Quantum correction to the linear response for a magnetized electron gas

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It is shown how the fully relativistic quantum expression for the response of an arbitrary magnetized electron (plus positron) gas reproduces its nonquantum counterpart. In the relativistic quantum case the dispersion is due to both gyromagnetic absorption and one-photon pair creation. Although one-photon pair creation has no classical counterpart, somewhat surprisingly it needs to be retained to reproduce the nonquantum limit correctly. For unpolarized electrons it is shown that the first quantum correction (order $\hbar$) to the nonquantum limit for the dispersion vanishes when one sums over all excited states, as for an unmagnetized electron gas. However, in the magnetized case there is a contribution of order $\hbar$ from the ground state, which is the only state with a specific spin. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1515271]

I. INTRODUCTION

The linear response tensor for a magnetized, relativistic quantum electron gas was first calculated by Svetozarova and Tsytovich[1] and discussed further by later authors.[2–6] However, it has not been shown in detail how the relativistic quantum case reproduces its well-known nonquantum limit. Moreover, the first quantum corrections to the nonquantum limit have not been determined. Our purposes in this paper are to show explicitly that the relativistic quantum expression for the linear response tensor reproduces the (relativistic) nonquantum limit and to evaluate the first quantum correction to it for unpolarized electrons, where "unpolarized" is explained below.

Five intrinsically quantum effects need to be considered in a general treatment of dispersion in a magnetized electron gas. Three of these are included in the detailed discussion here: the quantum recoil; dispersion associated with one-photon pair annihilation; and dependence on the ratio B/B_c, where $B_c = m^2c^2/e\hbar = 4.4\times 10^9$ T is the so-called critical magnetic field. The critical field is that for which the cyclotron energy, $\hbar\Omega$, $\Omega = eB/m$, is equal to the rest energy, mc^2. The dependence on B/B_c arises from quantization of the gyromagnetic motion and the inclusion of the spin. Gyromagnetic motion is simple harmonic, described by an orbital quantum number, $l = 0,1,\ldots$. The fourth quantum effect is the spin, described by a spin quantum number, $s = \pm 1$. Spin-orbit coupling is an intrinsically relativistic effect and the quantum numbers l and s are combined into the Landau quantum number, $n = l + \frac{1}{2}(1+s) = 0,1,\ldots$, such that the perpendicular momentum is quantized as $(2neB\hbar)^{1/2}$. The energy eigenvalues for a particle with parallel momentum p_z are $\varepsilon_n = [m^2c^4(1+2nB/B_c) + p_z^2c^2]^{1/2}$. The ground state, or lowest Landau level, $n=0$, $l=0$, $s=-1$, is nondegenerate and each excited state is doubly degenerate. By "unpolarized" electrons we imply that each excited state is equally populated for $s=\pm 1$. The fifth quantum effect is the vacuum polarization, which is ignored in this paper.

Dispersive effects are described by the Hermitian part of the response tensor, and dissipative effects by the anti-Hermitian part. There is a one-to-one relation (e.g., Kramers–Kronig) between dispersive effects and dissipative effects.[7,8] The only dispersive effect included in a nonquantum treatment is that associated with gyromagnetic absorption. The generalization to the relativistic quantum case introduces three additional dissipative processes. One of these may be treated classically: dispersion associated with gyromagnetic absorption by positrons, whose presence needs to be included in a relativistic quantum treatment. The other two are one-photon pair creation, which is most familiar in the context of the vacuum polarization,[9] and gyromagnetic absorption involving a spin–flip transition. The spin–flip contribution disappears in the nonquantum limit.[10] However, at least in the approach adopted here, it turns out to be necessary to retain the terms that describe the dispersive effects due to pair creation in the nonquantum limit; these terms cancel otherwise spurious terms that arise from the nonquantum limit of the gyromagnetic absorption terms.

The nonquantum limit involves the zeroth order terms in an expansion in B/B_c. The dependence on B/B_c appears both through the energy eigenvalues, and through the dependence of the wave functions of functions $J^n_{n-n'}(x)$ that are related to Laguerre polynomials with the argument $x = \hbar k_\perp^2/2eB = (\hbar k_\perp/mc)^2/2(B/B_c)$, with $k_\perp^2 = \mathbf{k}^2 - k_z^2$. Dispersion is associated with virtual transitions between states, and this decomposes into a sum over all pairs of Landau states, labeled n, n' here. The function $J^n_{n-n'}(x)$ reduces to a Bessel function, $J_{n-n'}(z)$, $z = k_\perp p_\perp/eB$, in the nonquantum limit, $\hbar\to 0$, $n\to\infty$, $\hbar n \to p_\perp^2/2eB$.

The main new result in this paper, apart from the explicit demonstration that the requirement that the known nonquantum limit be reproduced, is that the first quantum correction vanishes when one averages over the spin states for all excited states. This result is assumed implicitly in semiclassical treatments of gyromagnetic emission and absorption, in which the first quantum correction is included through the quantum recoil, with emission and absorption contributing to the dispersion with opposite sign of the recoil so that there is no correction of order $\hbar$. However, this cancellation of first order quantum corrections does not apply to a spin-dependent electron gas, and it does not apply to the ground state which is nondegenerate and which does give a contribution of order $\hbar$.

This presence of a correction of order $\hbar$ for electrons (and positrons) in their ground state is relevant to an application to pulsars; in a pulsar magnetosphere all the electrons are expected to be in their ground state, due to rapid gyromagnetic losses depopulating all excited Landau levels. The dispersion in such a one-dimensional electron gas is conventionally treated using a one-dimensional ($p_\perp=0$) classical model[11–17] for which $B/B_c\ll 1$ applies. Use of the nonquantum result neglects corrections only of order $(B/B_c)^2$ for excited states, but it neglects corrections of order B/B_c for the ground state, which is the case of relevance to pulsars. With $B/B_c\gtrsim 0.1$ in many pulsars, it is important to retain corrections of order B/B_c. (There are some pulsars with $B/B_c\gtrsim 1$, when an expansion in B/B_c is not justified.)

The formalism used here to describe the response of the electron gas is covariant, with Greek indices taking on the values 0,1,2,3, and Latin indices describing the space component 1,2,3. The metric tensor, $g^{\mu\nu}$, is diagonal $(1,-1,-1,-1)$. SI units are used, with $c=1$ henceforth, implying $\mu_0=1/\epsilon_0$ ($\mu_0\to 4\pi$ in Gaussian units). The plasma response is described in terms of the four-dimensional Fourier transform, $J^\mu(k)$ of the four-current density, with $k=[\omega,\mathbf{k}]$ the wave four-vector. The electromagnetic disturbance is described by the four-potential, $A^\mu(k)$, and the linear response tensor, $\alpha^{\mu\nu}(k)$, is defined by writing $J^\mu(k)=\alpha^{\mu\nu}(k)A_\nu(k)$. Charge continuity and gauge invariance require $k_\mu\alpha^{\mu\nu}(k)=0$ and $k_\nu\alpha^{\mu\nu}(k)=0$, respectively.

In Sec. II we write down general expressions for $\alpha^{\mu\nu}(k)$ in the relativistic quantum and nonquantum cases. In Sec. III we show how the nonquantum limit is reproduced starting from the relativistic quantum form. In Sec. IV we outline the proof that the only nonzero first quantum correction to the response tensor for unpolarized electrons arises from the nondegenerate ground state. Our results are discussed briefly in Sec. V.

II. GENERAL EXPRESSIONS FOR THE RESPONSE TENSOR

The method used here to calculate the relativistic quantum form of $\alpha^{\mu\nu}(k)$ is a generalization of the calculation of the vacuum polarization tensor in terms of the amplitude for the Feynman "bubble" diagram; the electron propagator *in vacuo* is replaced by a statistically averaged propagator in a magnetic field. The classical calculation that is most closely analogous to this relativistic quantum calculation is the forward-scattering method.[18] However, in taking the nonquantum limit of the relativistic quantum form, the classical form that is reproduced most directly is that derived using the Vlasov approach. The Vlasov form, which involves derivatives of the particle distribution function, is related to the forward-scattering form by a partial integration.

The relativistic quantum form is written down in this section, and the sum over spin states is performed assuming that the electrons are unpolarized. The Vlasov and forward-scattering forms for the response tensor are then written down.

A. The response tensor derived using QED

The derivation of the response tensor involves starting from the Dirac wave functions for an electron in a magnetic field. We assume plane-wave functions $\propto\exp[-i\epsilon(\varepsilon t-p_z z)/\hbar]$, where the magnetic field is along the z axis, and with $\epsilon=1$ for electrons and $\epsilon=-1$ for positrons. The parallel momentum, p_z, is then the physical parallel momentum for either an electron or a positron. For formal purposes we denote the set of quantum numbers by q, with the energy eigenvalues $\epsilon\varepsilon_q$, and the occupation number n_q^ϵ. In this notation, a general expression for the linear response tensor is[4]

$$\alpha^{\mu\nu}(k)=-\frac{e^3B}{2\pi\hbar}\sum_{\epsilon,q,\epsilon',q'}\int\frac{dp_z}{2\pi\hbar}\int\frac{dp'_z}{2\pi\hbar}2\pi\hbar$$
$$\times\delta(\epsilon' p'_z-\epsilon p_z+\hbar k_z)\,\frac{\frac{1}{2}(\epsilon'-\epsilon)+\epsilon n_q^\epsilon-\epsilon' n_{q'}^{\epsilon'}}{\hbar\omega-\epsilon\varepsilon_q+\epsilon'\varepsilon_{q'}+i0}$$
$$\times[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})]^\mu[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})]^{*\nu},\tag{1}$$

where the vertex function, $[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})]^\mu$, depends on the choice of spin operator. The response tensor (1) includes the response of the magnetized vacuum, which corresponds to the term $\frac{1}{2}(\epsilon'-\epsilon)$ in the numerator. The resulting expression diverges and a finite result for the vacuum response tensor is derived from it by a regularization procedure.[19,20] The vacuum response is neglected here.

There are two sets of quantum numbers in (1), denoted by ϵ,q and ϵ',q'. The set q includes the Landau quantum number $n=0,1,2,\ldots$, p_z and the spin $s=\pm 1$, and similarly q' includes n',p'_z,s'. The energy eigenvalues depend only on n,p_z, and we adopt the notation $\varepsilon_n=(m^2+p_z^2+2\hbar neB)^{1/2}$, $\varepsilon'_{n'}=(m^2+p_z'^2+2\hbar n'eB)^{1/2}$.

B. The vertex function

An explicit form for the vertex function can be written down once a specific choice of spin operator is made. There is a uniquely preferred spin operator when considering gyromagnetic effects, and this is the magnetic-moment operator,[4,21–23] which has eigenvalues $s\varepsilon_n^0$, with $s=\pm 1$ and $\varepsilon_n^0=(m^2+2\hbar neB)^{1/2}$. For this choice of spin operator, the vertex operator has the explicit form,

$$\begin{aligned}[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})]^\mu=b'^{*}_{n'}b_n(&[\lambda'_{\epsilon's'}\lambda_{\epsilon s}+P'P\lambda'_{-\epsilon's'}\lambda_{-\epsilon s}][\beta'_{s'}\beta_s J_{n'-n}^{n-1}+s's\beta'_{-s'}\beta_{-s}J_{n'-n}^{n}],\\ &-s'[\lambda'_{s'}\lambda_s-P'P\lambda'_{-s'}\lambda_{-s}][\beta'_{-s'}\beta_s e^{-i\psi}J_{n'-n+1}^{n-1}+s's\beta'_{s'}\beta_{-s}e^{i\psi}J_{n'-n-1}^{n}],\\ &-is'[\lambda'_{s'}\lambda_s-P'P\lambda'_{-s'}\lambda_{-s}][\beta'_{-s'}\beta_s e^{-i\psi}J_{n'-n+1}^{n-1}-s's\beta'_{s'}\beta_{-s}e^{i\psi}J_{n'-n-1}^{n}],\\ &P[\lambda'_{s'}\lambda_{-s}+P'P\lambda'_{-s'}\lambda_s][\beta'_{s'}\beta_s J_{n'-n}^{n-1}+s's\beta'_{-s'}\beta_{-s}J_{n'-n}^{n}]),\end{aligned}$$
$$b_n=\frac{(ie^{i\psi})^n}{(2\varepsilon_n^0 2\varepsilon_n)^{1/2}},\quad \lambda_s=(\varepsilon_n+\epsilon s\varepsilon_n^0)^{1/2},\quad \beta_s=(\varepsilon_n^0+sm)^{1/2},\tag{2}$$

with $P=p_z/|p_z|$, and similarly for the primed quantities. In (2) the wave vector is written $\mathbf{k}=(k_\perp\cos\psi,k_\perp\sin\psi,k_z)$, and one is free to choose $\psi=0$ by rotating the axes so that $\mathbf{k}$ is in the $x-z$ plane. The J-functions are related to generalized Laguerre polynomials,

$$J_\nu^n(x)=(-)^\nu J_{-\nu}^{n+\nu}(x)=\left(\frac{n!}{(n+\nu)!}\right)^{1/2}e^{-x/2}x^{\nu/2}L_n^\nu(x), \tag{3}$$

with argument $x=\hbar k_\perp^2/2eB$. It is implicit that the function $J_\nu^n(x)$ is identically zero for negative n. Relevant properties of $J_\nu^n(x)$ are written down in Appendix A.

C. Sum over spin states

When the occupation number is independent of spin, s, or when one averages over any s dependence, only the sum over the spins, s,s', of the product of vertex functions appears in (1). The sum is written here as

$$[\Pi_{n'n}^{\epsilon'\epsilon}(\mathbf{k})]^{\mu\nu}=\sum_{s,s'}[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})]^\mu[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})]^{*\nu}. \tag{4}$$

The explicit form of $[\Pi_{n'n}^{\epsilon'\epsilon}(\mathbf{k})]^{\mu\nu}$, which is independent of the choice of spin operator, is written down in Appendix B.

D. The response tensor derived classically

Two general methods for calculating the response tensors for a magnetized plasma are the Vlasov method and the forward-scattering method.

The classical covariant derivation using the Vlasov method leads to the following response tensor:[18]

$$\begin{aligned}\alpha^{\mu\nu}(k)=\sum_\epsilon e^2\int\frac{d^3\mathbf{p}}{\gamma}\bigg[u_z^\mu\frac{\partial}{\partial p_{z\nu}}+\frac{u_\parallel^\mu p_\parallel^\nu}{p_\perp}\frac{\partial}{\partial p_\perp}\\ -\sum_{a=-\infty}^{\infty}\frac{[U^\epsilon]^\mu(a,k)[U^\epsilon]^{*\nu}(a,k)}{(ku)_\parallel-a\Omega}\\ \times\left(\frac{(ku)_\parallel}{u_\perp}\frac{\partial}{\partial p_\perp}+k_z\frac{\partial}{\partial p_z}\right)\bigg]f^\epsilon(\mathbf{p}),\end{aligned} \tag{5}$$

where $u^\mu=p^\mu/m$ is the four-velocity, a is an integer and where $f^\epsilon(\mathbf{p})$ are the classical distribution functions for electrons ($\epsilon=1$) and positrons ($\epsilon=-1$). The invariant $(ku)_\parallel=\gamma(\omega-k_zv_z)$ is defined by $(ku)_\parallel=g_\parallel^{\mu\nu}k_\mu u_\nu$, where the metric tensor $g^{\mu\nu}=g_\perp^{\mu\nu}+g_\parallel^{\mu\nu}$ is separated into its projections onto the $x-y$ and $0-z$ planes, respectively. Formally, this is achieved by writing the Maxwell tensor for the background magnetic field in the form $Bf^{\mu\nu}$, and using $f^{\mu\nu}$ to construct $g_\perp^{\mu\nu}=-f^\mu{}_\alpha f^{\alpha\nu}$ and hence $g_\parallel^{\mu\nu}$. The projections $k_\parallel^\mu=g_\parallel^{\mu\nu}k_\nu$, $u_\parallel^\mu=g_\parallel^{\mu\nu}u_\nu$ are also four-vectors. The remaining quantity in (5) is the four-vector,

$$\begin{aligned}[U^\epsilon]^\mu(a,k)=\gamma(J_a,\tfrac{1}{2}v_\perp[e^{-i\epsilon\psi}J_{a-1}+e^{i\epsilon\psi}J_{a+1}],\\ i\epsilon\tfrac{1}{2}v_\perp[e^{-i\epsilon\psi}J_{a-1}-e^{i\epsilon\psi}J_{a+1}],v_zJ_a),\end{aligned} \tag{6}$$

with $\gamma=(m^2+p_\perp^2+p_z^2)^{1/2}/m$. The argument of the Bessel functions is $k_\perp R$, with $R=p_\perp/eB=\gamma v_\perp/\Omega_0$ the radius of gyration.

The classical response tensor for a magnetized gas derived using the forward-scattering method can be written in the form,

$$\begin{aligned}\alpha^{\mu\nu}(k)=-\frac{q^2}{m}\int\frac{d^3\mathbf{p}}{\gamma}f(\mathbf{p})\sum_{a=-\infty}^{\infty}\bigg\{g_\parallel^{\mu\nu}J_a^2\\ -\frac{k_\parallel^\mu U^{*\nu}+k_\parallel^\nu U^\mu}{(ku)_\parallel-a\Omega_0}J_a+\frac{(k^2)_\parallel U^\mu U^{*\nu}}{[(ku)_\parallel-a\Omega_0]^2}\\ +\frac{[(ku)_\parallel-a\Omega_0]^2}{[(ku)_\parallel-a\Omega_0]^2-\Omega_0^2}\bigg[g_\perp^{\mu\nu}J_a^2\\ -\frac{k_\perp^\mu U^{*\nu}+k_\perp^\nu U^\mu}{(ku)_\parallel-a\Omega_0}J_a+\frac{(k^2)_\perp U^\mu U^{*\nu}}{[(ku)_\parallel-a\Omega_0]^2}\bigg]\\ +\frac{i\eta\Omega_0[(ku)_\parallel-a\Omega_0]}{[(ku)_\parallel-a\Omega_0]^2-\Omega_0^2}\bigg[f^{\mu\nu}J_a^2\\ +\frac{k_G^\mu U^{*\nu}-k_G^\nu U^\mu}{(ku)_\parallel-a\Omega_0}J_a\bigg]\bigg\},\end{aligned} \tag{7}$$

with $k_G^\mu=f^{\mu\nu}k_\nu$ and $(k^2)_\parallel=g_\parallel^{\mu\nu}k_\mu k_\nu=\omega^2-k_z^2$. The form (7) may be obtained from (5) by a cumbersome partial integration, and rearrangement using the recursion formulas and sum rules for Bessel functions.

III. REDUCTION TO THE NONQUANTUM LIMIT

In this section we show how the relativistic form (1) for the response tensor reproduces the classical expressions (5) and (7). For the nonquantum limit it is convenient to concentrate on the Vlasov form (5), and when considering the first quantum corrections it is more convenient to consider the forward-scattering form (7). To reproduce (5) for electrons one approximates the difference $n_q^+-n_{q'}^+$ by the first term in a Taylor series expansion, and to reproduce (7) one treats n_q^+ and $n_{q'}^+$ separately, relabeling the sum of q' as one over q.

A. The nonquantum limit

The nonquantum limit involves expanding in $\hbar|k_z|\ll|p_z|$, $\hbar\omega\ll m$, $n-n'\ll n$, and it also involves treating n as a continuous variable, with $p_n\to p_\perp$ or $n\to p_\perp^2/2eB\hbar$. The function $J_{n'-n}^n(\hbar k_\perp^2/2eB)$ for large n is approximated by an expansion in Bessel functions, the zeroth and first order terms of which are

$$\begin{aligned}\lim_{n\to\infty}J_{n'-n}^n(\hbar k_\perp^2/2eB)\\ =J_{n'-n}(p_\perp k_\perp/eB)\\ +\frac{\hbar k_\perp}{2p_\perp}(n'-n+1)J'_{n'-n}(p_\perp k_\perp/eB).\end{aligned} \tag{8}$$

The denominator in (1) is approximated by making a Taylor series expansion,

$$\varepsilon'_{n'}(p'_z)=\left[1-\left(a\frac{\partial}{\partial n}+\hbar k_z\frac{\partial}{\partial p_z}\right)+\frac{1}{2}\left(a\frac{\partial}{\partial n}+\hbar k_z\frac{\partial}{\partial p_z}\right)^2+\cdots\right]\varepsilon_n(p_z), \tag{9}$$

where the second order derivatives lead to the quantum recoil term. The resonant denominator in the first term of (1) for $\epsilon=\epsilon'=1$ then reduces according to $\hbar\omega-\varepsilon_n(p_z)+\varepsilon'_{n'}(p'_z)\to\omega-a\Omega/\gamma-k_zv_z$, which reproduces the denominator in the classical Vlasov form (5) of the response tensor.

Consider the term associated with gyromagnetic absorption by electrons ($\epsilon'=\epsilon=1$) in (1). In reproducing the Vlasov form (5), the occupation number for the primed state is expanded in a Taylor series,

$$\begin{aligned} n^+_{n'}(p'_z)&=n^+_{n-a}(p_z-\hbar k_z)\\ &=\left[1-\left(a\frac{\partial}{\partial n}+\hbar k_z\frac{\partial}{\partial p_z}\right)+\cdots\right]n^+_n(p_z),\\ a\frac{\partial}{\partial n}&\to\frac{\hbar aeB}{p_\perp}\frac{\partial}{\partial p_\perp}\\ &=\frac{\hbar[-(\omega-a\Omega_0/\gamma-k_zv_z)+\omega-k_zv_z]}{v_\perp}\frac{\partial}{\partial p_\perp}, \end{aligned}\tag{10}$$

where a is the harmonic number, and where $2n\hbar eB\to p_\perp^2$ is used in the latter relation. The occupation number may then be rewritten in terms of the classical distribution function,

$$\frac{2n^+_n(p_z)}{(2\pi\hbar)^3}\to f^+(p_\perp,p_z),\quad n\to\frac{p_\perp^2}{2\hbar eB},\tag{11}$$

where $n=p_\perp^2/2\hbar eB$ is assumed large.

The vertex function in (1) is approximated by the leading term in (8). Transitions without a spin–flip, $s=s'=\pm1$ reduce to the same nonquantum limit, and transitions with a spin–flip are intrinsically quantum mechanical and do not contribute in the nonquantum limit. (Spin–flip transitions can be neglected entirely in the derivation of the Vlasov result, but to obtain the forward-scattering result one needs to include them and sum over them.) To lowest order one finds

$$[\Gamma^{\epsilon\epsilon}_{q'q}(\mathbf{k})]^\mu=(-ie^{i\epsilon\psi})^a[U^\epsilon]^\mu(a,k)/\gamma,\tag{12}$$

where $[U^\epsilon]^\mu(a,k)$ is given by (6). In deriving (12) the identity $J_{-a}(z)=(-1)^aJ_a(z)$ is used. For positrons one needs to interchange the primed and unprimed quantities in (1) in order for the contribution to the response tensor to appear in the same form as for electrons. This interchange in the role of the primed and unprimed states implies that the absorption process for a positron is from the state with occupation number $n^-_{n'}(p_z-\hbar k_z)$ to the state with occupation number $n^-_n(p_z)$, with $n=n'+a$. This interchanges the roles of initial and final states compared with electrons. With this change, one find that (12) also applies for positrons, $\epsilon'=\epsilon=-1$.

The nonquantum approximation to (4) for $\epsilon'=\epsilon$ in the Vlasov approach then becomes

$$[\Pi^{\epsilon\epsilon}_{n'n}(\mathbf{k})]^{\mu\nu}=2\frac{[U^\epsilon]^\mu(a,k)[U^\epsilon]^{*\nu}(a,k)}{\gamma^2},\tag{13}$$

where the factor of 2 arises from the sum over s',s with only $s'=s$ contributing.

The derivation of the forward-scattering form (7) in the nonquantum limit is more tedious than for the Vlasov form (5) because one needs to retain the first order term in (8), and hence also in (12) and (13), and also the second order terms in the Taylor expansion (9).

B. The Vlasov form for the response tensor in the nonquantum limit

After summing over the spin states, the general expression for the linear response tensor of (1) for electrons becomes

$$\begin{aligned} \alpha^{\mu\nu}(k)=&-\pi e^2\sum_a\int dp_z\\ &\times\int dp_\perp p_\perp\left(\frac{f^+(p_\perp,p_z)-f^+(p'_\perp,p_z-\hbar k_z)}{\hbar\omega-\varepsilon_n+\varepsilon'_{n'}}\right.\\ &\times2[\Pi^{++}]^{\mu\nu}+\frac{f^+(p_\perp,p_z)}{\hbar\omega-\varepsilon_n-\varepsilon_{n'}}[\Pi^{-+}]^{\mu\nu}\\ &\left.-\frac{f^+(p'_\perp,-p_z-\hbar k_z)}{\hbar\omega+\varepsilon_n+\varepsilon_{n'}}[\Pi^{+-}]^{\mu\nu}\right), \end{aligned}\tag{14}$$

with $[\Pi^{++}]^{\mu\nu}$ evaluated at $p'_z=p_z-\hbar k_z$, $[\Pi^{-+}]^{\mu\nu}$ at $p'_z=-p_z+\hbar k_z$, and $[\Pi^{+-}]^{\mu\nu}$ at $p'_z=-p_z-\hbar k_z$, and where the sums over n and n' are replaced by a sum over $n-n'=a$ and an integral over $n=p_\perp^2/2\hbar eB$. Using (10) and (13) in the first term in (14) and changing the variable of integration from p_z to $-p_z$ in the last term, one obtains

$$\begin{aligned} \alpha^{\mu\nu}(k)=&-\pi e^2\int dp_z\int dp_\perp\left(\frac{-2m}{\gamma}\frac{\partial f^+(p_\perp,p_z)}{\partial p_\perp}\right.\\ &\times\sum_a U^\mu U^{*\nu}+2\sum_a\frac{p_\perp U^\mu U^{*\nu}}{\gamma^2(\omega-a\Omega_0/\gamma-k_zv_z)}\\ &\times\left(\frac{\omega-k_zv_z}{v_\perp}\frac{\partial}{\partial p_\perp}+k_z\frac{\partial}{\partial p_z}\right)f^+(p_\perp,p_z)\\ &\left.-v_\perp f^+(p_\perp,p_z)\sum_a[\Pi^{-+}]^{\mu\nu}\right). \end{aligned}\tag{15}$$

In the last term, which is associated with dispersion due to pair creation, it is noted that, for $\mu\nu=00,ij$, if one changes the sign of p_z in $[\Pi^{+-}]^{\mu\nu}$, explicitly and implicitly within p'_z and $\varepsilon_{n'}$, it is equal to $[\Pi^{-+}]^{\mu\nu}$, and that for $\mu\nu=i0,0j$, $[\Pi^{+-}]^{\mu\nu}$ and $[\Pi^{-+}]^{\mu\nu}$ can be neglected in the nonquantum limit. The sums over a can be performed over the Bessel functions in the first and last terms using

$$\begin{aligned} &\sum_{a=-\infty}^{\infty}J_a^2(z)=1,\quad\sum_{a=-\infty}^{\infty}a^2J_a^2(z)=\tfrac{1}{2}z^2,\\ &\sum_{a=-\infty}^{\infty}J'^2_a(z)=\tfrac{1}{2}, \end{aligned}\tag{16}$$

together with $J_{-a}(z)=(-1)^aJ_a(z)$. The results of these sums are $\Sigma_aU^\mu U^{*\nu}=\varepsilon_n^2/m^2$, ε_np_z/m^2, $p_\perp^2/2m^2$, p_z^2/m^2 for

$\mu\nu=00$, 03 and 30, 11 and 22, 33, respectively and $\Sigma_a[\Pi^{-+}]^{\mu\nu}=2[1-p_\perp^2/2\varepsilon_n^2]$, $2[1-p_z^2/\varepsilon_n^2]$ for $\mu\nu=11$ and 22, 33, respectively.

The middle term of (15) reproduces the final term in the classical response tensor. The remaining term can be reproduced as follows. On partially integrating the 11 and 22 components of the gyromagnetic term leaving an expression proportional to $n_n^+(p_z)$, these components cancel with the 11 and 22 components of the dispersion term due to pair creation. On partially integrating the remaining 33 component of the dispersion term, one obtains the term proportional to $\partial f_n^+/\partial p_z$ present in the classical response tensor. In this way (1) reduces to (5) for electrons in the limit $\hbar\to 0$.

C. The forward-scattering form of the nonquantum limit of the response tensor

To obtain the response tensor in forward-scattering form (7) for electrons, one writes (1) as follows:

$$\begin{aligned}\alpha^{\mu\nu}(k)=&-\frac{e^3B}{(2\pi\hbar)^2}\int_{-\infty}^{+\infty}dp_z\sum_{n,a}\left(\frac{n_n^+(p_z)[\Pi^{++}]^{\mu\nu}}{\hbar\omega-\varepsilon_n+\varepsilon_{n'}'}\right.\\&\left.+\frac{n_n^+(p_z)[\Pi^{-+}]^{\mu\nu}}{\hbar\omega-\varepsilon_n-\varepsilon_{n'}}\right)\\&+\frac{e^3B}{(2\pi\hbar)^2}\int_{-\infty}^{+\infty}dp_z'\sum_{n',a}\left(\frac{n_{n'}^+(p_z')[\Pi^{++}]^{\mu\nu}}{\hbar\omega-\varepsilon_n+\varepsilon_{n'}}\right.\\&\left.+\frac{n_{n'}^+(p_z')[\Pi^{+-}]^{\mu\nu}}{\hbar\omega+\varepsilon_n+\varepsilon_{n'}}\right),\end{aligned}\tag{17}$$

with $[\Pi^{-+}]^{\mu\nu}$ in the first term evaluated at $p_z'=p_z-\hbar k_z$, $[\Pi^{-+}]^{\mu\nu}$ evaluated at $p_z'=-p_z+\hbar k_z$, $[\Pi^{-+}]^{\mu\nu}$ in the third term evaluated at $p_z=p_z'+\hbar k_z$, and $[\Pi^{+-}]^{\mu\nu}$ evaluated at $p_z=-p_z'+\hbar k_z$. After replacing the sum over n by an integral over $p_\perp$ in the first two terms, and the sum over n' by an integral over $p_\perp'$ in the remaining two terms, and dropping the primes on the variables of integration p_z', $p_\perp'$, it is straightforward to carry out the expansions in $\hbar$. A lengthy calculation reproduces (7).

IV. THE FIRST QUANTUM CORRECTION TO THE RESPONSE TENSOR

In this section we outline a proof that the quantum correction to the response tensor vanishes to first order in $\hbar$ when one averages over the spins of the excited states. The nondegenerate ground state, however, is shown to have a contribution of order $\hbar$. Only electrons are considered explicitly; the extension of the proof to unpolarized positrons is trivial. The ground state needs to be treated separately, due to it being intrinsically spin dependent; it is easier to treat the ground state explicitly using the forward-scattering form than the Vlasov form.

A. Specific quantum corrections

Extending the derivation of either the Vlasov or the forward-scattering forms to first order in $\hbar$ is straightforward but tedious. If one ignores the ground state, either derivation leads to a null result: the first quantum corrections vanish. In our calculation of the forward-scattering form, after replacing the sums over n,n' by integrals over $p_\perp$, $p_\perp'$ in (17) the corrections terms originating from the sum over n' cancels with the corrections terms arising from the sum over n. Similarly, the first quantum corrections to the derivation of the Vlasov form give zero provided that one ignores the ground state.

The treatment of the ground state needs to be special because it is nondegenerate, whereas all the excited states are doubly degenerate. The replacement of the sums over n and n' in (17) by integrals over $p_\perp=(2\hbar neB)^{1/2}$ and $p_\perp'=(2\hbar n'eB)^{1/2}$, which involves $\hbar$ explicitly, is not relevant for $n=0$, $p_\perp=0$. The replacement (11) of the occupation number by the classical distribution function is also not correct for the ground state and one must use the replacement

$$n_0^\epsilon(p_z)=\frac{(2\pi\hbar)^2}{eB}f^\epsilon(p_z).$$

As already noted the first quantum corrections give zero after replacing the sums over n,n' by integrals, and we need consider only this extra contribution from the ground state (gs). For soft photons, this is of the form,

$$\alpha_{\rm gs}^{\mu\nu}=-\frac{e^3B}{(2\pi\hbar)^2}\int_{-\infty}^{\infty}dp_z n_0^+(p_z)d^{\mu\nu},$$

$$\begin{aligned}d^{00}=&-\left[\frac{k_z^2(1-v_z^2)}{\varepsilon_0(\omega-k_zv_z)^2}+\frac{k_\perp^2}{\varepsilon_0[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]}\right]\\&+\hbar\left[-\frac{k_\perp^2k_z^2(1-v_z^2)}{2\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2](\Omega/\gamma)}\right.\\&+\frac{k_\perp^2(\omega^2-k_z^2)(\Omega/\gamma)}{\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]^2}\\&\left.-\frac{k_\perp^4}{\varepsilon_0^2[(\omega-k_zv_z)^2-(2\Omega/\gamma)^2](2\Omega/\gamma)}\right],\end{aligned}$$

$$\begin{aligned}d^{01}=d^{10}=&-\frac{k_\perp(\omega-v_zk_z)}{\varepsilon_0[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]}\\&+\hbar\left[\frac{k_\perp(\omega^2-k_z^2)(\omega-k_zv_z)(\Omega/\gamma)}{\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]^2}\right.\\&\left.-\frac{k_\perp^3(\omega-k_zv_z)}{\varepsilon_0^2[(\omega-k_zv_z)^2-(2\Omega/\gamma)^2](2\Omega/\gamma)}\right],\end{aligned}$$

$$\begin{aligned}d^{02}=-d^{20}=&\frac{ik_\perp\Omega/\gamma}{\varepsilon_0[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]}\\&+\hbar\left[-\frac{ik_\perp(\omega^2-k_z^2)[(\omega-k_zv_z)^2+(\Omega/\gamma)^2]}{2\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]^2}\right.\\&\left.+\frac{ik_\perp^3}{\varepsilon_0^2[(\omega-k_zv_z)^2-(2\Omega/\gamma)^2]}\right],\end{aligned}$$

$$d^{03}=d^{30}=-\left[\frac{k_z\omega(1-v_z^2)}{\varepsilon_0(\omega-k_zv_z)^2}+\frac{v_zk_\perp^2}{\varepsilon_0[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]}\right]+\hbar\times\left[-\frac{k_\perp^2\omega k_z(1-v_z^2)}{2\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2](\Omega/\gamma)}+\frac{k_\perp^2v_z(\omega^2-k_z^2)(\Omega/\gamma)}{\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]^2}-\frac{k_\perp^4v_z}{\varepsilon_0^2[(\omega-k_zv_z)^2-(2\Omega/\gamma)^2](2\Omega/\gamma)}\right],$$

$$d^{11}=d^{22}=-\frac{(\omega-k_zv_z)^2}{\varepsilon_0[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]}+\hbar\left[\frac{(\omega^2-k_z^2)(\omega-k_zv_z)^2(\Omega/\gamma)}{\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]^2}-\frac{k_\perp^2(\omega-k_zv_z)^2}{\varepsilon_0^2[(\omega-k_zv_z)^2-(2\Omega/\gamma)^2](2\Omega/\gamma)}\right],$$

$$d^{12}=-d^{21}=\frac{i(\omega-k_zv_z)\Omega/\gamma}{\varepsilon_0[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]}+\hbar\times\left[-\frac{i(\omega-k_zv_z)(\omega^2-k_z^2)[(\omega-k_zv_z)^2+(\Omega/\gamma)^2]}{2\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]^2}+\frac{i(\omega-k_zv_z)k_\perp^2}{\varepsilon_0^2[(\omega-k_zv_z)^2-(2\Omega/\gamma)^2]}\right],$$

$$d^{13}=d^{31}=-\frac{v_zk_\perp(\omega-v_zk_z)}{\varepsilon_0[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]}+\hbar\left[\frac{v_zk_\perp(\omega^2-k_z^2)(\omega-k_zv_z)(\Omega/\gamma)}{\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]^2}-\frac{v_zk_\perp^3(\omega-k_zv_z)}{\varepsilon_0^2[(\omega-k_zv_z)^2-(2\Omega/\gamma)^2](2\Omega/\gamma)}\right],$$

$$d^{23}=-d^{32}=-\frac{iv_zk_\perp\Omega/\gamma}{\varepsilon_0[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]}+\hbar\left[\frac{iv_zk_\perp(\omega^2-k_z^2)[(\omega-k_zv_z)^2+(\Omega/\gamma)^2]}{2\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]^2}-\frac{iv_zk_\perp^3}{\varepsilon_0^2[(\omega-k_zv_z)^2-(2\Omega/\gamma)^2]}\right],$$

$$d^{33}=-\left[\frac{\omega^2(1-v_z^2)}{\varepsilon_0(\omega-k_zv_z)^2}+\frac{v_z^2k_\perp^2}{\varepsilon_0[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]}\right]+\hbar\left[-\frac{k_\perp^2\omega^2(1-v_z^2)}{2\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2](\Omega/\gamma)}+\frac{k_\perp^2v_z^2(\omega^2-k_z^2)(\Omega/\gamma)}{\varepsilon_0^2[(\omega-k_zv_z)^2-(\Omega/\gamma)^2]^2}-\frac{k_\perp^4v_z^2}{\varepsilon_0^2[(\omega-k_zv_z)^2-(2\Omega/\gamma)^2](2\Omega/\gamma)}\right], \tag{18}$$

where $\psi=0$ is assumed and $\varepsilon_0=(m^2+p_z^2)^{1/2}$. The appearance of a correction of order $\hbar$ is characteristic of a spin dependence, and the first quantum corrections in (18) may be attributed to the ground state having a definite spin.

B. Comparison with the unmagnetized case

For an unmagnetized (unpolarized) electron gas, it is already known that the first quantum correction to the response tensor vanishes[24] and the foregoing proof generalizes this result to the magnetic case, provided that one ignores the ground state. In contrast to the unmagnetized case, the ground state is nondegenerate for magnetized electrons and gives rise to a nonzero first quantum correction. In the unmagnetized case, the second order quantum correction can be written in a simple form, as shown explicitly in Appendix C. However, we have been unable to identify an analogous simple form for the second order corrections associated with the excited states in the magnetized case.

V. DISCUSSION AND CONCLUSIONS

One of the two main results of this paper is the explicit demonstration that the relativistic quantum expression for the response tensor for a magnetized electron gas reduces to its known nonquantum counterpart in the nonquantum limit. This result is obviously required for self-consistency of the theory, but there is one surprising feature in the derivation. In the nonquantum limit it is obvious that one is to ignore intrinsically quantum effects, which include the spin of the electron and the vacuum polarization. However, one cannot ignore another quantum effect: dispersion due to one-photon pair creation. In the theory this effect leads to a nondispersive contribution that cancels a nondispersive contribution from dispersion due to gyromagnetic absorption. In principle, one could reformulate the relativistic quantum theory to exclude these canceling contributions, but *a priori* there is no reason to do so.

The other main result is that the first quantum corrections to the response tensor, obtained by expanding to first order in $\hbar$, are zero when one averages over the spins of the degenerate excited states. This generalizes a known result for an unmagnetized relativistic quantum gas[24] to the magnetized case. This result is of interest from both a formal and a practical viewpoint.

From a formal viewpoint it is ''known'' that a semiclassical derivation reproduces the first quantum correction to gyromagnetic emission and absorption correctly; starting from the classical result for gyromagnetic emission (e.g., the emissivity by an electron), the gyroresonance condition is interpreted in terms of conservation of energy and (parallel) momentum on emission of a photon, with the quantum recoil being the first quantum correction to the gyroresonance condition. This "known" result implies that the dispersion vanishes to first order in $\hbar$ because transitions between any pair of states contribute equally apart from the quantum recoil terms, which contribute with opposite signs for emission and absorption, so that the sum has no term of first order in $\hbar$. A formal proof of this "known" result requires that it be shown that first quantum correction to the dispersion is indeed zero, and to our knowledge ours is the first derivation of this result. The derivation depends explicitly on the assumption that the electrons are unpolarized. The result does not apply to polarized electrons; the response tensor has a nonzero first quantum correction that depends explicitly on the difference in the occupation number for the two spin states for polarized electrons. The result also does not apply to the ground state, which is nondegenerate and gives a contribution to the dispersion of order $\hbar$, which is written down in (18).

A practical application of the foregoing result is to radio-wave dispersion in a pulsar magnetosphere.[25] In conventional treatments of the dispersion, the only "quantum" feature taken into account is that all the electrons are in their ground state, that is, in their lowest Landau level. This results from gyromagnetic emission, which causes all electrons to radiate away their energy perpendicular to the field lines on a time much shorter than other relevant times. The dispersion is conventionally treated in terms of a one-dimensional, relativistic, classical electron gas.[11–17] This approach ignores intrinsically quantum effects, and so is justified only for $B/B_c \ll 1$. Radio pulsars have a range of surface fields with most having $B/B_c \lesssim 0.1$, a significant fraction having $B/B_c \gtrsim 0.1$, and some strong-field pulsars having $0.3 \lesssim B/B_c \lesssim 1$. (A subclass, called magnetars,[26] has even stronger fields, $B/B_c \gg 1$, and for this subclass it is essential to use the relativistic quantum theory; however, magnetars are not known to be radio emitters.) We conclude that intrinsically quantum effects should be taken into account for stronger-field radio pulsars; we propose to discuss this application in detail elsewhere.

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APPENDIX A: EXPANSION OF THE *J*-FUNCTIONS

The functions $J_\nu^n(x)$, defined by (3), satisfies recursion formulas, which follow from the recursion formulas satisfied by the Laguerre polynomials (as defined in Refs. 27 and 28). A basic pair of recursion formulas can be written in a variety of ways. Three related pairs of relations are

$$\begin{aligned}
&n^{1/2}J_\nu^{n-1}(x)=(n+\nu)^{1/2}J_\nu^n(x)-x^{1/2}J_{\nu-1}^n(x),\\
&n^{1/2}J_\nu^{n}(x)=(n+\nu)^{1/2}J_\nu^{n-1}(x)-x^{1/2}J_{\nu+1}^{n-1}(x),\\
&x^{1/2}J_{\nu+1}^{n-1}(x)=(n+\nu)^{1/2}J_\nu^{n-1}(x)-n^{1/2}J_\nu^n(x),\\
&x^{1/2}J_{\nu-1}^{n}(x)=-n^{1/2}J_\nu^{n-1}(x)+(n+\nu)^{1/2}J_\nu^n(x),\\
&\nu J_\nu^{n-1}(x)=x^{1/2}[n^{1/2}J_{\nu+1}^{n-1}(x)+(n+\nu)^{1/2}J_{\nu-1}^n(x)],\\
&\nu J_\nu^{n}(x)=x^{1/2}[(n+\nu)^{1/2}J_{\nu+1}^{n-1}(x)+n^{1/2}J_{\nu-1}^n(x)].
\end{aligned}\tag{A1}$$

Another pair of relations is

$$\begin{aligned}
(x+\nu)J_\nu^n(x)&=[x(n+\nu)]^{1/2}J_{\nu-1}^n(x)\\
&\quad+[x(n+\nu+1)]^{1/2}J_{\nu+1}^n(x),\\
2x\frac{d}{dx}J_\nu^n(x)&=[x(n+\nu)]^{1/2}J_{\nu-1}^n(x)\\
&\quad-[x(n+\nu+1)]^{1/2}J_{\nu+1}^n(x).
\end{aligned}\tag{A2}$$

These become the recurrence relations for Bessel functions in the limit $n\to\infty$, cf. (8).

The expansion of $J_\nu^n(x)$ in Bessel functions applies for large n, $x \propto 1/n$ and $\nu \ll n$. This follows from the expansion of the Laguerre polynomials in Bessel functions in the limit $n\to\infty$,

$$\begin{aligned}
&J_\nu^n\left(\frac{z^2}{4n}\right)=\left[\frac{(n+\nu)!}{n!\,n^\nu}\right]^{1/2}\sum_{a=0}^{\infty} b_a\left(\frac{z}{2n}\right)^a J_{\nu+a}(z),\\
&b_0=1,\quad b_1=-\tfrac{1}{2}(\nu+1),\quad b_2=\tfrac{1}{8}(\nu+1)(\nu+2),\\
&(a+1)b_{a+1}=-\tfrac{1}{2}(\nu+1)b_a+\tfrac{1}{4}(\nu+a)b_{a-1}-\tfrac{1}{4}nb_{a-2}.
\end{aligned}\tag{A3}$$

The leading term and the first correction are included in (8).

APPENDIX B: SUM OVER SPIN STATES

Explicit evaluation of the tensor $[\Pi_{n'n}^{\epsilon'\epsilon}(\mathbf{k})]^{\mu\nu}$ may be derived using the explicit form (2) for the vertex function in the definition (4). The same result is derived for any other choice of spin operator, and the result may also be derived without introducing any spin operator by generalizing a method detailed in Refs. 29–31. The resulting explicit form is

$$\begin{aligned}
2\varepsilon'_{n'}\varepsilon_n[\Pi_{n'n}^{\epsilon'\epsilon}(\mathbf{k})]^{00}&=(\varepsilon'_{n'}\varepsilon_n+p'_zp_z+\epsilon'\epsilon m^2)\\
&\quad\times[(J_{n'-n}^{n-1})^2+(J_{n'-n}^{n})^2]\\
&\quad+2\epsilon'\epsilon\, p_{n'}p_nJ_{n'-n}^{n-1}J_{n'-n}^{n},\\
2\varepsilon'_{n'}\varepsilon_n[\Pi_{n'n}^{\epsilon'\epsilon}(\mathbf{k})]^{01}&=-\epsilon'\varepsilon_np_{n'}[J_{n'-n}^{n-1}e^{i\psi}J_{n'-n+1}^{n-1}\\
&\quad+J_{n'-n}^{n}e^{-i\psi}J_{n'-n-1}^{n}]\\
&\quad-\epsilon\varepsilon'_{n'}p_n[J_{n'-n}^{n}e^{i\psi}J_{n'-n+1}^{n-1}\\
&\quad+J_{n'-n}^{n-1}e^{-i\psi}J_{n'-n-1}^{n}],\\
2\varepsilon'_{n'}\varepsilon_n[\Pi_{n'n}^{\epsilon'\epsilon}(\mathbf{k})]^{02}&=i\epsilon'\varepsilon_np_{n'}[J_{n'-n}^{n-1}e^{i\psi}J_{n'-n+1}^{n-1}\\
&\quad-J_{n'-n}^{n}e^{-i\psi}J_{n'-n-1}^{n}]\\
&\quad+i\epsilon\varepsilon'_{n'}p_n[J_{n'-n}^{n}e^{i\psi}J_{n'-n+1}^{n-1}\\
&\quad-J_{n'-n}^{n-1}e^{-i\psi}J_{n'-n-1}^{n}],
\end{aligned}$$

$$
\begin{aligned}
2\varepsilon'_{n'}\varepsilon_n[\Pi^{\epsilon'\epsilon}_{n'n}(\mathbf{k})]^{03} &= (\varepsilon'_{n'}p_z+\varepsilon_n p'_z)\\
&\quad\times[(J^{n-1}_{n'-n})^2+(J^{n}_{n'-n})^2],\\
2\varepsilon'_{n'}\varepsilon_n[\Pi^{\epsilon'\epsilon}_{n'n}(\mathbf{k})]^{11} &= (\varepsilon'_{n'}\varepsilon_n-p'_zp_z-\epsilon'\epsilon m^2)\\
&\quad\times[(J^{n-1}_{n'-n+1})^2+(J^{n}_{n'-n-1})^2]\\
&\quad+2\epsilon'\epsilon p_{n'}p_n\\
&\quad\times\cos(2\psi)\,J^{n-1}_{n'-n+1}J^{n}_{n'-n-1},\\
2\varepsilon'_{n'}\varepsilon_n[\Pi^{\epsilon'\epsilon}_{n'n}(\mathbf{k})]^{22} &= (\varepsilon'_{n'}\varepsilon_n-p'_zp_z-\epsilon'\epsilon m^2)\\
&\quad\times[(J^{n-1}_{n'-n+1})^2+(J^{n}_{n'-n-1})^2]\\
&\quad-2\epsilon'\epsilon p_{n'}p_n\\
&\quad\times\cos(2\psi)\,J^{n-1}_{n'-n+1}J^{n}_{n'-n-1},\\
2\varepsilon'_{n'}\varepsilon_n[\Pi^{\epsilon'\epsilon}_{n'n}(\mathbf{k})]^{33} &= (\varepsilon'_{n'}\varepsilon_n+p'_zp_z-\epsilon'\epsilon m^2)[(J^{n-1}_{n'-n})^2\\
&\quad+(J^{n}_{n'-n})^2]\\
&\quad-2\epsilon'\epsilon p_{n'}p_nJ^{n-1}_{n'-n}J^{n}_{n'-n},\\
2\varepsilon'_{n'}\varepsilon_n[\Pi^{\epsilon'\epsilon}_{n'n}(\mathbf{k})]^{12} &= -i(\varepsilon'_{n'}\varepsilon_n-p'_zp_z-\epsilon'\epsilon m^2)\\
&\quad\times[(J^{n-1}_{n'-n+1})^2-(J^{n}_{n'-n-1})^2]\\
&\quad+2\epsilon'\epsilon p_{n'}p_n\\
&\quad\times\sin(2\psi)\,J^{n-1}_{n'-n+1}J^{n}_{n'-n-1},\\
2\varepsilon'_{n'}\varepsilon_n[\Pi^{\epsilon'\epsilon}_{n'n}(\mathbf{k})]^{13} &= -\epsilon' p_{n'}p_z[J^{n-1}_{n'-n}e^{-i\psi}J^{n-1}_{n'-n+1}\\
&\quad+J^{n}_{n'-n}e^{i\psi}J^{n}_{n'-n-1}]\\
&\quad-\epsilon p_n p'_z[J^{n}_{n'-n}e^{-i\psi}J^{n-1}_{n'-n+1}\\
&\quad+J^{n-1}_{n'-n}e^{i\psi}J^{n}_{n'-n-1}],\\
2\varepsilon'_{n'}\varepsilon_n[\Pi^{\epsilon'\epsilon}_{n'n}(\mathbf{k})]^{23} &= -i\epsilon' p_{n'}p_z[J^{n-1}_{n'-n}e^{-i\psi}J^{n-1}_{n'-n+1}\\
&\quad-J^{n}_{n'-n}e^{i\psi}J^{n}_{n'-n-1}]\\
&\quad-i\epsilon p_n p'_z[J^{n}_{n'-n}e^{-i\psi}J^{n-1}_{n'-n+1}\\
&\quad-J^{n-1}_{n'-n}e^{i\psi}J^{n}_{n'-n-1}].
\end{aligned}\tag{B1}
$$

The tensor (4) is Hermitian, which determines the remaining terms in terms of those written in (B1). The space-components of (B1) were written down, in a slightly different notation, by Pavlov *et al.*,[32] who summed over the spin in evaluating the relativistic quantum expression for gyromagnetic emission.

APPENDIX C: THE UNMAGNETIZED CASE

The response tensor for an unmagnetized, unpolarized electron gas may be written in the form[24]

$$
\begin{aligned}
\alpha^{\mu\nu}(k) &= -2e^2\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\,\frac{\overline{n}(\mathbf{p})}{\varepsilon}\,\frac{(pk)^2a^{\mu\nu}(k,p/m)}{(pk)^2-(\hbar k^2/2)^2},\\
a^{\mu\nu}(k,u) &= g^{\mu\nu}-\frac{k^\mu u^\nu+k^\nu u^\mu}{ku}+\frac{k^2u^\mu u^\nu}{(ku)^2},
\end{aligned}\tag{C1}
$$

with $\overline{n}(\mathbf{p})=n^+(\mathbf{p})+n^-(\mathbf{p})$ and $\varepsilon=(m^2+\mathbf{p}^2)^{1/2}$. The nonquantum resonance condition is $pk=p^\mu k_\nu=0$, or $ku=0$, and the term $k^2/2=k^\mu k_\mu/2$ in the denominator is then interpreted as the quantum recoil term. When quantum effects are not necessarily small, $(pk)^2-(\hbar k^2/2)^2=0$ has a solution corresponding to the resonance condition for one-photon pair creation. The first quantum correction to the unmagnetized case vanishes, and the next order term, obtained by expanding in $(\hbar k^2/2)^2/(pk)^2$, is of order $\hbar^2$.

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