

On the recurrence phenomenon of a resonant spring pendulum

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(Received 13 May 1983; accepted for publication 12 July 1983)

Olsson's coupled equations for a resonant spring pendulum are solved. To-and-fro transfer of energy from one mode to another is shown, the period for such a recurrence is derived, and the patterns of the mass trajectory are obtained. Experimental data are reported and good agreement is found with the theory.

I. INTRODUCTION

When the length and the mass of a spring pendulum are chosen so that the frequency of the spring-type oscillation is twice that of the pendulum-type oscillation, it is long known^{1,2} that the motion initially in one mode will change to the other and back and forth. This *resonant* spring pendulum and its "peculiar" behavior has recently attracted much attention³⁻⁹ because it serves as a very good example for demonstrating the phenomena of parametric instability and nonlinear resonant coupling widely studied in nonlinear optics⁹ and nonlinear plasma physics.¹⁰

Theoretically, this is a two-dimensional mechanical problem. Keeping terms up to the third order in the Lagrangian, Minorsky¹ obtained two coupled equations of motion in terms of polar coordinates while Olsson³ got an *equivalent* but *simpler* set of equations in terms of Cartesian coordinates. They both pointed out that the startup of the sideways pendulum-type mode from the vertical spring-type mode is due to a parametric instability¹¹ whereas the reverse is due to a resonance effect of the linear type. However, their analysis (and some others⁴) is restricted to the initial stage of each process and therefore cannot explain the recurrence phenomenon or the to-and-fro transfer of energy from one mode to another. A multiple-time-scale perturbation method has been tried by Falk⁶ to account for the recurrence; the correct result was however not obtained due to a serious error in his analysis. Rusbridge⁸ has studied Olsson's equations extensively and has also presented experimental and computational results indicating the contribution from higher-order effects. However, he did not obtain enough quantitative results concerning the recurrence phenomenon to make an explicit comparison between theory and experiments. Furthermore, although his equations for the slow-varying amplitudes and phases are correct, the derivation contains an improper assumption.⁸

Nayfeh¹² has also derived the equations for the slow-varying amplitudes and phases from the less-simple formulation in terms of polar coordinates.¹

In this paper, we solve Olsson's coupled equations of motion through a perturbation method which assumes that

the amplitudes and phases of the two oscillating modes are slowly varying. This method is essentially that used by Rusbridge,⁸ which turns out to be simpler than the multiple-time-scale perturbation method¹³ widely used in solving such problems.¹⁴ Equations for the slow quantities are properly derived. Detailed analysis of these equations shows the to-and-fro transfer of energy from one mode to another. Time variation of the amplitude in terms of an elliptic function is given and the period for the recurrence process is derived. Furthermore, patterns of the mass trajectory or the Lissajou's figures are obtained and their relation to Olsson's observation of stable modes of motion is discussed. In order to test the theory, we have constructed resonant spring pendulums and made measurements on the period of one complete to-and-fro transfer of energy. This is described in Sec. III.

II. ANALYSIS OF THE PROBLEM

We start with Olsson's coupled equations of motion³:

$$\ddot{x} + \omega_p^2 x = 3\omega_p^2 xz/L, \quad (1)$$

$$\ddot{z} + 4\omega_p^2 z = 3\omega_p^2 x^2/2L, \quad (2)$$

where x and z are, respectively, the horizontal and the vertical components of the mass position relative to the equilibrium point, L is the length of the spring (usually plus a

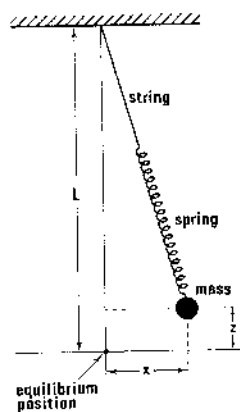


Fig. 1. A spring pendulum.

connecting string) at equilibrium (Fig. 1), and ω_p is the angular frequency of the pendulum-type oscillation. Here we have set the angular frequency of the spring-type oscillation equal to $2\omega_p$. Note that the validity of (1) and (2) is limited to $x/L \ll 1$ and $z/L \ll 1$.

It is worth pointing out that the coupled equations (1) and (2) describe a special case of the more general resonant interaction between three oscillators or three waves studied in nonlinear optics⁹ and nonlinear plasma physics.¹⁰

A. Equations for amplitudes and phases

Since x/L and z/L are small compared to unity, we may, to the lowest order, set the quantities on the right-hand side of (1) and (2) equal to zero. This leads to two sinusoidal solutions, one oscillating at frequency ω_p and the other at $2\omega_p$, both with *constant* amplitude and phase. The inclusion of the next-order terms in (1) gives rise to two small driving forces, one at frequency ω_p and the other at $3\omega_p$. The small force oscillating at $3\omega_p$ is of no interest because it cannot pump the system resonantly. The force oscillating at ω_p , on the other hand, can change the behavior substantially. But this resonant force is small, the rate of change is thus slow. Similar argument holds for (2), though here the two small driving forces are separately of zero frequency and $2\omega_p$ frequency. We therefore assume that x and z oscillate with *slow-varying* amplitudes and phases, i.e.,

$$x = A(t) \cos[\omega_p t + \phi(t)], \quad (3)$$

$$z = B(t) \cos[2\omega_p t + \psi(t)]. \quad (4)$$

Substituting (3) and (4) into (1), neglecting terms containing \dot{A} , $\dot{\phi}$ or $\dot{\phi}\dot{A}$ which are of second-order smallness and ignoring the small nonresonant driving force, we obtain an equation which essentially consists of two terms, one linear in $\sin(\omega_p t + \phi)$ and the other linear in $\cos(\omega_p t + \phi)$. Since the coefficients of these terms contain only slow-varying quantities, they must separately equal to zero. We thus have

$$\dot{A} = -3\omega_p AB \sin(2\phi - \psi)/4L, \quad (5)$$

$$\dot{\phi} = -3\omega_p B \cos(2\phi - \psi)/4L. \quad (6)$$

In a similar way, Eq. (2) leads to

$$\dot{B} = 3\omega_p A^2 \sin(2\phi - \psi)/16L, \quad (7)$$

$$\dot{\psi} = -3\omega_p A^2 \cos(2\phi - \psi)/16LB. \quad (8)$$

Noting that the right-hand sides in (5)–(8) depend on the phase angles through the same $2\phi - \psi$, we define

$$\chi \equiv 2\phi - \psi, \quad (9)$$

and combine (6) and (8) into

$$\dot{\chi} = -(3\omega_p/2L)(B - A^2/8B) \cos \chi. \quad (10)$$

B. Constants of motion

We now seek solutions to (5)–(8). Eliminating $\sin \chi$ from (5) and (7), we easily obtain

side a $B \sin \chi$ term and an $A^2 \sin \chi$ term which can be expressed in terms of A and B through (5) and (7). This eventually leads to

$$(\tan \chi) \dot{\chi} = 2\dot{A}/A + \dot{B}/B, \quad (13)$$

giving the second constant of motion

$$A^2 B \cos \chi = N_0. \quad (14)$$

With (12), this equation describes how χ varies with A or B and it will be shown useful in explaining why the system possesses relatively stable configurations of the particle trajectory as noted by Olsson.³

C. Time-varying amplitude

To find how A varies with time, we must eliminate B and χ from (5), (7), and (10). To this end, we multiply (5) with A and then take another time derivative on the whole resulting equation. Elimination of \dot{A} , \dot{B} , and $\dot{\chi}$ can be done with the aid of (5), (7), and (10). We therefore arrive at

$$\frac{d^2}{dt^2} A^2 = \frac{9\omega_p^2}{32L^2} A^2 (2M_0^2 - 3A^2), \quad (15)$$

where (12) has been used to eliminate B . This resembles a one-dimensional Newtonian equation of motion of a particle in a potential field. After integration once, we have

$$\frac{1}{2} \left(\frac{d\alpha}{dt} \right)^2 + V(\alpha) = E, \quad (16)$$

where

$$\alpha \equiv A^2/M_0^2, \quad (17)$$

$$\tau \equiv 3\omega_p t M_0 / (4\sqrt{2}L), \quad (18)$$

$$V(\alpha) \equiv -\alpha^2 + \alpha^3, \quad (19)$$

and E is a constant. The “potential energy” function $V(\alpha)$ is plotted against the “position” α in Fig. 2. By (12) and (17) we need only consider the domain of α between 0 and 1. The “energy” E must be negative in order to have a meaningful solution. Suppose such a negative E value intersects with $V(\alpha)$ at α_0 and α_1 where $0 < \alpha_0 < \alpha_1 < 1$ (Fig. 2). This limits α to vary between the two turning points α_0 and α_1 . This in turn means that the corresponding amplitude A of the horizontal oscillation, if initially equal to $M_0 \sqrt{\alpha_0}$, in-

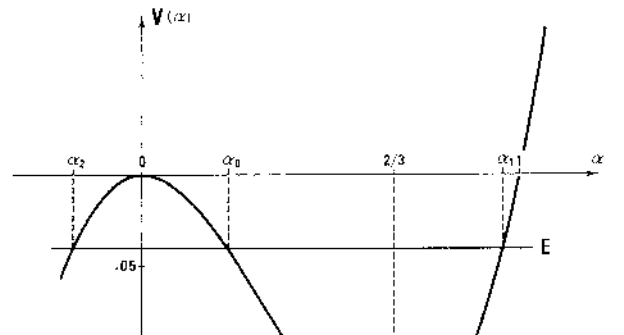


Table I. Experimental data for the three different equilibrium length L 's. The initial vertical displacement z_0 was always set at -10 cm.

$L = 86$ cm		$L = 112$ cm		$L = 138$ cm	
$L/ z_0 = 8.6$	T/T_p	$L/ z_0 = 11.2$	T/T_p	$L/ z_0 = 13.8$	T/T_p
$ z_0 /x_0$		$ z_0 /x_0$		$ z_0 /x_0$	
1	6.5	1	9	1	10.5
2	9.5	2	12	2	13.75
3	10.5	3	14	3	17
5	12.75	5	15.5	5	20

counted in one cycle, and one-half of the number gave T/T_p . The experimental data are collected in Table I. Each T/T_p value is in fact the mean value resulted from at least two releases, one at positive x_0 and the other at negative x_0 .

To make a comparison with (25), we first note that, in terms of the initial amplitudes x_0 and z_0 , we have

$$\alpha_0 = 1/(1 + 4z_0^2/x_0^2), \quad (28)$$

and (25) may be rewritten as

$$\frac{T}{T_p} = \frac{4}{3\pi} \frac{L}{|z_0|} \left(\frac{1 - \alpha_0}{1 + 3\alpha_0} \right)^{1/4} F\left(\frac{\pi}{2}, k\right), \quad (29)$$

where we have made use of (12), (17), and (21). From (21) and (24), we may obtain a useful expression for k :

$$k = 1 - \alpha_0 + \alpha_0^2, \quad (30)$$

valid for $\alpha_0 \ll 1$. With $F(\pi/2, k)$ determined from the tables in Ref. 16, T/T_p may be plotted against $|z_0|/x_0$ at a constant value of $L/|z_0|$. The three curves in Fig. 4 are such theoretical results corresponding to $L/|z_0| = 8.6, 11.2,$ and 13.8 . Experimental data points according to Table I are also shown for comparison. Judging from the fact that a

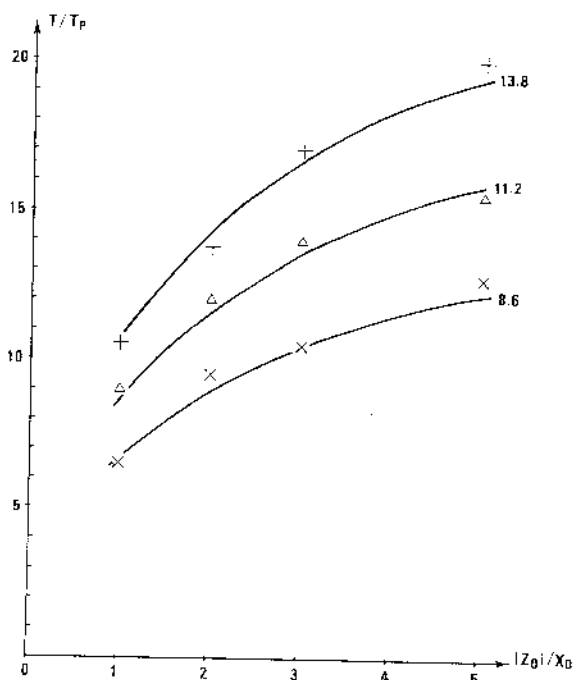


Fig. 4. T/T_p vs $|z_0|/x_0$ at $L/|z_0| = 13.8, 11.2,$ and 8.6 . Curves are obtained from the theory [Eq. (29)]. The experimental data points are from Table I.

complete cycle is not easy to be determined accurately, we conclude that the experimental results are in good agreement with the theory.

We have also studied the figures of mass trajectory and the orientation of motion. The features observed well agree with those predicted by the theory in Sec. II D.

Before leaving this section, we note that the mass of the spring is about or less than $\frac{1}{3}$ that of the weight and therefore contributes at most $\frac{1}{3}$ to the equivalent mass in relation to the spring-type oscillation. Furthermore, in relation to the pendulum-type oscillation, the effect of the spring mass may be estimated by treating the system as a physical pendulum and by comparing the distance of the center of oscillation from the pivot with the L 's in our experiments. We have found that the difference is less than $L/50$. We thus conclude that the finite mass of the spring in our experiments should not affect our theoretical solutions for an idealized spring pendulum.

IV. CONCLUSION AND DISCUSSION

In this paper, we have solved the coupled equations as formulated by Olsson in describing the "peculiar" behavior of a resonant spring pendulum. The to-and-fro transfer of energy from one mode of oscillation to another has been shown to take place, the period for such a recurrence has been derived and the patterns of mass trajectory have been obtained. Experiments have been carried out and the comparison with the theory has been found in good agreement. It is felt that both the standard analysis and the simple experiments presented here could help a great deal in understanding the physics of nonlinear resonant coupling between several oscillators or waves.

Finally, we want to point out that $F(\pi/2, k)$ in (25) approaches infinity as k tends to 1 or α_0 tends to 0. This means that the sidewise oscillation does not occur if the mass is initially displaced with *exactly* $x_0 = 0$. However, because it is very difficult, if not impossible, to have such an ideal initial condition and because there always exist small influences from the environment, the mass will begin to swing sideways no matter how carefully it was vertically displaced initially.

ACKNOWLEDGMENT

The author acknowledges a fruitful correspondence with Professor J. Walker of Cleveland State University. He is also grateful to Chun-sen Cheng for experimental assistance.

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