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Mutual Helicity between Reconnecting Flux Loops

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Abstract. To treat conservation of helicity in a quadrupolar model for a solar flare requires expressions for the self and mutual helicities of the reconnecting loops. Available formulas for the mutual helicity do not satisfy the requirement that they be symmetric in the interchange of the loops, and that the mutual helicity be proportional to the mutual inductance for force-free loops. The first step in a possible alternative derivation of the mutual helicity is outlined.

1. Introduction

Solar flares are attributed to magnetic reconnection reducing the stresses in the pre-flare magnetic field. A favored class of models for the pre-flare configuration consists of an emerging magnetic loop pushing into a pre-existing loop, leading to reconnection between the two loops (e.g. Machado et al. 1988). Such models are referred to here as 'quadrupolar' models (e.g. Aschwanden et al. 1999; Uchida et al. 2003). The reconnection must involve transfer of both magnetic flux and current between two pairs of loops. During a flare there is insufficient time for the magnetic field structure, including the current, to change below the photosphere, and the appropriate photospheric boundary condition is conservation of both the fluxes and currents at each of the four footpoints. These boundary conditions provide constraints on an acceptable model for energy release (Melrose 1997).

It is now widely accepted that magnetic helicity plays an important role in magnetic evolution in the corona. The role of sub-photospheric motions in injecting helicity into the corona has received considerable attention recently (Berger & Ruzmikin 2000; Kusano et al. 2002; Magara & Longcope 2003; Welsch & Longcope 2003; Zhang & Low 2003). The injected helicity is assumed to escape from the corona, either in CMEs (Rust 1999) or through the photosphere (van Ballegooijen & Martens 1989). Magnetic helicity is conserved during reconnection (Taylor 1986). The time for transport out of the corona is long compared with the flare timescale, and hence release of energy during a flare occurs at constant helicity. Thus, during a flare the helicity can only be transferred between magnetic structures or transformed from one scale to another or between twist and writhe (Yousel & Brandenburg 2003).

Conservation of helicity provides an additional constraint on magnetic energy release in a quadrupolar flare model, cf. Fig. 1. It is shown in Sect. 2 how, for a force-free field, the helicity integral may be expressed in terms of the current. This allows one to relate the self and mutual helicities in a multi-loop model to the self and mutual inductances of the loops. Conservation of helicity in a quadrupolar model requires that the reconnecting loops have different αs

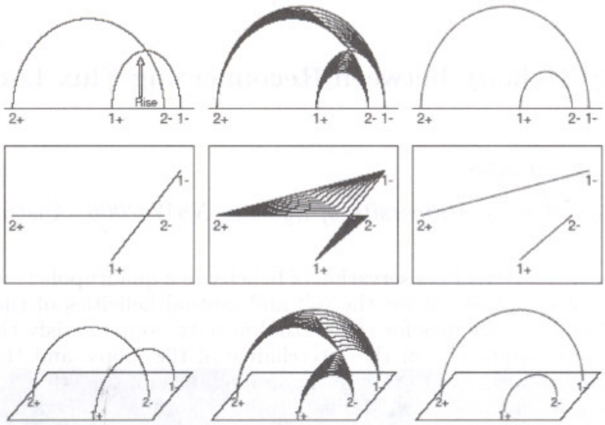

Figure 1. Schematic illustration of two-loop reconnection after Aschwanden et al. (1999). Columns: left = pre-connection, middle = during reconnection, right = post-connection. Rows: top = side view, middle = top view, bottom = perspective view.

(Melrose 2004). This is not surprising in view of the well-known result that the minimum energy state resulting from reconnection has constant-α (Taylor 1986): a constant-α model has no free magnetic energy that can be released through reconnection.

Two aspects of the model are discussed in this paper: expressions for the mutual helicity in a multi-loop model, and the neglect of current-current interactions with other loops in a quadrupolar model.

2. Magnetic Helicity for a Force-Free Field

In this section it is shown how the helicity in a force-free magnetic field may be expressed in terms a double integral over the current density. In a multi-loop model the integral may be written in terms of self and mutual helicities that are related to the self and mutual inductances.

The magnetic helicity, H, is defined by

$$H = \int d^3\boldsymbol{x}\, \boldsymbol{B}(\boldsymbol{x}) \cdot \boldsymbol{A}(\boldsymbol{x}), \qquad \boldsymbol{A}(\boldsymbol{x}) = \mu_0 \int d^3\boldsymbol{x}'\, \frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{1}$$

where $\boldsymbol{A}(\boldsymbol{x})$ is the solution of Poisson's equation for the vector potential (in the Coulomb gauge) in terms of the current density, $\boldsymbol{J}(\boldsymbol{x})$. For a force-free magnetic field, one has $\boldsymbol{J}(\boldsymbol{x}) = \alpha(\boldsymbol{x})\, \boldsymbol{B}(\boldsymbol{x})/\mu_0$, where $\alpha(\boldsymbol{x})$ is a constant along each magnetic field line. Then Eq. (1) becomes

$$H = \mu_0^2 \int d^3\boldsymbol{x} d^3\boldsymbol{x}'\, \frac{\boldsymbol{J}(\boldsymbol{x}) \cdot \boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\, \frac{\alpha(\boldsymbol{x}) + \alpha(\boldsymbol{x}')}{2\alpha(\boldsymbol{x})\alpha(\boldsymbol{x}')}, \tag{2}$$

where the integral is symmetrized without loss of generality.

In a multiple circuit model, the current density is approximated by a collection of current lines, each of which represents the current in a magnetic loop. The current, I_i, and flux, F_i, in the i-th line, are related by

$$I_i = \frac{\alpha_i}{\mu_0} F_i, \tag{3}$$

where α_i is an appropriate average of $\alpha(\boldsymbol{x})$ across the loop. The helicity (2) may be written in terms of the fluxes in each of the loops (Berger 1998):

$$H = \sum_{ij} \mathcal{L}_{ij} F_i F_j = \sum_i T_i F_i^2 + 2 \sum_{i<j} \mathcal{L}_{ij} F_i F_j, \tag{4}$$

with $T_i = \mathcal{L}_{ii}$ describing the self helicity and $\mathcal{L}_{ij} = \mathcal{L}_{ji}$ for $i \neq j$ describing the mutual helicity. The quantity T_i may be interpreted as the number of twists (angle of the twist divided by 2π) along the i-th loop. The coefficients in Eq. (4) are related to the self inductances L_i and the mutual inductances M_{ij} by

$$T_i = \frac{2\alpha_i}{\mu_0} L_i, \qquad \mathcal{L}_{ij} = \frac{\alpha_i + \alpha_j}{\mu_0} M_{ij}. \tag{5}$$

3. Conservation of Helicity in a Quadrupolar Model

One version of a quadrupolar model involves flux and current being transferred from two initial loops to two final loops, with current and flux conserved at the four footpoints (Melrose 1997; Aschwanden et al. 1999). In analyzing such a model it is assumed that the geometry is pre-determined, in the sense that the self and mutual inductances are specified. It is straightforward to include conservation of helicity in this model (Melrose 2004).

One result that emerges immediately is that if all loops have the same α, then their contribution to the helicity is proportional to their contribution to the magnetic energy, and conservation of helicity implies conservation of energy. Hence, no energy release is possible when both loops have the same α. Suppose the two initial loops are labeled 1 and 2, and the two final loops 3 and 4, Assume a transfer of flux ΔF and current $\Delta I = \alpha_0 \Delta F/\mu_0$. Further suppose that loops 3 and 4 are created by the transfer, so that they have $\alpha_3 = \alpha_4 = \alpha_0$. One can then show that the energy released is maximized at constant ΔF for $\alpha_1 = 2\alpha_2$. With the further simplifying assumption that loop 1 has a much larger current than loop 2, the change in magnetic energy is dominated by the change $L_1 I_1 \Delta I$ of the energy in loop 1. Up to one eighth of this energy can be released in this simple model (Melrose 2004).

The model implies that, for $\alpha_1 > \alpha_2$, the final αs, denoted by primes, satisfy $\alpha_1' > \alpha_1$, $\alpha_2' < \alpha_2$. For example, after the transfer, loop 1 is left with current $I_1' = I_1 - \Delta I$ and flux $F_1' = F_1 - \Delta F$, and this implies $\alpha_1' - \alpha_1 = (\alpha_1 - \alpha_0)\Delta F/F_1'$, which is necessarily positive. Although the assumptions in the model require $\alpha_1' > \alpha_1$, $\alpha_2' < \alpha_2$, this is contrary to the expectation that reconnection should reduce the difference in the αs. This highlights a weakness in the model, in that the interaction with other loops is ignored in estimating the change in energy and helicity, cf. Sect. 5.

4. The Mutual Helicity

There is an inconsistency between an expression for the mutual helicity derived by Berger (1998, 1999) and Priest (1999) and properties implied by the foregoing discussion. This inconsistency is discussed here.

The expression for $\mathcal{L}_{ij}$ is derived by integrating

$$\frac{dH_{ij}}{dt} = -\frac{2}{\pi}\,\Omega_{ij}\,F_iF_j, \tag{6}$$

for some specific motion starting from an initial condition where H_{ij} is zero. In Eq. (6), Ω_{ij} is the angular velocity of the j-th loop about the i-th (Berger 1998; Welsch & Longcope 2003). The specific motions considered are rotations of one loop relative to the other, and the movement of the two loops together, starting from an infinite separation where their mutual helicity is zero. Berger (1999) found

$$H_{ij} = \frac{F_iF_j}{\pi}(\theta_{i+j-} + \theta_{i-j+} - \theta_{i+j+} - \theta_{i-j-}), \tag{7}$$

where the angles are between an arbitrary fixed direction and the line that passes through the two footpoints labeled by the subscripts. Priest (1999) illustrated Eq. (7) in the cases where the loops do and do not cross: for example, when they do not cross, $\theta_{i+j-} - \theta_{i+j+}$ and $\theta_{i-j-} - \theta_{i-j+}$ are the angles subtended by loop j at footpoints $i+$ and $i-$, respectively, and H_{ij} is proportional to the difference between these angles.

The result (7) is not symmetric in the interchange of loops i and j, $H_{ji} \neq H_{ij}$. It follows from Eq. (4) that only the symmetric part is physically relevant. Hence, one might expect to derive an appropriate symmetrized form simply by symmetrizing Eq. (7). However, on interchanging i and j in Eq. (7), the angles change according to $\theta_{i+j-} \to \theta_{j+i-} = \pi - \theta_{i-j+}$, $\theta_{i+j+} \to \theta_{j+i+} = \pi - \theta_{i+j+}$, implying $H_{ji} = -H_{ij}$. Hence, symmetrizing Eq. (7) gives zero. This inconsistency is inherent in the asymmetric procedure used to derive Eq. (7).

A symmetric procedure for a special case is as follows. Consider an initial single loop, with current I and twist T, that is regarded as a sum of two loops, with currents I_i, I_j, with $I = I_i + I_j$. The helicity, $H = \frac{1}{2}T(I_i + I_j)^2$ implies the self helicities $T_i = T_j = T$, and the mutual helicity $\mathcal{L}_{ij} = \mathcal{L}_{ji} = T$. Now apply Eq. (6) for a motion that rotates loop 2 about the joint centers of the loops, keeping loop 1 fixed. A rotation through π makes loop 2 antiparallel to loop 1, giving a loop with net current $I_i - I_j$, and implying $\mathcal{L}_{ij} = \mathcal{L}_{ji} = -T$. A rotation through $\pi/2$, so that the two loops are orthogonal, leads to a configuration with zero mutual helicity. The same results follow by rotating loop 1 about the joint center, keeping loop 2 fixed. Thus for two loops of equal length with a common center, one has

$$eq.8H_{ij} = -\frac{2F_iF_j\theta_{ij}}{\pi}, \tag{8}$$

where θ_{ij} is defined as $-\pi/2$ plus the positive angle between the lines, j_+j_- and i_+i_-. By construction Eq. (8) is symmetric in the two loops.

The interpretation of Eq. (7) in terms of the difference between the angles subtended by loop j at footpoints $i+$ and $i-$ leads to a dependence on the separation between the loops, when this separation is large, that reproduces the known dependence for the mutual inductance. The form (8) does not generalize in any obvious way to reproduce this dependence. A semi-empirical expression for the mutual inductance (Melrose 1997) corresponds to a mutual helicity

$$\mathcal{L}_{ij} = -(T_i T_j)^{1/2} \cos\theta_{ij} \left(\frac{4a_i a_j}{(a_i + a_j)^2 + d_{ij}^2} \right)^{3/2} . \tag{9}$$

where a_i, a_j are the lengths of the loops and d_{ij} is the distance between their centers. For semi-quantitative estimates, the factor $\cos\theta_{ij}$ can be replaced by $2\theta_{ij}/\pi$ so that, for $a_i = a_j$, $d_{ij} = 0$, Eq. (9) reproduces Eq. (8).

The form (9) does not take account of the twisting of field lines. In fact, all available results for the mutual inductance assume relatively simple geometric structures for the current lines. A specific calculation for a force-free field is needed to complement the known results. One may conclude that each of Eqs. (7), (8), and (9) has some features that should be present in a general form, but that the appropriate general form has yet to be identified.

5. Further Development of the Model

Several questions are raised by the inclusion of helicity in the quadrupolar model for a flare: the formation of intrinsically new loops, the conversion of helicity into other forms (between twist and writhe or between large and small scales), and the escape of helicity (e.g. in CMEs). A weakness in the existing model is that it is based on only two initial loops, neglecting the current-current interactions with other loops. This neglect is not obviously justified, and it is discussed here in the context of a specific problem.

The specific problem is that the model implies $\alpha_1' > \alpha_1$, $\alpha_2' < \alpha_2$ for $\alpha_1 > \alpha_2$, and this is contrary to the expectation that reconnection should reduce the difference between the αs. The assumption that flux and current are conserved at all four footpoints requires that the same flux, ΔF, and current, ΔI, be transferred from both initial loops to both final loops. For $\alpha_1 > \alpha_2$ this requires that both loops 1 and 2 must split into two parts, one part of each loop must have flux ΔF, current ΔI and $\alpha_0 = \mu_0 \Delta I / \Delta F$, with these parts transferred to the final loops, leaving the other parts with $\alpha_1' > \alpha_1$ and $\alpha_2' < \alpha_2$, respectively. Consider a variant of the model in which only the parts of the initial loops with ΔF, ΔI, and α_0 are retained. This corresponds to ignoring the interaction with the other parts of the initial loops, as well as with other loops. However, in this variant, the parts transferred correspond to a constant-α_0 model, for which there is no free energy to drive the flare. It follows that the variant is unacceptable, and that one must include the interaction between the two parts of the current in each initial loop: it is the change in these current-current interaction energies that provides the free energy to drive the flare. The counter-intuitive implication, $\alpha_1' > \alpha_1$, $\alpha_2' < \alpha_2$, reflects the fact that the current-current interaction between the reconnecting loops and other loops cannot be neglected. Inclusion of other loops, and the formation of new (CME-associated) loops with no footpoints in the photosphere are natural extensions of the model.

6. Conclusions

The quadrupolar model with flux and current conserved at the four footpoints, is useful in estimating the available free energy for release in a flare (Melrose 1997; Aschwanden et al. 1999). The inclusion of conservation of helicity further constrains the model, but also simplifies it by constraining the final configuration for a given initial configuration. Specifically, conservation of helicity requires that the initial loops have different αs, with the maximum energy release, subject to a given flux transfer for $\alpha_1 = 2\alpha_2$. For one dominant loop reconnecting with a smaller loop, up to one eighth of the energy, $L_1 I_1 \Delta I$, lost by the initial loop is released to power the flare.

One difficulty identified here is the form for the mutual helicity of force-free loops : the mutual helicity should be symmetric in the interchange of the loops, whereas in simple cases the available expressions are anti-symmetric. Weaknesses in the existing quadrupolar model are the oversimplified treatment of the current-current interaction between flaring and non-flaring loops, and the neglect of the possibility that new loops with no footpoints in the photosphere might be formed during reconnection.

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