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One-photon pair production in pulsars: non-relativistic and relativistic regimes

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ABSTRACT

The process in which photons, emitted by primary particles, decay into pairs is usually assumed to be responsible for populating a pulsar magnetosphere with highly relativistic secondary pairs. Synchrotron emission from these secondary pairs is then assumed to contribute to the high-energy emission from some pulsars. We show that the perpendicular motion of the secondary pairs is highly relativistic only if the magnetic field is sufficiently weak. We consider pair production as a function of the energy of the curvature photons emitted by the primary electron (or positron). The secondary electrons and positrons are in their lowest Landau levels for strong magnetic fields, $B \gtrsim 0.5B_{\rm cr}$, and all have non-relativistic perpendicular momenta for $B \gtrsim 0.1B_{\rm cr}$. The conventional treatment of one-photon pair creation in terms of the asymptotic limit, where the electron and positron have highly relativistic perpendicular momenta, is invalid for such fields. For weaker fields, the asymptotic theory applies for the higher energy portion of the photon spectrum, and it is valid for the entire spectrum only for $B \lesssim 0.02B_{\rm cr}$. Synchrotron emission by secondary pairs cannot be relevant to high-energy emission from pulsars for $B \gtrsim 0.1B_{\rm cr}$.

Key words: radiation mechanisms: non-thermal – pulsars: general – gamma-rays: theory.

1 INTRODUCTION

The magnetospheres of isolated pulsars are thought to be populated with highly relativistic electron–positron pairs created by the decay of photons. Any such pair creation process requires photons with energies in excess of the threshold of twice the rest energy of the electron. In the polar cap model (Sturrock 1971), the photons are generated through curvature emission by 'primary' electrons, and they produce pairs through magnetic one-photon decay. The primary electrons are accelerated to ultrarelativistic energies (Lorentz factor $\gamma \sim 10^7$) just above the surface of the magnetized rotating neutron star. (The polar cap region is defined by the region of 'open' field lines that extend beyond the light cylinder, where the corotation speed equals the speed of light.) Subsequent models for pulsars developed this 'inner gap' model further (Ruderman & Sutherland 1975; Arons 1983; Mestel 1998), by invoking emission arising from inverse Compton scattering as well as curvature radiation (Luo 1996), and included pair creation in an 'outer gap' arising from photon–photon interations (Cheng, Ho & Ruderman 1986). We concentrate on magnetic pair creation in an inner gap.

Magnetic one-photon pair creation (Toll 1952; Klepikov 1954; Erber 1966; Tsai & Erber 1974; Daugherty & Harding 1983) may be treated in an inertial frame, which we call the perpendicular frame, in which the photon is propagating perpendicular to the magnetic field ($\theta = \pi/2$). Let ω be the photon energy in this frame (we use natural units, $\hbar = c = 1$ except where stated otherwise) and ε_γ be the photon energy in the pulsar frame. With θ the angle between the direction of propagation of the photon and the magnetic field in the pulsar frame, the photon energy in the perpendicular frame is $\omega = \varepsilon_\gamma \sin\theta$. The threshold condition is $\omega > 2m$. Photons emitted by highly relativistic particles are confined to a forward cone, $\theta \lesssim 1/\gamma$, and such photons do not satisfy the threshold condition at the point of emission (Usov & Melrose 1995). Refraction is negligible for the photons that propagate along straight lines so that the angle θ increases as the curved field lines deviate away from the ray path. This causes the photon energy in the local perpendicular frame to increase monotonically, $d\omega/dt = (d\theta/dt)\varepsilon_\gamma \cos\theta \approx (d\theta/dt)\varepsilon_\gamma$, with $d\theta/dt$ the rate of increase of the angle resulting from the deviation of field lines away from the ray path. Once the threshold is exceeded, the excess energy $\omega - 2m$ becomes available as kinetic energy of the resulting electron and positron. Just above threshold, the electron and positron must be non-relativistic in the perpendicular frame, and pairs that are highly relativistic in the perpendicular frame result only if the photon does not decay until $\omega - 2m$ greatly exceeds $2m$. In detailed treatments of magnetic one-photon pair production in pulsars, it is usually assumed that the pairs are highly relativistic, and formulae that apply in the asymptotic limit are used. For example, the well-known formulae of Erber (1966) and Tsai & Erber (1974) apply in the asymptotic limit. However, the

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asymptotic results apply only if (i) the photon does not decay into a non-relativistic pair just above the threshold for $\omega - 2m \ll 2m$, and (ii) the photon does decay further from the star, where $\omega \gg 2m$ is satisfied, rather than escaping to infinity.

In this paper we discuss the decay of a photon into a non-relativistic pair, and consider the conditions when this occurs rather than the usually assumed decay into a highly relativistic pair. It has long been recognized that production of non-relativistic pairs should not be neglected (Daugherty & Harding 1983). In particular, the rate of decay of a photon into a pair is singular 'at threshold', and in fact at a sequence of thresholds corresponding to the Landau levels of the electron and positron. The energy eigenstates of an electron or a positron in a magnetic field, B, are $\varepsilon_n = (m^2 + p_z^2 + p_n^2)^{1/2}$, with p_z the component of momentum along the magnetic field lines and $p_n = (2neB)^{1/2}$ the perpendicular component of momentum, with $n = 0, 1, \ldots$ denoting the Landau levels. Let j, k denote the Landau levels for the positron and electron. There are thresholds at which the pair production rate is singular for every value of j, k. In the asymptotic limit j, k are treated as continuous numbers and the pair production rate is averaged such that the densely packed singular spikes are smoothed into a continuum. The singular nature of the pair production rate at threshold suggests that the photon may be captured near the first threshold that it encounters, provided that the thresholds are sufficiently well separated. If this is the case, then the conventional use of the asymptotic limit is unjustified. Whether the pairs are non-relativistic or highly relativistic (in the perpendicular frame) has important implications in connection with γ-ray production. The observed γ-rays, from a subset of observed radio pulsars, are attributed to either synchrotron emission or inverse Compton emission (Daugherty & Harding 1994, 1996; Sturner, Dermer & Michel 1995). Synchrotron emission can be excluded if the pairs are non-relativistic.

The strength of the magnetic field relative to the critical magnetic field, $B_{\mathrm{cr}} = m^2/e = 4.4. \times 10^9$ T, plays an important role in the theory. We write $B' = B/B_{\mathrm{cr}}$, so that the energy eigenstates are $\varepsilon_n = [m^2(1 + 2nB') + p_z^2]^{1/2}$. The asymptotic formulae apply only for $2nB' \gg 1$, and the pairs are non-relativistic, from the point of view of not emitting synchrotron radiation, in the opposite limit $2nB' \ll 1$. In strong fields, $B' \gtrsim 1$, only the ground state $(j, k = 0)$ is non-relativistic, and in weak fields, $B' \ll 1$, there are a large number of non-relativistic states, $j + k \lesssim 1/B'$. This leads one to expect that the asymptotic limit should apply only if the field is relatively weak. We consider magnetic one-photon pair production for photons with the curvature radiation spectrum over the range from threshold up to the energy of the primary electron. Our primary objective is to determine the conditions under which the pairs are non-relativistic and the most likely state (values of j, k) that results when they are non-relativistic. A significant complication is that the photon may travel a sufficiently large distance, between the point of emission and the point where it decays into a pair, that the change in the value of B needs to be taken into account. In what follows, we consider two complementary models: a local model, in which we ignore any change in the magnetic field between the point of emission and absorption, and a non-local model, in which the change in the magnetic field between the point of emission and absorption is taken into account assuming a dipolar field.

In treating pair creation it is necessary to take into account the polarization of the photons. There are two natural wave modes of the magnetized vacuum (which is birefringent), corresponding to parallel and perpendicular polarization, where these refer to the direction of the electric vector in the wave relative to the ambient magnetic field. Absorption of perpendicular-polarized photons differs from that of parallel-polarized photons in that the electron and positron have opposite spins, rather than the same spins (where we assume that the spin operator is the component of the magnetic moment operator along $\boldsymbol{B}$). The ground state has a definite spin $(s = -1)$ and so absorption with both electron and positron in their ground states $(j = k = 0)$ is forbidden for perpendicular-polarized photons.

The general theory of pair production is outlined in Section 2 following the notation of Daugherty & Harding (1983). The effect of the pair production process on the curvature radiation spectrum is investigated near threshold in Section 3 and in the asymptotic limit in Section 4. The results are discussed in Section 5, with a conclusion in Section 6.

2 GENERAL THEORY OF PAIR PRODUCTION

Suppose we have a photon with energy ε_γ propagating at an angle θ to the magnetic field $\boldsymbol{B}$, assumed to be in the z-direction. Following the notation of Daugherty & Harding (1983) and writing the wavevector of this photon as $\boldsymbol{k} = \varepsilon_\gamma(0, \sin\theta, \cos\theta)$, the conservation laws for momentum and energy for pair production become

$$\varepsilon_\gamma = E_j + E_k, \qquad \varepsilon_\gamma \cos\theta = p + q, \tag{1}$$

where the energy quantum numbers $j, k = 0, 1, 2, \ldots$ and the z-momentum components p, q refer to the positron and electron respectively. The Landau energy levels for the two particles are of the form

$$E_j = \sqrt{p^2 + m^2(1 + 2jB')}, \qquad E_k = \sqrt{q^2 + m^2(1 + 2kB')}, \tag{2}$$

where $B' = B/B_{\mathrm{cr}}$. Choosing a frame in which $\boldsymbol{k}\cdot\boldsymbol{B} = 0$ so that $p = -q$ and solving equation (1) for p, one obtains $p = \pm p_{jk}$ with

$$p_{jk} = \pm m\sqrt{\omega'^2 - 1 - (j + k)B' + (j - k)^2\frac{B'^2}{4\omega'^2}}, \tag{3}$$

where $\omega' = \varepsilon_\gamma/2m$. As $p^2 \geq 0$, only those (j, k) values that give a non-negative p^2 are allowed.

The attenuation coefficients for pair creation by photons, the electric vectors of which are polarized parallel and perpendicular to the direction of the magnetic field, are of the form

$$R_\parallel(\omega', B') = \frac{\alpha_0}{2\xi}\sum_j\sum_k \frac{1}{|p_{jk}|}\{(E_jE_k + m^2 - p^2) \times [|M(j,k)|^2 + |M(j-1,k-1)|^2] + 2(jk)^{1/2}B'm^2 \times [M\dagger(j,k)M(j-1,k-1) + M\dagger(j-1,k-1)M(j,k)]\},$$

$$R_\perp(\omega', B') = \frac{\alpha_0}{2\xi}\sum_j\sum_k \frac{1}{|p_{jk}|}\{(E_jE_k + m^2 + p^2) \times [|M(j-1,k)|^2 + |M(j,k-1)|^2] - 2(jk)^{1/2}B'm^2 \times [M\dagger(j-1,k)M(j,k-1) + M\dagger(j,k-1)M(j-1,k)]\}, \tag{4}$$

where

$$\xi = \frac{2\omega'^2}{B'},$$

$$M(j,k) = (-\mathrm{i})^{G-S}\sqrt{\frac{S!}{G!}}\exp^{-\xi/2}\xi^{(G-S)/2}L_S^{G-S}(\xi),$$

$$G = \max(j,k), \qquad S = \min(j,k), \tag{5}$$

α_0 is the fine structure constant and $L_S^{G-S}(\xi)$ is the generalized Laguerre polynomial. As the attenuation coefficients in equation (4) have $|p_{jk}|$ in the denominator, there are singular absorption edges whenever p_{jk} of equation (3) vanishes, that is, whenever

$$\omega'(j,k) = \frac{1}{2}\left[\sqrt{1+2jB'} + \sqrt{1+2kB'}\right]. \tag{6}$$

These absorption edges correspond to the different thresholds for pair production. When the photon wavevector makes an angle θ with the magnetic field, the attenuation coefficients can be obtained from $R_{\parallel}$ and $R_{\perp}$ in equation (4) by making the following transformation: $R_{\parallel,\perp}(\varepsilon_\gamma) \rightarrow \sin\theta R_{\parallel,\perp}(\varepsilon_\gamma \sin\theta) \equiv \bar{R}_{\parallel,\perp}$.

3 PAIR PRODUCTION BY CURVATURE RADIATION

Radiation emitted by a particle with Lorentz factor γ is tangential to the magnetic field lines to within an angle $\theta \lesssim 1/\gamma$. The photon propagates in a straight line, and the angle θ increases because of the curvature of the magnetic field lines. Pair production becomes possible, for a parallel-polarized photon, when the threshold $\varepsilon_\gamma \sin\theta = 2m$ for the ground state ($j = k = 0$) is reached. The lowest threshold for perpendicular-polarized photons is $\varepsilon_\gamma \sin\theta = m(1 + \sqrt{1+2B'})$, corresponding to one of the pair being in its ground state Landau level (j or $k = 0$) and the other in its first excited Landau level (k or $j = 1$).

3.1 Attenuation along curved field lines

Using the attenuation coefficient $\bar{R}$ for arbitrary θ, one can evaluate the optical depth for a photon of energy ε_γ travelling a distance ds between d_0 and d_1, where d_0 and d_1 are the distances from the point of emission of the photon to two consecutive thresholds ($d_0 < d_1$), namely

$$\tau = \int_{d_0}^{d_1} \mathrm{d}s\bar{R} = \int_{t_0}^{t_1} \mathrm{d}t\bar{R} = \int_{\omega_0}^{\omega_1} \frac{\mathrm{d}t}{\mathrm{d}\omega}\mathrm{d}\omega\bar{R}, \tag{7}$$

where the distances ds, d_0 and d_1 are transformed to times dt, t_0 and t_1 ($t = d/c$ with $c = 1$ here) and then energies dω, ω_0 and ω_1. The definition

$$\omega \equiv \varepsilon_\gamma \sin\theta \tag{8}$$

implies $\mathrm{d}\omega/\mathrm{d}t = \mathrm{d}(\varepsilon_\gamma \sin\theta)/\mathrm{d}t = \varepsilon_\gamma \cos\theta\dot{\theta}$. The time derivative of the angle θ can be expressed in terms of the curvature radius of the magnetic field lines, ρ (assumed here to be constant), and θ itself as follows. The relation between the distance the photon travels d and the angle θ it makes with $\boldsymbol{B}$ is given by

$$d \simeq 2\rho\sin\theta, \tag{9}$$

where d is the chord subtended by the angle 2θ of a circle of radius ρ. The photon takes the time t to travel this distance where

$$t \simeq d \simeq 2\rho\sin\theta. \tag{10}$$

One then obtains

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} \simeq 2\rho\cos\theta. \tag{11}$$

It follows that one has

$$\dot{\theta} \simeq \frac{1}{2\rho\cos\theta}, \tag{12}$$

and hence

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} \simeq \varepsilon_\gamma \cos\theta \frac{1}{2\rho\cos\theta} \simeq \frac{\varepsilon_\gamma}{2\rho}. \tag{13}$$

Therefore τ can be written as

$$\tau \simeq \frac{2\rho}{\varepsilon_\gamma}\int_{\omega_0}^{\omega_1} \mathrm{d}\omega\bar{R}(\omega, B) = \frac{2\rho 2m}{\varepsilon_\gamma}\int_{\omega'_0}^{\omega'_1} \mathrm{d}\omega'\bar{R}(\omega', B'), \tag{14}$$

with $\omega' = \omega/2m$, $B' = B/B_{\mathrm{cr}}$, and where $\omega'_0 = \omega_0/2m$, $\omega'_1 = \omega_1/2m$ are cut-offs for the integral.

3.2 Attenuation between the two lowest thresholds for parallel-polarized photons

Consider absorption into the ground state $k = j = 0$. The absorption coefficient is non-zero for $\omega \gtrsim 2m$, and formally we could integrate from the point where $\omega_0 = 2m$ to $\omega_1 \rightarrow \infty$ in evaluating τ. However, when the threshold corresponding to $k + j = 1$ is reached, the absorption into this state dominates over the absorption into $k = j = 0$. In addition, ω' must not exceed ε_γ, as $\sin\theta$ cannot be greater than unity. Hence, in considering absorption into the ground state, it is appropriate to cut the integral off at the threshold for the first excited state. We refer to this as the $00 \rightarrow 01$ interval, and it corresponds to the range $\omega'_0 \le \omega' \le \omega'_1$, $\omega'_0 = 1$, $\omega'_1 = (1 + \sqrt{1+2B'})/2$.

For $k = j = 0$ only $R_{\parallel}$ is non-zero, and the optical depth becomes

$$\tau = \frac{(2m)^2}{\varepsilon_\gamma^2}\rho\alpha_0 m\int_1^{(1+\sqrt{1+2B'})/2} \mathrm{d}\omega' B'\frac{\exp(-2\omega'^2/B')}{\omega'\sqrt{\omega'^2 - 1}}, \tag{15}$$

where we substitute

$$\bar{R}_{\parallel} = \frac{\alpha_0 B' m\sin\theta}{2\omega'^2\sqrt{\omega'^2 - 1}}\exp(-2\omega'^2/B'), \tag{16}$$

with $\sin\theta = \omega/\varepsilon_\gamma$, into equation (14) for τ. Separating out the integral part, one obtains the following for τ:

$$\tau = \frac{(2m)^2}{\varepsilon_\gamma^2}\rho\alpha_0 mI, \tag{17}$$

with

$$I = \int_1^{(1+\sqrt{1+2B'})/2} \mathrm{d}\omega' B'\frac{\exp(-2\omega'^2/B')}{\omega'\sqrt{\omega'^2 - 1}}. \tag{18}$$

The dimensionless constant

$$\rho\alpha_0\frac{mc}{\hbar} = 1.8876 \times 10^{15}\left(\frac{\rho}{10^5\ \mathrm{m}}\right) \tag{19}$$

is common to all our expressions for optical depths.

3.3 Local and non-local absorption

If the value of τ is such that $\tau > 1$, then the medium is optically thick and we assume that a photon of energy ε_γ satisfying $\tau > 1$, that is a photon with energy (in MeV) satisfying

$$\varepsilon_\gamma < \sqrt{\rho\alpha_0 m I}, \tag{20}$$

cannot traverse the entire medium [from d_0 where $\omega_0' = 1$ to d_1 where $\omega_1' = (1 + \sqrt{1 + 2B'})/2$] without being absorbed and converted into an electron–positron pair. If the point of absorption is close to the point of emission, then B' can be assumed constant along the ray path between emission and absorption and the absorption is local. However, if the point where $\tau = 1$ is far from the point of emission then the absorption is non-local and the change of B' with height above the star needs to be taken into account. Let us assume that the field is B'_* at the point of emission, and that the field is dipolar so that the variation with radial distance, r, from the centre of the star is $B' \propto 1/r^3$. For a photon that travels a distance d between emission and absorption, the strength of the field at the absorption point is

$$B'(d) = B'_* \left(\frac{R_*}{R_* + d}\right)^3, \tag{21}$$

where R_* is the radius of the point of emission. The less energetic the curvature radiation, the greater the distance the photon has to travel, and hence the greater the angle θ, before the threshold $2m$ is reached. The greater the distance d travelled from the point of emission of the curvature radiation, the weaker the magnetic field that is present when pair production takes place. This behaviour of magnetic field strength B'/B'_* with photon energy ε_γ is exhibited in Fig. 1 where we assume that $R_* \sim R_{\rm star} = 10^4$ m. (The actual point of emission may be some distance out from the surface of the star, $\sim 10^2$ m say, and it is model-dependent. However, in what follows we assume that the emission point is near the surface of the star.) For $\varepsilon \gtrsim 2000$ MeV, B/B'_* is approximately unity and the absorption is local.

Using equation (20) and plotting this energy versus B', one obtains a curve representing $\varepsilon_{\gamma\rm max}$ at a particular field B', such that photons with energies below this $\varepsilon_{\gamma\rm max}$ produce a pair. In Fig. 2, these $\varepsilon_{\gamma\rm max}$ values are plotted as a function of B' for $\rho = 10^5$ m and $j = k = 0$ (solid line). The value of ε_γ must be greater than $2m$ and less than or equal to the energy of the primary electron; values of ε_γ outside these limits are unphysical. In this solid curve where $b = B'$, B' represents the magnetic field at the point where the absorption of the photon takes place. The escape of photons with $\varepsilon_\gamma \gtrsim \varepsilon_{\gamma\rm max}$ is attributed to the optical depth for absorption decreasing with increasing ε_γ (see equation 15).

There is also a minimum energy, $\varepsilon_{\gamma\rm min}$, such that photons with $\varepsilon_\gamma \lesssim \varepsilon_{\gamma\rm min}$ do not pair produce. Pair production is forbidden for $\varepsilon_\gamma < 2m$, implying $\varepsilon_{\gamma\rm min} \geq 2m$. For ε_γ just above $2m$, the lowest threshold, $\varepsilon_\gamma \sin\theta = 2m$, can be reached only when the photon is very far from the point where it is emitted (with $\theta \lesssim 1/\gamma$). Thus, for such photons any pair production is non-local, with the absorption coefficient decreasing with decreasing B. For $B'_* \gg 1$, the value of B' where ε_γ is just above $2m$ is still large enough for the optical depth to satisfy $\tau \sim 1$. However, for $B'_* \ll 1$, the value of B' is so small that $\tau \ll 1$. Hence as B'_* decreases, the value of $\varepsilon_{\gamma\rm min}$ increases. The dashed curve in Fig. 2, in which $b = B'_*$, takes this effect into account, with the change in B' with distance from the point of emission being modelled as dipolar. This curve is obtained by substituting equation (21) into equation (18) at each B'_* assuming $d = 2\rho(2m/\varepsilon_\gamma)$, $R_* = 10^4$ m (the radius of the star) and

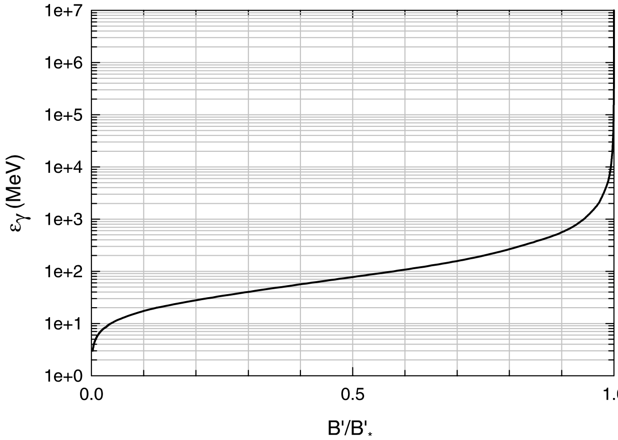

Figure 1. The variation of the dipolar magnetic field, in units of B'_*, with curvature photon energy ε_γ, for $\varepsilon_\gamma \sin\theta = 1$.

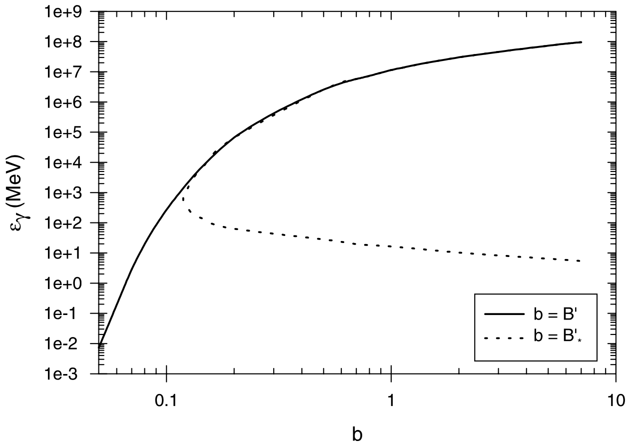

Figure 2. The values of $\varepsilon_{\gamma\rm max}$, with $b = B'$, and $\varepsilon_{\gamma\rm max}$, $\varepsilon_{\gamma\rm min}$, for $b = B'_*$, as a function of b for the creation of pairs with j, $k = 0$ by parallel-polarized photons. For $b = B'$, pairs are created by photons with energies $1 \leq \varepsilon_\gamma \leq \varepsilon_{\gamma\rm max}$ (or $1 \leq \varepsilon_\gamma \leq E_{\rm e}$ if $\varepsilon_{\gamma\rm max} > E_{\rm e}$). For $b = B'_*$, pairs are created by photons with energies $\varepsilon_{\gamma\rm min} \leq \varepsilon_\gamma \leq \varepsilon_{\gamma\rm max}$ (where $\varepsilon_{\gamma\rm min} = 1$ if $\varepsilon_{\gamma\rm min} < 1$ and $\varepsilon_{\gamma\rm max} = E_{\rm e}$ if $\varepsilon_{\gamma\rm max} > E_{\rm e}$).

$\rho = 10^5$ m. Except at the turning point, two values of ε_γ give an optical depth of unity. Only photons with $\varepsilon_{\gamma\rm min} \lesssim \varepsilon_\gamma \lesssim \varepsilon_{\gamma\rm max}$ pair produce. The dashed curve coincides with the solid curve in Fig. 2 at the higher ε_γ where the absorption is local.

For parameters to the left of the dashed curve in Fig. 2 there is no absorption of the photons into the ground state, $j = k = 0$. For a given ε_γ this corresponds to a lower limit on the magnetic field B'_* for which absorption into the ground state is significant. It is only for $B'_* \lesssim 0.1$ that absorption into the ground state can be ignored. For $B'_* > 0.1$ all the parallel-polarized photons with $\varepsilon_{\gamma\rm min} \lesssim \varepsilon_\gamma \lesssim \varepsilon_{\gamma\rm max}$ produce a pair with $j = k = 0$. With our choice of parameters the maximum energy of the photon is the energy, 5×10^6 MeV, of the emitting ultrarelativistic electron, and from Fig. 2, one has $\varepsilon_{\gamma\rm max} \gtrsim 5 \times 10^6$ MeV for $B'_* \gtrsim 0.5$. It follows that for $B'_* \gtrsim 0.5$, effectively all parallel-polarized photons with $\varepsilon_\gamma \gtrsim \varepsilon_{\gamma\rm min}$ produce a pair in the ground state Landau levels. For $0.1 \lesssim B'_* \lesssim 0.5$ the only photons that are not absorbed into the ground state are those with $\varepsilon_\gamma \lesssim \varepsilon_{\gamma\rm min}$ and those with $\varepsilon_{\gamma\rm max} \lesssim \varepsilon_\gamma \lesssim 5 \times 10^6$ MeV.

3.4 Absorption intervals between thresholds

Our detailed treatment of the absorption is based on ordering all the thresholds, denoted by jk, according to increasing photon energy, and calculating the absorption coefficient between one threshold and the next. Given a model for the magnetic field, and we assume a dipolar model, each such interval corresponds to an interval of length along the ray path. The four lowest intervals for $B' < 5$ are $00 \to 01$, $01 \to 02$, $02 \to 11$, $11 \to 03$. Those photons that do not form a pair in the $00 \to 01$ interval either produce a pair in higher Landau levels or escape. Before treating absorption in this sequence of levels, let us comment on absorption of perpendicular-polarized photons in the lowest allowed level.

3.5 Attenuation between the two lowest thresholds for perpendicular-polarized photons

One can follow a similar procedure for the perpendicular-polarized photons to that for parallel-polarized photons. The lowest threshold is $\omega = m(1 + \sqrt{1 + 2B'})$, corresponding to $j + k = 1$, with the next highest threshold corresponding to $\omega = m(1 + \sqrt{1 + 4B'})$ in which either $j = 2$, $k = 0$ or $j = 0$, $k = 2$. We refer to this as the $01 \to 02$ interval, and it corresponds to the range $\omega'_0 \le \omega' \le \omega'_1$, with $\omega'_0 = (1 + \sqrt{1 + 2B'})/2$, $\omega'_1 = (1 + \sqrt{1 + 4B'})/2$. In this case, the absorption coefficient is

$$\bar{R}(\omega', B') = \frac{\alpha_0 B' m \sin\theta}{2\omega' \sqrt{\omega'^4 - (1 + B')\omega'^2 + B'^2/4}} \times (2\omega'^2 - B') \exp(-2\omega'^2/B'), \qquad (22)$$

which is substituted into equation (14) to evaluate the optical depth. The $\varepsilon_{\gamma\max}$ and $\varepsilon_{\gamma\min}$ values one obtains are a little higher and a little lower respectively than the $\varepsilon_{\gamma\max}$ and $\varepsilon_{\gamma\min}$ values presented in Fig. 2 for the $00 \to 01$ interval for parallel-polarized photons. Nevertheless, the conclusions are much the same. For magnetic fields $B'_* \gtrsim 0.1$ all perpendicular-polarized photons with $\varepsilon_{\gamma\min} \lesssim \varepsilon_\gamma \lesssim \varepsilon_{\gamma\max}$ produce a pair with $j + k = 1$, and for $B'_* \gtrsim 0.5$ one has $\varepsilon_{\gamma\max} \gtrsim E_e$, so that all perpendicular-polarized photons with $\varepsilon_\gamma \gtrsim \varepsilon_{\gamma\min}$ produce a pair with one of the pair in its ground state Landau level and the other in the first excited Landau level. In magnetic fields of this order, those secondary electrons/positrons that are in their first excited Landau levels quickly decay to their ground state Landau levels, emitting photons with the Doppler-shifted cyclotron frequency.

3.6 Absorption into excited states

The curvature radiation spectrum includes both perpendicular and parallel polarizations (e.g. Baring & Harding 2001), and both polarizations need to be taken into account when discussing pair creation. In the remainder of this section, we concentrate on parallel-polarized photons; the results for perpendicular-polarized photons are similar.

The optical depths for the $00 \to 02$, $00 \to 11$ and $00 \to 03$ intervals are calculated by integrating the absorption coefficients for all allowed absorptions in each interval. For a given interval, the allowed values are the values corresponding to the threshold for that interval, plus the values for all the lower thresholds. Thus for the $00 \to 02$, the new threshold is $j = 0$, $k = 1$ and $j = 1$, $k = 0$ with $j = k = 0$ also allowed, for $00 \to 11$ the new threshold $j = 0$, $k = 2$ and $j = 2$, $k = 0$ is added, for $00 \to 03$ the new threshold $j = k = 1$ is added, and so on.

As in Fig. 2 we evaluate the parameters for which the optical depth is unity in each interval and plot the result as a graph of ε_γ versus B' for local absorption and B'_* for non-local absorption, in Fig. 3(a) and (b) respectively. The curves in Fig. 3(a) represent the $\varepsilon_{\gamma\max}$ values for local absorption of photons for the $00 \to 01$, $00 \to 02$, $00 \to 11$ and $00 \to 03$ intervals. As each higher interval is included, a few more of the higher energy photons are converted into pairs. The curves in Fig. 3(b) take non-local absorption into account, leading to the additional lower ($\varepsilon_{\gamma\min}$) bound of the energies of the photons absorbed in the $00 \to 01$, $00 \to 02$, $00 \to 11$ and $00 \to 03$ intervals. The regions between the curves corresponding to the successive intervals define the parameters for which absorption in that interval is important, that is for which photons neither are absorbed in lower intervals nor pass through to higher intervals. The curves corresponding to higher intervals extend to lower values of B'_*. Thus, as B'_* decreases, higher and higher intervals become important. This presupposes that the secondary electrons and positrons are non-relativistic, which is the case for $j, k \ll 1/2B'_*$. The conventional treatment using the asymptotic formulae applies only when the pair is highly relativistic, $j, k \gg 1/2B'_*$, and is therefore invalid for all photons that produce non-relativistic pairs.

3.7 Energy distribution of the secondary electrons and positrons

As a means of determining the energy distribution of the secondary electrons and positrons produced as the photons reach the first few successive thresholds, we evaluate the fraction of photons absorbed

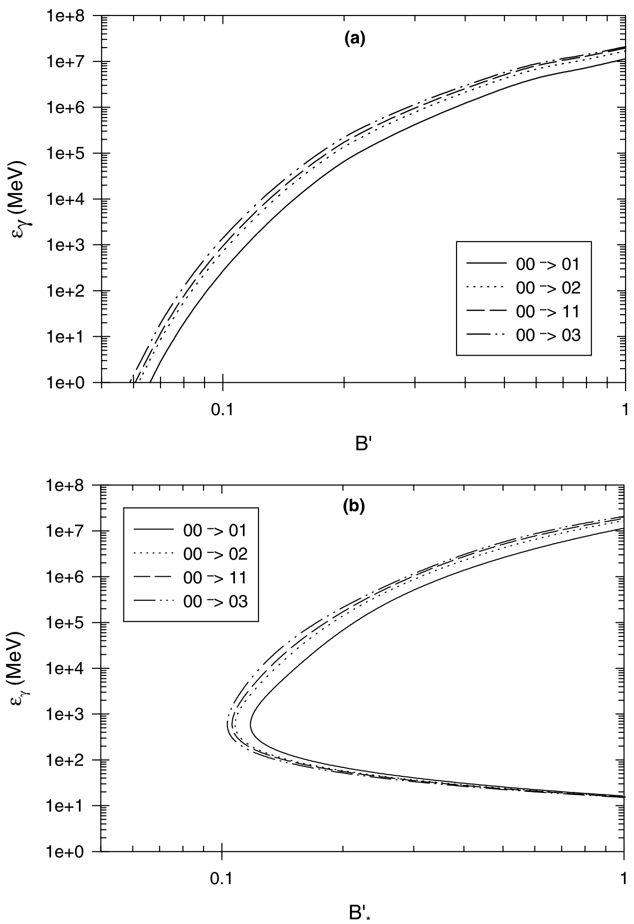

Figure 3. The $\varepsilon_{\gamma\max}$ values as a function of B' for local absorption (panel a) and the $\varepsilon_{\gamma\max}$, $\varepsilon_{\gamma\min}$ values as a function of B'_* for non-local absorption (panel b) for the $00 \to 01$, $00 \to 02$, $00 \to 11$ and $00 \to 03$ intervals.

into each of the intervals $00 \rightarrow 01$, $01 \rightarrow 02$, $02 \rightarrow 11$ and $11 \rightarrow 03$ for a range of B' and B'_* between 0.05 and 1. These fractions are obtained as follows. For a given photon energy ε_γ and magnetic field, the fraction of photons unabsorbed in the $00 \rightarrow 01$ interval is $\exp(-\tau^{00}_{00\rightarrow 01})$, where the superscript on the optical depth, τ, denotes the value of jk and the subscript denotes the interval. The fraction absorbed in the $00 \rightarrow 01$ interval is $1 - \exp(-\tau^{00}_{00\rightarrow 01})$. Over the next interval $01 \rightarrow 02$, those unabsorbed photons may form a pair in which one of the pair is in its ground state Landau level and the other in the first excited Landau level (01, $01 \rightarrow 02$) with optical depth $\tau^{01}_{01\rightarrow 02}$, or they may form a pair in which both the electron and positron are in their ground state Landau levels (00, $01 \rightarrow 02$) with optical depth $\tau^{00}_{01\rightarrow 02}$, or they may not form a pair at all. The fraction of photons of energy ε_γ unabsorbed in the $00 \rightarrow 02$ interval is $\exp(-\tau^{00}_{00\rightarrow 01})\exp[-(\tau^{00}_{01\rightarrow 02} + \tau^{01}_{01\rightarrow 02})]$, from which the fraction of the photons unabsorbed in the $00 \rightarrow 01$ interval that are absorbed into the $01 \rightarrow 02$ interval is $\exp(-\tau^{00}_{00\rightarrow 01})1 - \{\exp[-(\tau^{00}_{01\rightarrow 02} + \tau^{01}_{01\rightarrow 02})]\}$. In this manner, the fraction of available photons of energy ε_γ absorbed into pair production, locally at a magnetic field B' or non-locally for emission at a magnetic field B'_*, is evaluated for the first few thresholds for parallel-polarized photons.

These results are different representations of Fig. 3(a) and (b). For example, in Fig. 4(a) and (b), the fraction absorbed in the production of electron–positron pairs in a particular interval is presented as a function of ε_γ for $B' = 0.1$ and $B'_* = 0.1$, respectively. The dominant pair production to the $00 \rightarrow 01$ interval exhibited in Fig. 3(a) at $B' = 0.1$, and equivalently in Fig. 4(a), is suppressed in Figs 3(b) and 4(b), for $B'_* = 0.1$, as the dipolar magnetic field dependence of the absorption coefficient of the photons is taken into account in the latter B'_* figures. In the same manner as the regions between the curves corresponding to successive intervals define the parameters for which absorption in that interval is important in Fig. 3, in this alternative representation of Fig. 4, once a fraction of photons of a certain energy is absorbed in a particular interval, that fraction is no longer available for absorption at the higher intervals. Hence in Fig. 4, for a particular magnetic field and photon of a given energy, the sum of all the curves cannot be greater than unity.

The advantage of representing the results in this fraction form is that for a particular interval, one may dissect the fraction further into the particular j, k at which the absorption takes place. Such a dissection is exhibited in Fig. 5 for the same B' and B'_* of 0.1 of Fig. 4. By repeating the foregoing calculation for a range of values of B' and B'_*, we find that the dominant absorption occurs to the threshold that defines that interval. This is not surprising: an interval defined by a threshold with a given j, k has a singular absorption coefficient for that j, k at threshold, and the optical depth over the interval is dominated by the (integrable) contribution from this singular function. Thus pairs produced in the $01 \rightarrow 02$ interval predominantly have $j + k = 1$ rather than $j = k = 0$, those produced in the $02 \rightarrow 11$ interval predominantly have $j = 0$, $k = 2$ or $j = 2$, $k = 0$ rather than $j + k = 1$ or $j = k = 0$, and so on.

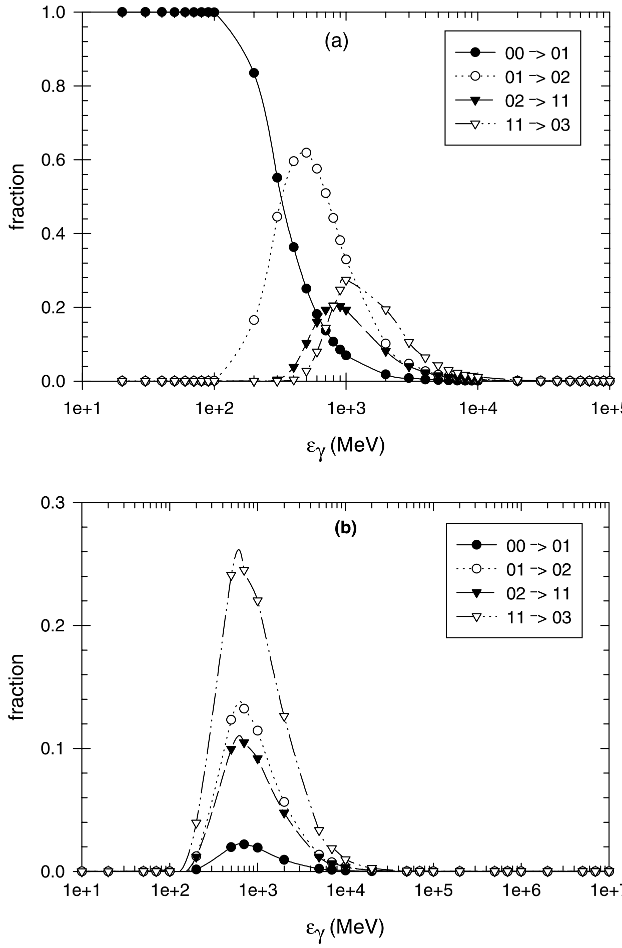

Figure 4. Fraction of parallel-polarized photons absorbed in the production of electron–positron pairs in the intervals $00 \rightarrow 01$, $01 \rightarrow 02$, $02 \rightarrow 11$ and $11 \rightarrow 03$ as a function of ε_γ for $B' = 0.1$ (panel a) and $B'_* = 0.1$ (panel b).

4 ASYMPTOTIC LIMIT

In this section we treat pair creation using the conventional asymptotic formula, which is based on the assumption that both the electron and positron have highly relativistic perpendicular momenta. The asymptotic limit for pair creation is analogous to the synchrotron limit of gyromagnetic emission; it applies for highly relativistic particles for which the numbers j, k are sufficiently large that they can be replaced by continuous variables.

4.1 Optical depth in the asymptotic limit

Like their synchrotron counterparts, the asymptotic forms for the mean attenuation coefficient $[R_{\rm mean} = (R_\parallel + R_\perp)/2]$ involve making the Airy approximation to the generalized Laguerre polynomials in equation (5), resulting in expressions involving the Macdonald function K_ν with argument $\nu = 2/3$. As in the exact treatment, it is convenient to perform the detailed calculation in the frame in which the photon is propagating perpendicular to the magnetic field and to transform to the laboratory frame where the photon has energy ε_γ and is propagating at angle θ to the magnetic field. We write $\bar{R}_{\rm mean} = \sin\theta R_{\rm mean}(\omega', B')$, where $\sin\theta \simeq 2m\omega'/\varepsilon_\gamma$ with $2m\omega' < \varepsilon_\gamma$. The asymptotic result is

$$\bar{R}_{\rm mean} = \frac{2m}{\varepsilon_\gamma}\alpha_0 m \frac{1}{\sqrt{3}\pi}\int_0^1 {\rm d}v \frac{3/2 - v^2/6}{1 - v^2} K_{2/3}\left[\frac{4}{\lambda(1 - v^2)}\right], \quad (23)$$

with $\lambda = 3B'\omega'$ and subject to the conditions $\omega' \gg 0.5$ and $B' \ll 1$. Simplification occurs in the two limits of λ: for $\lambda \ll 1$ one has

$$\bar{R}_{\rm mean} = \frac{2m}{\varepsilon_\gamma}\alpha_0 m\, 0.23 \frac{\lambda}{3}\exp\left(-\frac{4}{\lambda}\right), \quad (24)$$

and for $\lambda \gg 1$ one has

$$\bar{R}_{\rm mean} = \frac{2m}{\varepsilon_\gamma}\alpha_0 m\, 0.3\left(\frac{\lambda}{3}\right)^{2/3}. \qquad (25)$$

These two limiting cases give the same value of $\bar{R}_{\rm mean}$ for $\lambda = 15$. Comparing these approximations with the exact (asymptotic) expression (23), one finds that (25) is useful only for $\lambda > 15$ and that (24) is by far the better of the two for $\lambda < 15$, with the only significant difference being in the region $5 \lesssim \lambda \lesssim 15$ where it overestimates (23) by 30 to 50 per cent. The range $\lambda > 15$ is not relevant in the following discussion, and the approximate form (24) suffices for our purposes. The optical depth for arbitrary θ is

$$\tau = 0.46\,\rho\alpha_0 m\left(\frac{2m}{\varepsilon_\gamma}\right)^2 B'\int {\rm d}\omega'\,\omega' \exp\left(-\frac{4}{3B'\omega'}\right), \qquad (26)$$

when equation (24) is adopted for the mean attenuation coefficient.

4.2 The frequency $\omega'_{\rm top}$

Just above the threshold, $\omega' = 1$, for pair creation the pairs are non-relativistic, and one does not expect the asymptotic formula to be valid. The asymptotic formula is valid only for sufficiently large ω', and the value above which it is reliable may be estimated by comparing the optical depths evaluated over the ω' interval from 1 to $\omega'_{\rm top}$ as a function of the variable $\omega'_{\rm top}$ using both the exact (from equation 4) and asymptotic (using equation 24) forms of the mean attenuation coefficients. The asymptotic formula is valid to a specified accuracy at frequencies above the value of $\omega'_{\rm top}$ at which the ratio of the asymptotic and exact results differ from unity by less than a specified amount.

In Fig. 6, the ratios of these mean optical depths, $\tau_{\rm asym}/\tau_{\rm exact}$, are plotted as a function of $\omega'_{\rm top}$ for three magnetic fields, $B' = 0.1$, 0.07 and 0.05. For $\omega'_{\rm top}$ near threshold, the ratios are large, and they decrease to $\lesssim 1.5$ for $\omega'_{\rm top} \gtrsim 2.5$. As argued above, one has $\tau \ll 1$ for $\omega'_{\rm top}$ near threshold for $B' < 0.1$, and τ approaches unity only at high Landau levels. As $\omega'_{\rm top}$ increases, both the exact and asymptotic optical depths increase by large factors compared with their values near threshold. As a consequence, the poor correspondence between $\tau_{\rm exact}$ and $\tau_{\rm asym}$ near threshold is physically unimportant. For larger values of B', for which the optical depth is non-negligible near threshold, use of the asymptotic approximation seriously overestimates the optical depth. In the following discussion we assume that the asymptotic form is valid for $\omega' \gtrsim 2.5$ and $B' \lesssim 0.1$. Under these conditions, the resulting electron and positron are highly relativistic provided that the perpendicular energy of the photon is approximately equally shared, which is the case, cf. fig. 7 of Daugherty & Harding (1983).

Although it is inappropriate to use the asymptotic formula to treat pair creation for $\omega' < 2.5$, nevertheless it is instructive to estimate the optical depth using it in this range. For a given magnetic field, if the asymptotic formula implies no significant absorption in the interval $1 \le \omega' \le 2.5$, where it overestimates $\bar{R}_{\rm mean}$, then clearly the correct formula also implies no absorption in this range. Furthermore, if there is no absorption in the interval $1 \le \omega' \le 2.5$, then the only absorption that could conceivably occur is for $\omega' > 2.5$, where the asymptotic approximation is valid.

We used the asymptotic formula to evaluate the optical depth (26) by integrating over the range $1 \le \omega' \le \omega'_{\rm top}$, where $\omega'_{\rm top}$ is an adjustable parameter. In an analogous way to that for the low-lying levels, and for a range of B' and B'_* from 0.001 to 0.1, the photon energy ε_γ for which the optical depth is unity is determined for a given $\omega'_{\rm top}$. The angle θ at which the optical depth became unity is then found from this $\omega'_{\rm top}$ through $\theta = \arcsin(\omega'_{\rm top}/\varepsilon_\gamma)$. For local absorption the optical depth (26) applies. To allow for non-local

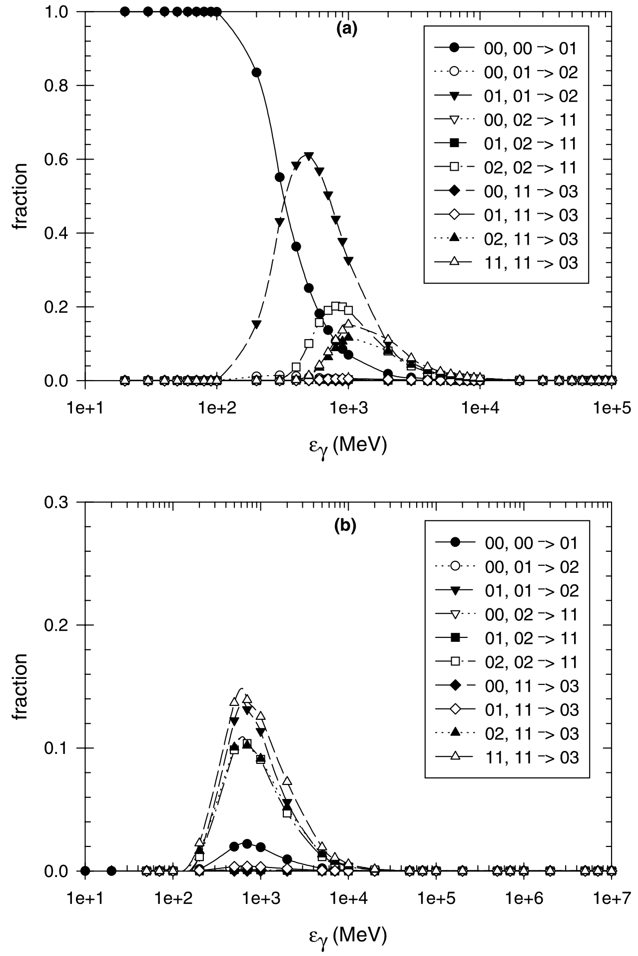

Figure 5. Fraction of parallel-polarized photons absorbed in the production of electron–positron pairs with a particular jk in the intervals $00 \rightarrow 01$, $01 \rightarrow 02$, $02 \rightarrow 11$ and $11 \rightarrow 03$ as a function of ε_γ for $B' = 0.1$ (panel a) and $B'_* = 0.1$ (panel b).

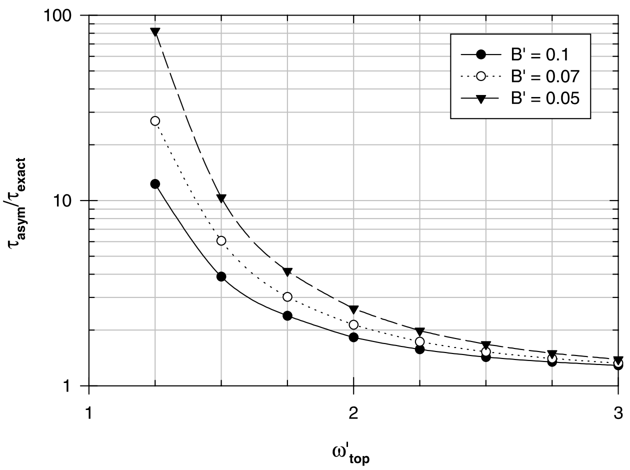

Figure 6. Ratio of the asymptotic to exact mean optical depths as a function of $\omega'_{\rm top}$.

absorption one substitutes (21) into (26) and uses the resulting expression,

$$\tau \simeq 0.46\,\rho\alpha_0 m\left(\frac{2m}{\varepsilon_\gamma}\right)^2 \times \int d\omega' \frac{\omega' B'_*}{(1+20\omega'/\varepsilon_\gamma)^3}\exp\left[-\frac{4(1+20\omega'/\varepsilon_\gamma)^3}{3B'_*\omega'}\right], \qquad (27)$$

to evaluate the absorption coefficient in terms of the magnetic field B'_* at the point of emission, assumed to be at $R_* \sim R_{\rm star} = 10^4$ m.

4.3 Validity of the asymptotic approximation

In Fig. 7(a) and (b), the results of the foregoing analysis are presented as a plot of the energy of the curvature photon against the angle θ where the optical depth is unity. Fig. 7(a) is for a range of B', the magnetic field at the point of absorption, and Fig. 7(b) is for a range of B'_*, the field at the point of emission of the photons. For $\varepsilon_\gamma > 10^4$ MeV, the two figures, 7(a) and 7(b), are much the same. However, in Fig. 7(b) the curves flatten off at low ε_γ values, such that for $B'_* = 0.001$, say, photons with energies below about 10^4 MeV do not produce pairs at all. This minimum energy decreases with increasing B'_* and is approximately given by $10/B'_*$ in MeV.

Included in Fig. 7(a) and (b) are the $\omega' = 1$, 2.5, 10 lines. The $\omega' = 1$ line represents the minimum angle θ that the trajectory of the photon has to make with the magnetic field line for pair production to be possible for a curvature photon of energy ε_γ. Those parts of the curves below this line are unphysical. The asymptotic formula overestimates the absorption for $1 < \omega' \lesssim 2.5$, and the curves in this region are useful only as limits on the actual absorption. Only for $\omega' \gtrsim 2.5$ is the asymptotic formula accurate.

As an example, consider Fig. 7(a) at a specific, relatively weak, magnetic field of $B' = 0.001$ and a curvature radiation photon of energy $\varepsilon_\gamma = 10^6$ MeV. The threshold for pair production ($\omega' = 1$) is then at $\theta \sim 6 \times 10^{-5°}$, but significant absorption occurs (optical depth greater than about unity) only for $\theta \sim 10^{-2°}$. Note that the position of a particular B' or B'_* curve represents essentially the value of ω' for a given ε_γ where the bulk of the absorption occurs.

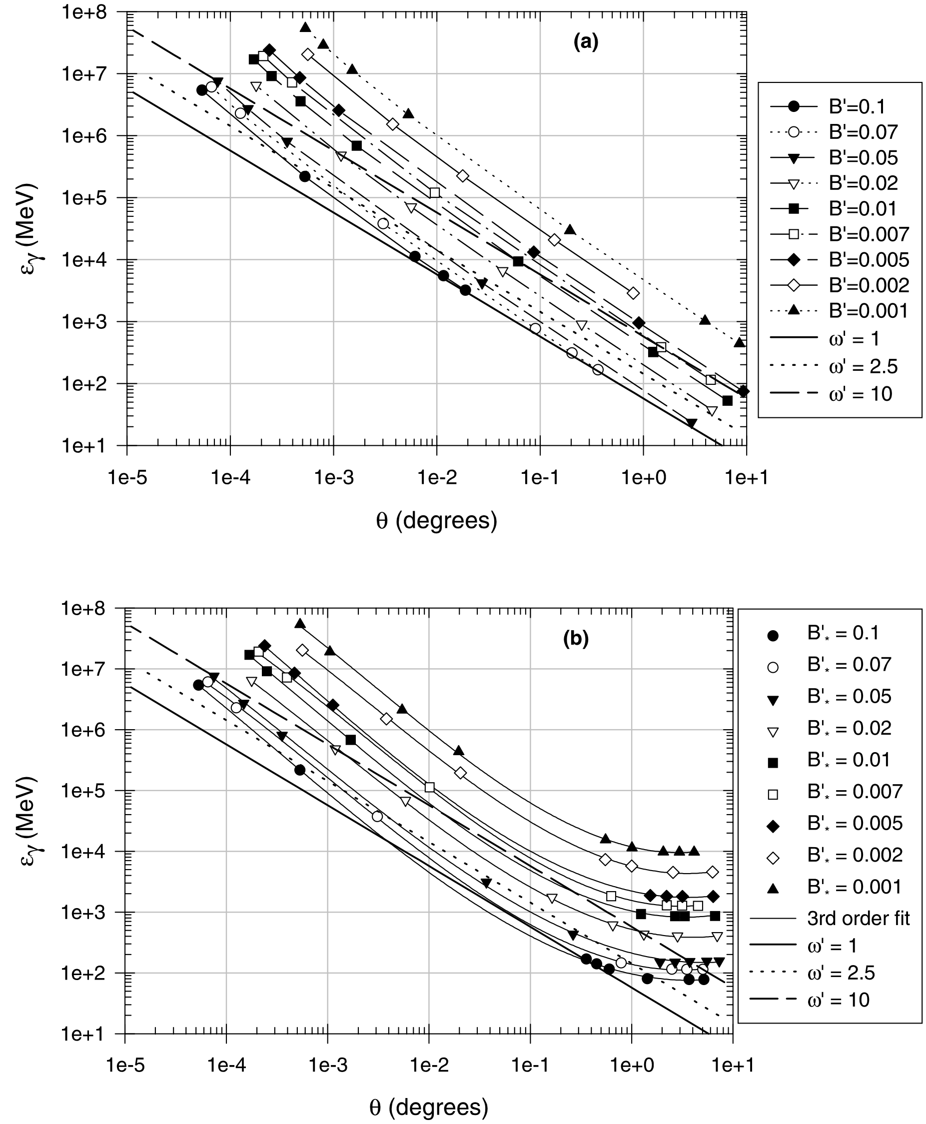

Figure 7. Energy of the curvature photon ε_γ versus angle θ where the optical depth is unity for a range of $B' \leq 0.1$ (panel a) and $B'_* \leq 0.1$ (panel b). Also shown are the $\omega' = 1, 2.5, 10$ lines.

Thus if the curves are above the $\omega' = 2.5$ line, then all the secondary electrons and positrons are clearly relativistic. Indeed only if these curves are above the $\omega' = 2.5$ line is the asymptotic form of the attenuation coefficient (used in the derivation of the curves) valid and the results reliable. If the curves lie below the $\omega' = 2.5$ line then the results are unreliable as the asymptotic form of the attenuation coefficient is not applicable.

These results imply that the asymptotic limit is not valid for a large fraction of normal pulsars. Specifically, referring to the curves in Fig. 7(a) and (b), and noting that the asymptotic limit is invalid for curves below the $\omega' = 2.5$ line, it is apparent that the asymptotic approximation is not valid for pulsars with $B' \gtrsim 0.02$ and $B'_* \gtrsim 0.02$ for local and non-local pair creation respectively. An implication is that it is only for pulsars with relatively weak magnetic fields, $B'_* \lesssim 0.02$, that the pairs produced by most of the curvature photons emitted near the polar cap have highly relativistic perpendicular motion. For $0.02 < B' \lesssim 0.1$ in Fig. 7(a), only the very high energy tail of the photon spectrum can produce relativistic pairs. For example, for $B' = 0.1$, only photons with $\varepsilon_\gamma \gtrsim 5 \times 10^5$ MeV pair-create into highly relativistic electrons and positrons that can produce synchrotron radiation. This minimum value of ε_γ for the photons to produce synchrotron-emitting pairs decreases with decreasing B', and it is only for $B' \lesssim 0.02$ that essentially all the curvature photons create secondary electrons and positrons that emit synchrotron radiation. For the curves in Fig. 7(b), the situation is similar except that a few of the lower energy photons in the proximity of $\varepsilon_{\gamma\min}$ lead to relativistic electrons and positrons for $0.02 < B'_* \lesssim 0.1$. However, it is only for $B'_* \lesssim 0.02$ that essentially all the secondary electrons and positrons are relativistic.

5 DISCUSSION

The results derived in this paper imply that one of the conventional assumptions made in polar-cap models for pulsars is not completely justified for pulsars with surface magnetic fields $B' = B/B_{\rm cr} \gtrsim 0.02$, $B_{\rm cr} = 4 \times 10^9$ T, and is invalid for $B' \gtrsim 0.1$. In the polar cap model it is assumed that primary particles, which are accelerated to ultrarelativistic energies ($\gamma \sim 10^7$) very close to the surface of the star, produce high-energy photons through curvature emission and that these populate the magnetosphere with secondary pairs through one-photon pair creation. The specific assumption that our results call into question concerns the perpendicular (to the magnetic field) momenta of these secondary pairs. The conventional treatment of pair creation is in the asymptotic limit, where the perpendicular momenta are assumed highly relativistic. This has an important implication in the model: the secondary pairs can then emit synchrotron radiation, which can either be a source of further pairs in a cascade process, if these photons are absorbed, or can contribute to observable high-energy radiation, if these photons escape. What we find is that this picture cannot be valid for $B' \gtrsim 0.1$ because all the photons that decay produce pairs with non-relativistic perpendicular momenta. In a specific decay, the perpendicular momenta are characterized by the Landau quantum numbers, k, j, for the electron and positron, and $k,\ j \ll B_{\rm cr}/2B$ is required for the perpendicular motion to be non-relativistic. The favoured values of k, j increase with decreasing B' at fixed ω. For increasing $B' \gtrsim 0.1$, there is an increasing preference for the pairs to be in their lowest allowed Landau states, $k + j = 0, 1$.

In the ultrarelativistic case, the synchrotron radiation emitted by the pairs can lead to a second generation of pairs, thereby initiating a pair cascade. It is of interest to consider whether a cascade can be initiated in the non-relativistic case. An electron or positron in a state j, $k \geq 1$ decays to its ground state Landau level, emitting cyclotron photons. If the cyclotron photon is to produce a pair then its perpendicular energy must exceed $2m$. Setting $p_z^2 = 0$, the transition from the nth to the $(n-1)$th excited state satisfies this criterion for $\sqrt{1 + 2nB'} - \sqrt{1 + 2(n-1)B'} > 2$, that is, for $B' > 4(2n - 1)$. Hence, even the highest frequency cyclotron photon (for $n = 1$) can pair-create only in a strongly supercritical magnetic field. In such strong fields, any pair creation strongly favours the lowest Landau levels, $k + j = 0, 1$, so at most one pair is created per photon. Thus, cyclotron photons at $B' < 4$ are never converted into pairs, and cyclotron photons at $B' > 4$ do not initiate a cascade.

Our results may be summarized in terms of the total optical depth, τ, for absorption of a photon. We regard τ as being made up from three possible contributions,

$$\tau = \tau_{\rm GS} + \tau_{\rm NR} + \tau_{\rm REL}. \qquad (28)$$

In equation (28), the contribution $\tau_{\rm GS}$ is the optical depth for a photon creating an electron and positron in their lowest possible Landau levels ($k = j = 0$ for parallel-polarized photons, $k + j = 1$ for perpendicular-polarized photons). For $\tau_{\rm GS} \gtrsim 1$, a photon produces a pair in the ground state and the other two contributions are irrelevant because the photon is already absorbed. The contribution $\tau_{\rm NR}$ is the optical depth for a photon creating an electron and positron with $k + j > 1$ and $2kB'$, $2jB' \lesssim 1$ (non-relativistic perpendicular momentum). For $\tau_{\rm GS} < 1$ and $\tau_{\rm NR} \gtrsim 1$ the photon produces a pair that is in a non-relativistic excited state, and the remaining contribution is irrelevant because the photon is absorbed before it reaches the threshold to produce a relativistic pair. The final contribution, $\tau_{\rm REL}$, is the optical depth for a photon creating an electron and positron that are relativistic with $k + j \gg 1$ and $2kB'$, $2jB' \gg 1$. It is only for $\tau_{\rm GS} + \tau_{\rm NR} \lesssim 1$ and $\tau_{\rm REL} \gtrsim 1$ that the conventional treatment of pair creation based on the asymptotic formula is valid.

Earlier authors, e.g. Daugherty & Harding (1994, 1996), have already noted this inadequacy of the asymptotic formula in high-field pulsars. Daugherty & Harding, in their numerical simulations, used smoothed-out fits to the exact sawtooth attenuation coefficients. Our analytic treatment makes it clear that the asymptotic formula applies under much more restrictive conditions than recognized previously.

We define local and non-local absorption according to whether the magnetic field, B', at the point of absorption is approximately equal to, or much less than, respectively, the magnetic field B'_* at the point of emission. We assume a dipolar magnetic field in treating non-local absorption. Our qualitative conclusions are not particularly sensitive to whether absorption is local or non-local.

Which of the contributions in (28) is relevant depends on the energy of the photon and the strength of the magnetic field. We consider a spectrum of photons resulting from curvature emission by the primary particles. For $B'_* \gtrsim 0.5$, one has $\tau_{\rm GS} \gtrsim 1$ for all photons with energies $\varepsilon_\gamma \gtrsim \varepsilon_{\gamma\min}$, all of which therefore decay into pairs in their ground state ($j = k = 0$ for parallel-polarized photons and $j + k = 1$ for perpendicular-polarized photons). For $0.1 < B'_* \lesssim 0.5$, the curvature photons with energies in the range $\varepsilon_{\gamma\min} \lesssim \varepsilon_\gamma \lesssim \varepsilon_{\gamma\max}$, cf. Fig. 3, are absorbed in creating pairs in the ground state, and consequently are not available to produce any pairs with $j + k > 1$. Comparison of Figs 3 and 7 shows that production of pairs in non-relativistic excited states occurs on the weak-field side of the region where all photons produce pairs in the ground state. As a function of decreasing B'_*, the relative importance of pairs in higher excited states increases. The role of highly relativistic pairs increases with decreasing B'_*, over the range $0.1 > B'_* \gtrsim 0.02$, and with increasing photon energy. For $B'_* \sim 0.1$ only the highest energy photons produce relativistic pairs, and for $B'_* \lesssim 0.02$ effectively all the photons that decay produce highly relativistic pairs. It is only for $B'_* \lesssim 0.02$ that the conventional treatment in terms of $\tau_{\rm REL}$, ignoring the production of non-relativistic pairs, is valid.

Consider the fate of a photon emitted along a dipolar field line at a point where the magnetic field is B'_*. For $B'_* \lesssim 0.02$, no significant conversion of photons to pairs with $j, k \ll 1/2B'$ occurs. The conventional theory then applies, with the photons either escaping from the pulsar (as do the photons with energies $\varepsilon_\gamma < \varepsilon_{\gamma\rm min}$) or being converted into relativistic pairs ($j, k \gg /2B'$). For $0.02 < B'_* < 0.1$, those photons with energies $\varepsilon_\gamma > \varepsilon_{\gamma\rm max}$ or $\varepsilon_\gamma < \varepsilon_{\gamma\rm min}$, where $\varepsilon_{\gamma\rm max}$ and $\varepsilon_{\gamma\rm min}$ are defined by the $\omega' = 2.5$ line in Fig. 7(b), produce relativistic pairs. Those photons with energies $\varepsilon_{\gamma\rm min} \lesssim \varepsilon_\gamma \lesssim \varepsilon_{\gamma\rm max}$ produce non-relativistic pairs. The values of these $\varepsilon_{\gamma\rm max}$ and $\varepsilon_{\gamma\rm min}$ decrease and increase respectively as B'_* increases until, for $B'_* \gtrsim 0.1$, any pairs created are non-relativistic with $j + k = 0, 1$ or $j, k \ll 1/2B'$. Around $B'_* = 0.2$, absorption into these low states becomes substantial with all photons with energies between $\varepsilon_{\gamma\rm min} \sim 100\,\rm MeV$ and $\varepsilon_{\gamma\rm max} \sim 10^5\,\rm MeV$ producing pairs with $j + k = 0, 1$. For higher B'_*, this conversion to $j + k = 0, 1$ pairs extends to both higher and lower photon energies, that is, $\varepsilon_{\gamma\rm min}$ decreases and $\varepsilon_{\gamma\rm max}$ increases with increasing B'_*. Those photons with energies $\varepsilon_\gamma > \varepsilon_{\gamma\rm max}$ or $\varepsilon_\gamma < \varepsilon_{\gamma\rm min}$ either escape or produce non-relativistic pairs. A similar finding as regards the number of Landau levels populated in the pair production process as a function of B'_* and ε_γ has been exhibited by Baring & Harding (2001) in their fig. 3.

These results have implications for theories of high-energy emission from pulsars. One of the processes assumed to contribute to the high-energy emission is synchrotron radiation by the secondary pairs. This is absent if the secondary pairs are produced in their ground state, as is the case when the magnetic field is relatively strong. Specifically, there can be no synchrotron emission from secondary pairs for pulsars with $B'_* \gtrsim 0.5$ and synchrotron emission is suppressed relative to predictions based on the asymptotic formula for pair creation for $0.1 \lesssim B'_* \lesssim 0.5$. Some high-energy emission can result from Doppler-shifted cyclotron emission when pairs are created in an excited state. In particular, a pair produced by a perpendicular-polarized photon is necessarily in an excited state and necessarily emits a cyclotron photon.

So far, we consider the fate of only the curvature photons emitted by the primary electrons. If pair production occurs, the secondary electrons can also emit curvature radiation, which in turn may produce tertiary pairs, and so on. The frequency distributions of the curvature photons produced by these secondary, tertiary, etc., electrons depend upon the electron energy. If the electrons are formed within the vacuum gap, where there is a parallel electric field, they are accelerated very quickly to $\gamma \sim 10^7$. Provided that the secondary electrons are in their ground state, they are then similar to the primary electrons and emit a similar spectrum of curvature radiation. If, on the other hand, the secondary electrons are formed outside the vacuum gap, where the parallel electric field is no longer present, then they have Lorentz factors that are much smaller than those of the primary electrons. The Lorentz factor of an electron with energy $E_{\rm elec}$ in MeV is $\gamma \sim 2E_{\rm elec}$. These secondary electrons still emit curvature radiation but with much lower energies than for the primary electrons. From the profile of the curvature radiation spectrum, obtained using the equations in Jackson (1975), say, one finds, for example, that all but one part in 10^4 of curvature photons emitted by a relativistic electron of energy $1.5\times10^5\,\rm MeV$ ($\gamma \sim 3\times10^5$) have energies $\lesssim 2m$, and consequently do not contribute to any further pair production. This proportion decreases with decreasing electron energy.

6 CONCLUSIONS

The main point made in this paper is that creation of pairs with non-relativistic rather than highly relativistic perpendicular momenta occurs when the magnetic field is sufficiently strong. This result is related to another effect: conversion of photons into positronium, that is into bound pairs. Bound and unbound non-relativistic pairs may be regarded as limiting cases of a single system, in a similar fashion to the relation between the bound and unbound states of the hydrogen atom. In view of this relation, it is not surprising that the formation of bound pairs also occurs for $B' \gtrsim 0.15$ (e.g. Usov & Melrose 1995, 1996). Whether or not the pairs are bound or unbound is of secondary importance compared to whether their perpendicular motion is relativistic, non-relativistic or zero ($j = k = 0$). Synchrotron emission is possible, and can contribute to a pair cascade or to escaping high-energy emission only if the perpendicular motion is highly relativistic. Cyclotron emission, which is strongly Doppler-shifted as a result of the parallel motion, should occur for $j, k \neq 0$, and its contribution to the high-energy spectrum needs to be taken into account. However, for $j, k = 0$ neither synchrotron nor cyclotron emission can occur, and high-energy emission can result only from a subsequent excitation to higher Landau levels through some independent process (Machabeli et al. 2000).

To illustrate the possible implications of our results, consider the isolated pulsar, PSR 1509 − 58, which has a surface magnetic field of $B'_* = 0.7$. In this case essentially all photons with energies above $\varepsilon_{\gamma\rm min} \sim 20\,\rm MeV$ are absorbed into the production of pairs predominantly with $j + k = 0, 1$. The spectrum of this pulsar covers the radio to the γ-ray bands, with a dramatic reduction in flux above 5 to 10 MeV. Our theoretical cut-off at $\varepsilon_{\gamma\rm min}$ is consistent with this observed cut-off being identified with creation of non-relativistic pairs. A possible interpretation of the cut-off above 5 to 10 MeV in this pulsar is that the higher energy photons are absorbed into the ground state, and that the observed photons are those with $\varepsilon_\gamma \lesssim \varepsilon_{\gamma\rm min}$ that escape this absorption because their energies are too low. Two processes, cyclotron emission from the decay of an electron or positron in its first excited Landau level to its ground state and inverse Compton scattering (see for example Baring & Harding 2001) may affect this simple picture. Both these processes may be important in the production of high-energy photons. Before applying this argument to other γ-ray pulsars that exhibit high-energy cut-offs, one also needs to consider the effect of the height of the inner gap. The γ-ray beams emitted by energetic electrons are detectable, having wider profiles, only by extending the acceleration zone (Daugherty & Harding 1996; Harding 2000). The effect of these extended inner gaps on the secondary, tertiary, etc. electrons and the subsequent cascades has not been considered in our simple model. Young pulsars with surface magnetic fields $B'_{\rm star} \gtrsim 0.1$, however, have thin inner gaps and consequently their cascades form closer to the star where $B'_*/B'_{\rm star}$ remains close to unity and $\varepsilon_{\gamma\rm min}$ is close to its value for $B'_* = B'_{\rm star}$. This is the case for PSR 1509 − 58. Older pulsars with

$B'_{\rm star} \lesssim 0.02$ have extended acceleration zones up to heights of a few stellar radii and consequently the pair production front where the cascades occur has $B'_{*} \ll B'_{\rm star}$. Consequently the high-energy cut-off is much greater than the value of $\varepsilon_{\gamma\rm min}$ estimated for $B'_{*} = B'_{\rm star}$ (cf. Fig. 7b). This is consistent with fig. 2 of Harding (2000).

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