr>
<tr>
<td>$P / \pi$</td>
<td>0</td>
<td>± 1.440</td>
<td>± 2.457</td>
<td>± 3.473</td>
<td>± 4.470</td>
</tr>
<tr>
<td>theory</td>
<td>1.0000</td>
<td>0.0472</td>
<td>0.0165</td>
<td>0.0083</td>
<td>0.0050</td>
</tr>
<tr>
<td>$I/I_o$</td>
<td>1.0000</td>
<td>0.0471</td>
<td>0.0165</td>
<td>0.0083</td>
<td>0.0050</td>
</tr>
</tbody>
</table>
Table 3. Zeros in the irradiance for the diffraction pattern

<table>
<thead>
<tr>
<th>zeros</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>theory</td>
<td>± 1</td>
<td>± 2</td>
<td>± 3</td>
<td>± 4</td>
</tr>
<tr>
<td>( v_P / \pi )</td>
<td>± 0.9981</td>
<td>± 1.9960</td>
<td>± 2.9937</td>
<td>± 3.9910</td>
</tr>
</tbody>
</table>

How finding the zero crossings (indexMin) and maxima (indexMax) in the Command Window where indexMin gives the position in an array for the zeros and indexMax gives the position in arrays for the maxima. Note the Matlab variable for the irradiance is \( WP \).

Type in the Command Window:

```matlab
yData = WP(indexyP,:); xData = xP(indexyP,:); [indexMin indexMax] = turningPoints(xData,yData);

indexMin \rightarrow 24 55 86 117 179 210 241 272

vPx(indexyP,indexMin)./pi \rightarrow
-3.9910 -2.9937 -1.9960 -0.9981 0.9981 1.9960 2.9937 3.9910

indexMax \rightarrow 9 40 72 104 148 192 224 256 287

vPx(indexyP,indexMax)./pi \rightarrow
-4.470 -3.473 -2.457 -1.4440 0.0000 1.4440 2.457 3.473 4.470

WP(indexyP,indexMax) \rightarrow
0.0050 0.0083 0.0165 0.0471 1.0000 0.0471 0.0165 0.0083 0.0050
```

In a figure Window you can use the Data Cursor to find values for the zero crossings and maxima.

There is excellent agreement between the predictions of the Fraunhofer approximation and the calculations performed by evaluating the Rayleigh-Sommerfeld diffraction integral.
Diffraction pattern in near field

Fig. 4. Diffraction pattern from a rectangular aperture. Observation plane close to aperture plane $z_p = 100\lambda$ and observation plane dimensions $10\lambda \times 10\lambda$. 
**Diffraction – single slit**

slit dimensions $a_x = 1 \times 10^{-4}$ m $a_y = 1 \times 10^{-6}$ m

![Diffraction pattern from a single slit.](image)

Fig. 5. Diffraction pattern from a single slit.
Diffraction – square aperture

slit dimensions $a_x = 1 \times 10^{-4}$ m $a_y = 1 \times 10^{-4}$ m

Fig. 6. Diffraction pattern from a square aperture with different scaling for the irradiance.
The parameters for the aperture and observation planes are all changed within the mscript. After the program is run, a summary of the input and output parameters is shown in the Command Window.

Table 1: Parameter summary  [SI units]

- Wavelength [m] = 6.33e-07
- nQ = 099
- nP = 201

Aperature Space
- X width [m] = 1.000e-04
- Y width [m] = 2.000e-04
- Energy density [W/m²] uQmax = 1.000e-03
- Energy from aperture [J/s] UQ(theory) = 2.000e-11
- Energy from aperture [J/s] UQ(calculated) = 2.000e-11

Observation Space
- X width [m] = 4.000e-02
- Y width [m] = 4.000e-02
- Distance aperture to observation plane [m] zP = 1.000e+00
- Fraunhofer: position of 1st min in X direction [m] = 6.328e-03
- Fraunhofer: position of 1st min in Y direction [m] = 3.164e-03
- Max energy density [W./m²] uPmax = 9.987e-07
- Energy to aperture [J/s] UP = 1.902e-11

Percentage energy enclosed within circle of radius (vPx = 3.14159) = 99.5

Elapsed time is 40.470176 seconds.
Fig. 7. The variation on the energy density along the X and Y directions. The plot was created in Matlab version 2014b which has improved graphics compared to earlier versions.

Fig. 8. Black and white photograph like time exposure of the diffraction pattern. The yellow circle has a radius equal to the position of the first zero along the X axis. 99.5% of the energy incident upon the observation screen is concentrated within this circle.
Fig. 9. A Matlab surf style plot showing the energy density in the XY observation plane.

**Investigations**

In the Fraunhofer region (far field) the inverse square law applies. This can be tested by doubling the aperture plane to the observation plane distance and noting that the maximum energy density at a point on the Z axis decreases by a factor of 4. For example using the data as shown in Table 1 and only varying the distance $z_P$:

\[
z_{P1} = 1
\]

Energy density  
$u_{Pmax1} = 9.987 \times 10^{-7}$ W.m$^{-2}$

$1^{st}$ min  
$x_{1min} = 6.328 \times 10^{-3}$ m  
$y_{1min} = 3.164 \times 10^{-3}$ m

\[
z_{P2} = 2
\]

Energy density  
$u_{Pmax2} = 2.3497 \times 10^{-7}$ W.m$^{-2}$

$1^{st}$ min  
$x_{2min} = 12.66 \times 10^{-3}$ m  
$y_{2min} = 6.328 \times 10^{-3}$ m

\[
\frac{z_{P2}}{z_{P1}} = 2
\]

$u_{Pmax2} / u_{Pmax1} = \frac{1}{4}$ inverse square law satisfied

\[
\frac{x_{2min}}{x_{1min}} = 2.0 \quad \frac{y_{2min}}{y_{1min}} = 2.0 \quad \text{pattern is wider}
\]
Fig. 10. Fraunhofer diffraction: upper plot $z_p = 1.0$ m and lower plot $z_p = 2.0$ m. As the aperture to observation plane distance increases the energy density distribution shape does not change but becomes wider and flatter.
Next, we can consider the effect of halving the area of the aperture by changing the width $a_y$ in the Y direction from $2 \times 10^{-4}$ m to $1.0 \times 10^{-4}$ m.

Width $a_{y1} = 2 \times 10^{-4}$ m $a_{y2} = 1 \times 10^{-2}$ m

Energy transferred from aperture to observation plane

$U_{Q1} = 2.0 \times 10^{-11}$ J.s$^{-1}$ $U_{Q2} = 1.0 \times 10^{-11}$ J.s$^{-1}$

$U_{Q2} / U_{Q1} = 0.50$

Maximum energy density in observation plane

$u_{Pmax1} = 9.987 \times 10^{-7}$ W.m$^{-2}$ $u_{Pmax2} = 2.497 \times 10^{-7}$ W.m$^{-2}$

$u_{Pmax2} / u_{Pmax1} = 0.25$

The effect of doubling the area of the aperture results in a doubling of the energy transferred from the aperture to the observation plane and a four-fold increase in the energy density.

**Fresnel Diffraction ⇔ Fraunhofer Diffraction**

Fraunhofer diffraction is only valid in the far field. There is a transition from Fraunhofer diffraction to Fresnel diffraction as the distance between the aperture and observation planes decreases. The distance dividing the two regimes is known as the Rayleigh distance $d_{RL}$

$$d_{RL} = \frac{a^2}{\lambda} \quad \text{where } a \text{ is the maximum of } a_x \text{ and } a_y$$

Fraunhofer diffraction (far field) $z_P > d_{RL}$

Fresnel diffraction (near field) $z_P < d_{RL}$

For an aperture $a_x = a_y = 1.0 \times 10^{-4}$ m and wavelength $\lambda = 6.33 \times 10^{-7}$ m, the Rayleigh distance is

$$d_{RL} = 1.58 \times 10^{-2} \text{ m}$$

The mscript *op_rs_rxy_02.m* can be run a few times to investigate the transition regime when $z_P \sim d_{RL}$ by changing the values for $z_P, x_P$ and $y_P$ in the mscript.
Rayleigh distance \( d_{RL} = 0.0158 \, \text{m} \)