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Approximate plasma dispersion functions at relativistic temperatures

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Abstract. Analytic approximations to relativistic plasma dispersion functions are derived in terms of the exponential integral function, Ei(x), for relativistic temperatures $T \gtrsim m_e c^2$ (where $m_e c^2$ is the electron rest energy). It is shown that a simpler, useful approximation to these functions can be obtained based on known approximations to the exponential integral function.

1. Introduction

Various relativistic plasma dispersion functions (RPDFs) have been introduced to describe dispersion at relativistic temperatures, for isotropic plasmas (e.g. Prentice (1968); Godfrey et al. (1975); Magneville (1990)), magnetized plasmas (e.g. Trubnikov (1958); Silin (1960, 1961); Dnestrovskii and Kostomarov (1961, 1962); Dnestrovskii et al. (1964); Shkarofsky (1966)), and highly magnetized plasmas (e.g. Gedalin et al. (1998); Kennett et al. (2000); Melrose et al. (1999)). For isotropic unmagnetized plasmas, one requires two different functions to describe the dispersion in general. One choice of two independent RPDFs consists of the function $T(z, \rho)$ and its derivative $T'(z, \rho) = \partial T(z, \rho)/\partial z$, introduced by Godfrey et al. (1975), where $\rho = mc^2/T$ is the inverse temperature in units of the electron rest energy (0.5×10^{10} K), and z is the ratio of the phase speed to the speed of light. The RPDFs for magnetized plasmas are qualitatively different from those for unmagnetized plasma (e.g. Robinson (1986)), except for pulsar plasmas where the superstrong magnetic field causes the distribution of particles to be one dimensional along the magnetic field lines, and the dispersion can then be expressed in terms of the same RPDFs as for an unmagnetized plasma. Only these RPDFs are discussed in this paper.

The functional form of RPDFs in relativistic plasmas, $\rho \lesssim 1$, exhibits features that are absent in the non-relativistic case, $\rho \gg 1$. The RPDFs can be expressed in terms of integrals over particle speed, β (in units of the speed of light), with the distribution function proportional to $\exp(-\rho\gamma)$, where $\gamma = 1/(1-\beta^2)^{1/2}$ is the Lorentz factor. The RPDFs contain a resonant denominator $z - \beta$. A characteristic feature of RPDFs for $\rho \lesssim 1$ is a sharp peak due to the resonance at $z = \beta$, which occurs near the value of β corresponding to $\rho\gamma \sim 1$. It is desirable to have analytic approximations that treat this characteristic feature accurately.

In this paper, we write the RPDFs as integrals over γ with the resonant denominator expressed in terms of Lorentz factors and evaluated in terms of the exponential integral function, Ei(x). Analytic approximations are then developed using known approximations to $\mathrm{Ei}(x)$. It is convenient to introduce a Lorentz factor, $\gamma_z = 1/(1-z^2)^{1/2}$, corresponding to the phase speed z. The resonance then occurs at $\gamma_z = \gamma$, and the peak in the RPDFs occur at $\gamma_z \sim 1/\rho$. The extension to superluminal phase speeds, $z > 1$ is achieved simply by allowing γ_z to be imaginary and using approximations to the exponential integral function for the imaginary argument.

The outline of the present paper is as follows. In Sec. 2 the specific RPDFs discussed in this paper are defined and their relation to other RPDFs are summarized. In Sec. 3 these RPDFs are written in terms of exponential integral functions, and approximate forms in the highly relativistic limit are derived in Sec. 4. The results are discussed in Sec. 5.

2. RPDFs

The RPDFs used to treat dispersion in isotropic, unmagnetized thermal plasmas, isotropic magnetized plasmas, and pulsar (strongly magnetized, one-dimensional) plasmas, are summarized in this section.

2.1. Dispersion in isotropic, unmagnetized plasmas

An isotropic Jüttner distribution normalized (by integration over $d^3\mathbf{p}$) to number density n is

$$f(\mathbf{p}) = \frac{n\rho e^{-\rho\gamma}}{4\pi(m_e c)^3 K_2(\rho)}, \tag{2.1}$$

where $|\mathbf{p}| = m_e c\gamma\beta$ and $K_n(\rho)$ is a modified Bessel function. For an unmagnetized electron gas with distribution (2.1), dispersion may be described in terms of the RPDF introduced by Godfrey et al. (1975)

$$T(z,\rho) = \int_{-1}^{1} d\beta \frac{e^{-\rho\gamma}}{\beta - z}, \tag{2.2}$$

and its derivative $T'(z,\rho) = \partial T(z,\rho)/\partial z$.

2.2. Dispersion in pulsar plasmas

One model for the electron or positron distributions in a pulsar plasma (e.g. Arendt and Eilek (2002)), is a one-dimensional (along the magnetic field lines) Jüttner distribution

$$f(p) = \frac{n e^{-\rho\gamma}}{2m_e c K_1(\rho)}. \tag{2.3}$$

Dispersion for such a distribution may be expressed in terms of the three RPDFs (Melrose et al. (1999))

$$W(z) = \left\langle \frac{1}{\gamma^3(\beta - z)^2} \right\rangle, \quad R(z) = \left\langle \frac{1}{\gamma(\beta - z)} \right\rangle, \quad S(z) = \left\langle \frac{1}{\gamma^2(\beta - z)} \right\rangle, \tag{2.4}$$

where the angular brackets denote an average over the distribution function (2.3). The average, $\langle K \rangle$, for any variable K is given by

$$\langle K \rangle = \frac{1}{2K_1(\rho)} \int_{-1}^{1} d\beta \gamma^3 K e^{-\rho\gamma}. \tag{2.5}$$

Regardless of the specific form of the distribution, $W(z)$ can be expressed in terms of $R(z)$,

$$W(z) = -2zR(z) + (1-z^2)R'(z) - \left\langle \frac{1}{\gamma} \right\rangle. \tag{2.6}$$

For the distribution (2.3), one has $\langle 1/\gamma\rangle = K_0(\rho)/K_1(\rho)$. In terms of the notation (2.4), the definition (2.2) corresponds to

$$T(z,\rho) = -2K_1(\rho) \left\langle \frac{1}{\gamma^3(z-\beta)} \right\rangle. \tag{2.7}$$

The RPDFs (2.4) can be expressed in terms of the RPDF (2.7):

$$\begin{gathered} W(z) = \frac{T'(z,\rho)}{2K_1(\rho)}, \quad R(z) = \frac{1}{1-z^2}\left[z\,\frac{K_0(\rho)}{K_1(\rho)} + \frac{T(z,\rho)}{2K_1(\rho)} \right], \\ S(z) = -\frac{1}{z}\left[1 + \frac{(1-z^2)}{\rho}\,\frac{T'(z,\rho)}{2K_1(\rho)} \right]. \end{gathered} \tag{2.8}$$

Functions (2.7)–(2.8) are well defined for all z. These three RPDFs are not independent of each other, and one can write $W(z)$ and $S(z)$ in terms of $R(z)$. One obtains

$$W(z) = -2zR(z) + (1-z^2)R'(z) - \frac{K_0(\rho)}{K_1(\rho)}, \tag{2.9}$$

$$S(z) = -\frac{1}{z}\left\{ 1 - \frac{2z(1-z^2)}{\rho} R(z) + \frac{(1-z^2)^2}{\rho} R'(z) - \frac{1-z^2}{\rho}\,\frac{K_0(\rho)}{K_1(\rho)} \right\}, \tag{2.10}$$

with $R'(z) = \partial R(z)/\partial z$.

The response tensor for a pulsar plasma involves the RPDFs $W(z)$, $R(z_\pm)$, $S(z_\pm)$, with

$$z = \frac{\omega}{k_\parallel c}, \quad y = \frac{\omega_B}{k_\parallel c}, \quad z_\pm = \frac{z \pm y(1+y^2-z^2)^{1/2}}{1+y^2}, \tag{2.11}$$

where $k_\parallel$ is the component of $\mathbf{k}$ along the direction of the magnetic field.

3. Expressions in terms of exponential integral functions

In developing the approximations we start with $R(z_\pm)$ and $S(z_\pm)$, whose arguments are restricted to $|z_\pm| \leqslant 1$.

3.1. The RPDFs $R(z_\pm)$, $S(z_\pm)$

The restriction $|z_\pm| \leqslant 1$ implies that the corresponding Lorentz factors,

$$\gamma_\pm = (1-z_\pm^2)^{-1/2} = \left(\frac{yz \pm (1+y^2-z^2)^{1/2}}{1-z^2} \right)^{1/2}, \tag{3.1}$$

always exist. Then (2.4) gives

$$R(z_\pm) = \frac{1}{2}\gamma_\pm^2 z_\pm \left\langle \frac{1}{\gamma-\gamma_\pm} + \frac{1}{\gamma+\gamma_\pm} \right\rangle, \tag{3.2}$$

$$S(z_\pm) = \frac{1}{2}\gamma_\pm z_\pm \left\langle \frac{1}{\gamma-\gamma_\pm} - \frac{1}{\gamma+\gamma_\pm} \right\rangle. \tag{3.3}$$

For convenience we write $z_\pm$ as z and $\gamma_\pm$ as γ_z. The RPDFs contain the integrals,

$$\int_0^1 \frac{d\beta\gamma^3 e^{-\rho\gamma}}{\gamma\pm\gamma_z} = \int_1^\infty \frac{d\gamma e^{-\rho\gamma}}{\beta(\gamma\pm\gamma_z)}$$

$$= \int_1^\infty d\gamma e^{-\rho\gamma}\left[\frac{1}{\gamma\pm\gamma_z} + \sum_{n=1}^{\infty}\frac{c_n}{\gamma_z^{2n}}\left(\frac{1}{\gamma\pm\gamma_z} \pm \sum_{s=1}^{2n}(\mp 1)^s\frac{\gamma_z^{s-1}}{\gamma^s}\right)\right], \quad (3.4)$$

where one changes the integral variable $d\beta = d\gamma/\beta\gamma^3$ and makes an expansion $1/\beta = 1 + \sum_{n=1}^{\infty} c_n/\gamma^{2n}$ with $c_1 = \frac{1}{2}$, $c_2 = \frac{1\cdot 3}{2\cdot 4}$, $c_2 = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}$, ... The integrals on the right-hand side can be expressed in terms of exponential integral functions,

$$\int_1^\infty \frac{d\gamma e^{-\rho\gamma}}{\gamma\pm\gamma_z} = -e^{\pm\rho\gamma_z}\,\mathrm{Ei}(\mp\rho(\gamma_z\pm 1)), \quad (3.5)$$

$$\int_1^\infty \frac{d\gamma e^{-\rho\gamma}}{\gamma^s} = \mathrm{E}_s(\rho). \quad (3.6)$$

The exponential integral functions are defined by (Abramowitz and Stegun 1970):

$$\mathrm{Ei}(x) = -\mathscr{P}\int_{-x}^{\infty} dt\frac{e^{-t}}{t}, \quad (3.7)$$

$$\mathrm{E}_n(x) = \int_1^\infty dt\frac{e^{-xt}}{t^n}, \quad x>0,\ n=0,1,2,\ldots, \quad (3.8)$$

where $\mathscr{P}$ denotes the Cauchy principal value. One may extend $\mathrm{Ei}(x)$ to negative values ($x<0$) in terms of $\mathrm{E}_1(x)$:

$$\mathrm{Ei}(-x) = -\mathrm{E}_1(x). \quad (3.9)$$

In terms of these functions, the RPDFs (3.2) and (3.3) become

$$R(z) = -\frac{\gamma_z^2 z}{2K_1(\rho)}[\Psi_+(\rho,\gamma_z) + \Phi_+(\rho,\gamma_z)], \quad (3.10)$$

$$S(z) = -\frac{\gamma_z z}{2K_1(\rho)}[\Psi_-(\rho,\gamma_z) + \Phi_-(\rho,\gamma_z)], \quad (3.11)$$

with

$$\Psi_\pm = e^{\rho\gamma_z}\,\mathrm{Ei}(-\rho(\gamma_z+1)) \pm e^{-\rho\gamma_z}\,\mathrm{Ei}(\rho(\gamma_z-1)), \quad (3.12)$$

$$\Phi_\pm = \sum_{n=1}^{\infty}\frac{c_n}{\gamma_z^{2n}}\left[\Psi_\pm - \sum_{s=1}^{2n}[(-1)^s \mp 1]\gamma_z^{s-1}\mathrm{E}_s(\rho)\right]. \quad (3.13)$$

At this stage no approximation has been made. It can be shown that in the relativistic approximation $\rho \ll 1$, $\Phi_\pm$ can be ignored (cf. Sec. 4).

3.2. Derivation of $W(z)$

The foregoing procedure may be applied to $W(z)$, $T(z,\rho)$ and $T'(z,\rho)$, which are all determined by $R(z)$ through (2.9) and (2.10). Here we consider only $W(z)$ explicitly.

For $|z| \leq 1$, substituting (3.10) into (2.6) or (2.9) one obtains

$$W(z) = -\frac{1}{K_1(\rho)}\left\{\gamma_z^2\left[e^{-\rho}+\frac{1}{2}\rho\gamma_z z^2\Psi_-\right]+K_0(\rho)+\sum_{n=1}^{\infty}\frac{c_n}{\gamma_z^{2(n-1)}}\right.$$
$$\left.\times\left[e^{-\rho}+\frac{1}{2}\rho\gamma_z z^2\Psi_- + \sum_{s=1}^{n}\left(\gamma_z^{-2}-2(n-s+1)z^2\right)\gamma_z^{2(s-1)}\mathrm{E}_{2s-1}(\rho)\right]\right\}, \quad (3.14)$$

where $\mathrm{E}_{2s-1}(\rho)$ is the exponential integral function defined by (3.8).

For $|z| > 1$, $\gamma_z = (1-z^2)^{-1/2}$ is imaginary. The poles at $\gamma = \pm\gamma_z$ are then not on the real axis, and the argument of the exponential integral functions is imaginary. In this case, one may extend (3.10) to $z > 1$ by writing the relevant integrals in terms of the exponential functions with a complex argument. Since the exponential integral function $\mathrm{E}_1(x)$ is well defined in the upper half complex plane (with $\arg(x) < \pi$) (see, e.g., Abramawitz and Stegun (1970, p. 228)), one may extend the definition (3.9) to include a complex $E_1(x)$. Then, one can derive $W(z)$ as (3.14) but with $\gamma_z = -i/(z^2-1)^{1/2}$, where $|z| > 1$.

4. Approximate RPDFs

One of the major advantages of expressing RPDFs in terms of $\mathrm{Ei}(x)$ is that some useful approximations can be readily obtained based on the known approximations to the exponential integral functions. Specifically, approximate expressions for the RPDFs in the limit $\rho \lesssim 1$ are derived in this section.

4.1. Approximate forms for $R(z)$ and $S(z)$

The general forms of $R(z)$ and $S(z)$ are similar: both tend to zero at $z=0$, $R(0) = S(0) = 0$, and both have negative peaks for z near the speed corresponding to the thermal peak in the distribution function. Notable differences are that for $\rho \ll 1$, $R(z)$ approaches zero from above and $S(z)$ from below for $z \to 0$, and the peak in $R(z)$ is much larger than the peak in $S(z)$. In approximating these functions the most important features are the peaks and the limiting values for $z \to 1$.

We first consider the large γ_z approximation, in which $\Phi_\pm$ in (3.10) and (3.11) can be ignored. Assuming $\mathrm{Ei}(\mp\rho(\gamma\pm1)) \approx \mathrm{Ei}(\mp\rho\gamma)$, one has

$$R(z) \approx -\frac{\gamma_z^2 z}{2K_1(\rho)}\Psi_+, \quad S(z) \approx -\frac{\gamma_z z}{2K_1(\rho)}\Psi_-, \quad (4.1)$$

with $\Psi_\pm \approx e^{\rho\gamma_z}\mathrm{Ei}(-\rho\gamma_z) \pm e^{-\rho\gamma_z}\mathrm{Ei}(\rho\gamma_z)$. The approximations (4.1) are derived for $\gamma_z \gg 1$, and they are reasonably accurate even for $\gamma_z \approx 1$ for $\rho \ll 1$. This is due to the approximate expressions being functions only of $\rho\gamma_z = \rho/(1-z^2)^{1/2}$, which varies slowly with z except for the peak near $z \sim 1$ (corresponding to a large γ_z). By decreasing ρ one essentially moves the peak toward $z=1$, and the validity of the approximation thus extends toward $\gamma_z \gtrsim 1$. Using $K_1(\rho) \approx 1/\rho$ for $\rho \ll 1$, one obtains

$$R(z) \approx -\tfrac{1}{2}\rho\gamma_z^2 z[e^{-\rho\gamma_z}\mathrm{Ei}(\rho\gamma_z) + e^{\rho\gamma_z}\mathrm{Ei}(-\rho\gamma_z)], \quad (4.2)$$

$$S(z) \approx -\tfrac{1}{2}\rho\gamma_z z[e^{-\rho\gamma_z}\mathrm{Ei}(\rho\gamma_z) - e^{\rho\gamma_z}\mathrm{Ei}(-\rho\gamma_z)]. \quad (4.3)$$

From these approximate expressions one may obtain a much simpler, useful approximation in two separate parameter regions, $\rho\gamma_z \leq 1$ and $\rho\gamma_z > 1$.

Two simple approximations are found by Taylor series expansions. First, expanding in powers of $\rho\gamma_z < 1$ and retaining only the leading terms gives the

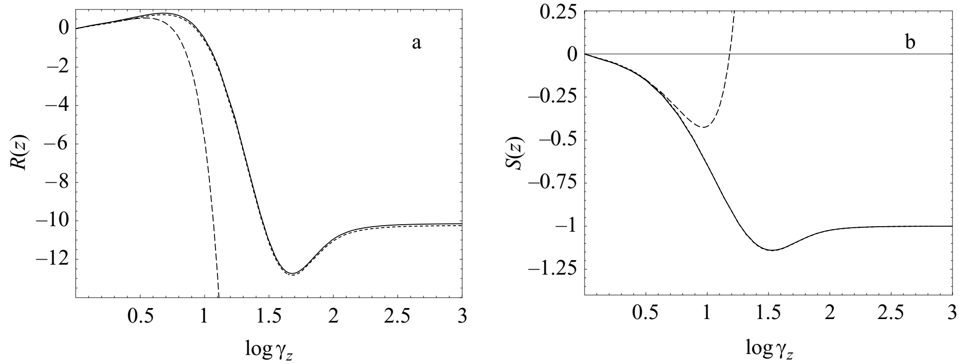

Figure 1. The functions $R(z)$ (a) and $S(z)$ (b) for $\rho = 0.1$, plotted as a function of $\gamma_z = 1/(1 - z^2)^{1/2}$. The solid curves are the plots of the functions (4.2) and (4.3) obtained in the $\rho \ll 1$ limit, the dashed curves are for approximations given by (4.4) and (4.5). The exact functions, obtained by numerical integration, are plotted as dotted curves that mostly cannot be distinguished from the solid curves.

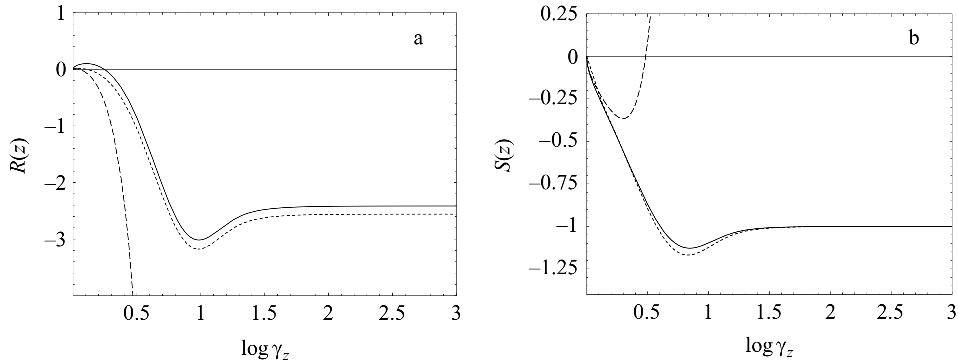

Figure 2. As Fig. 1, but for $\rho = 0.5$.

approximations

$$R(z) \approx -\rho\gamma_z^2 z[\ln(\rho\gamma_z) + C], \tag{4.4}$$

$$S(z) \approx (\rho\gamma_z)^2 z[\ln(\rho\gamma_z) + C - 1], \tag{4.5}$$

where $C \approx 0.577$ is Euler's constant. Second, expanding $\mathrm{Ei}(\pm\rho\gamma_z)$ in $1/\rho\gamma_z \ll 1$, one finds

$$R(z) \approx -\frac{z}{\rho}\left[1 + \frac{6}{(\rho\gamma_z)^2}\right], \tag{4.6}$$

$$S(z) \approx -z\left[1 + \frac{2}{(\rho\gamma_z)^2}\right]. \tag{4.7}$$

These approximations apply to (a) and (b), respectively, in Figs 1 and 2. However, neither is valid in the intermediate range $\rho\gamma_z \sim 1$ where $R(z)$ and $S(z)$ are peaked; the forms (4.2) and (4.3) cannot be further simplified without loss of accuracy in this range.

The approximations (4.2) and (4.3) are compared with the exact expressions in Fig. 1 for $\rho = 0.1$. For clarity in exhibiting the peaks, the functions are plotted versus $\log_{10}\gamma_z$, with $\gamma_z = (1-z^2)^{-1/2}$. The approximations given by the expansions (4.4), (4.5) break down below the peaks, $\rho\gamma_z \sim 1$. The approximations (4.2) and (4.3) match the exact expressions with high precision, with the differences being barely resolvable in the figure. Analogous plots are shown in Fig. 2 for $\rho = 0.5$. The approximations (4.2) and (4.3) give a good fit even in this case where the approximation $\rho \ll 1$ might be expected to be a poor one.

4.2. Approximation to $W(z)$

Proceeding as in the derivation of (4.2) and (4.3), one may first find the approximation to $W(z)$ for $\gamma_z \gg 1$. In this limit, only the first term ($\propto\gamma_z^2$) and the terms of $s = n$ in the double sum in (3.14) are important. Ignoring all terms with $s \neq n$, which are of higher order in the $1/\gamma_z$ expansion, the double sum can be approximated by the sum of an infinite series given by

$$-2\sum_{n=1}^{\infty} c_n \mathrm{E}_{2n-1}(\rho) = -K_2(\rho) - K_0(\rho) + 2\rho^{-1}(1+\rho^{-1})e^{-\rho}. \tag{4.8}$$

The expression (3.14) at $\gamma_z \gg 1$ is then reduced to

$$W(z) \approx -\frac{1}{K_1(\rho)}\left[\gamma_z^2\left(e^{-\rho} + \frac{1}{2}\rho\gamma_z z^2\Psi_-\right) - K_2(\rho) + \frac{2}{\rho}\left(1+\frac{1}{\rho}\right)e^{-\rho}\right]. \tag{4.9}$$

Similar to (4.2) and (4.3), this approximation can be extended to include $\gamma_z > 1$ in the relativistic limit $\rho \ll 1$, leading to the following approximate form:

$$W(z) \approx -\rho\gamma_z^2\{1 - \tfrac{1}{2}\rho\gamma_z z^2[e^{-\rho\gamma_z}\mathrm{Ei}(\rho\gamma_z) - e^{\rho\gamma_z}\mathrm{Ei}(-\rho\gamma_z)]\}. \tag{4.10}$$

For $z \to 1$, (4.10) gives $W(1) \approx 2/\rho$, which reproduces the known limit $W(1) = 2\langle\gamma\beta^2\rangle \approx 2/\rho$.

For $z > 1$, one replaces γ_z in (4.9) with $\gamma_z = -i\gamma_*$, where $\gamma_* = 1/(z^2-1)^{1/2}$. In the relativistic limit, the approximate expression (4.9) also extends to include $\gamma_* \ll 1$, which can be written as

$$W(z) \approx \rho\gamma_*^2\{e^{-\rho} - \tfrac{1}{2}i\rho\gamma_* z^2[e^{-i\rho\gamma_*}\mathrm{Ei}(\rho(i\gamma_*+1)) - e^{i\rho\gamma_*}\mathrm{Ei}(-\rho(i\gamma_*-1))]\}. \tag{4.11}$$

Note that (4.11) has a similar form to (4.10). However, the terms of the next order in ρ are retained here so that (4.11) can be regular even at $\gamma_* \to 0$. The exact expression for the large z limit, $z^2W(\infty) = \langle\gamma^{-3}\rangle = K_0(\rho)/K_1(\rho)$, can be obtained from (2.4). Since (4.11) can reproduce this limit, i.e. $z^2W(z) \approx \rho$, the validity of such an approximation including a large z (corresponding to a small γ_*) is justified.

The approximations (4.10) and (4.11) are compared with exact results for $W(z)$ for $\rho = 0.1$ in Fig. 3 and for $\rho = 0.5$ in Fig. 4. As for $R(z)$ and $S(z)$, the approximation is excellent for $\rho \ll 1$, reproducing the peak quite accurately, and the approximation is still relatively accurate for $\rho = 0.5$. As already noted, $W(z)$ is related to $R(z)$ through (2.6) or (2.9), and the foregoing approximation to $W(z)$ is not independent of that to $R(z)$.

One may expand (4.10) and (4.11) in power series. For $\rho|\gamma_z| \ll 1$, which includes both $|z| \ll (1-\rho^2)^{1/2}$ and $|z| \gg (1+\rho^2)^{1/2}$, one has

$$W(z) \approx \begin{cases} -\rho\gamma_z^2\{1 - \rho^2\gamma_z^2 z^2[C + \ln(\rho\gamma_z)]\}, & |z| \ll (1-\rho^2)^{1/2}, \\ \rho\gamma_*^2\{1 + \rho^2\gamma_*^2 z^2[C + \ln\rho]\}, & |z| \gg (1+\rho^2)^{1/2}. \end{cases} \tag{4.12}$$

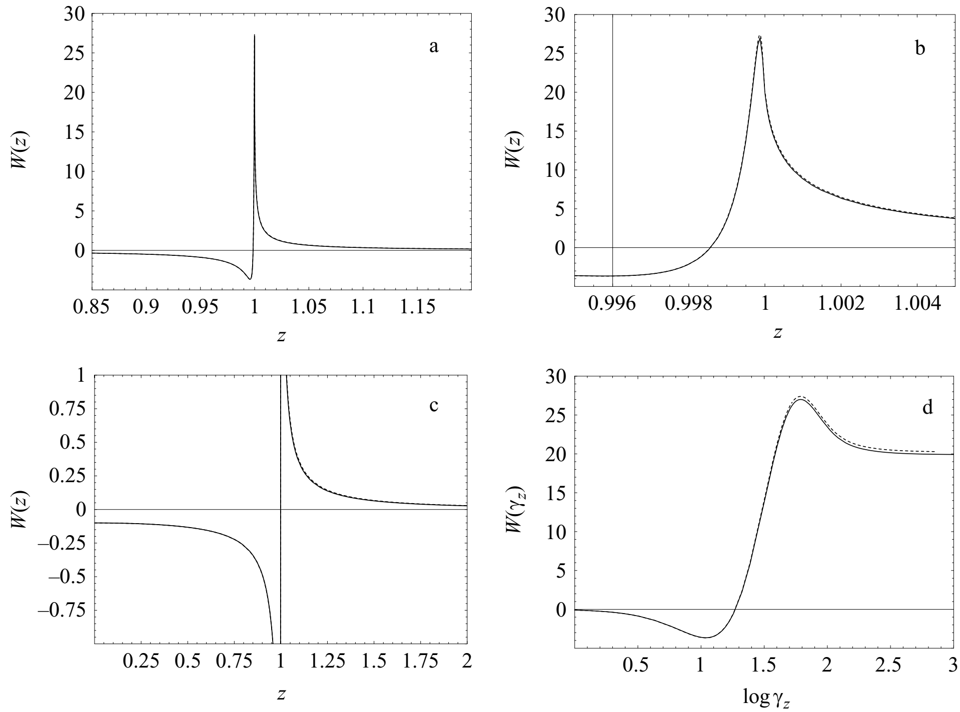

Figure 3. Plots of $W(z)$ for $\rho = 0.1$. The solid curves are for the approximations (4.10) and (4.11) and dotted curves, which mostly cannot be distinguished from the solid curves, are for the exact function: (a)–(c) correspond to the same plot with different plot ranges; (d) is the plot of $W(\gamma_z)$ with $|z| \leq 1$. The relativistic approximation gives a good fit to $W(z)$ obtained by the numerical integration.

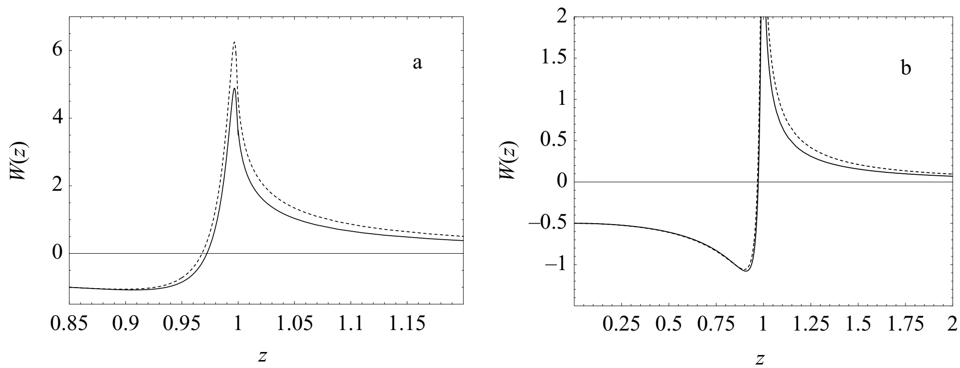

Figure 4. Plots of $W(z)$ as in Fig. 3 but with $\rho = 0.5$. Except for the region near the peaks, the accuracy of the highly relativistic approximation even in this mildly relativistic regime is reasonable.

For $\rho|\gamma_z| \gg 1$, an expansion of (4.10) and (4.11) around $1/\rho\gamma_z$ leads to the approximation

$$W(z) \approx \frac{2z^2}{\rho}\left[1 + \frac{12}{(\rho\gamma_z)^2}\right] - \rho. \tag{4.13}$$

The approximations (4.12) and (4.13) are related to (4.4) and (4.6), respectively; one may obtain them by substituting the latter two into (2.6) with (2.9).

5. Summary and conclusions

The main result of this paper is that useful approximations to RPDFs may be obtained in a two step process. First, the RPDFs are expressed in terms of the exponential integral function, which involves no approximation, and then the ultrarelativistic approximation is made, such that the Lorentz factor, γ, is assumed large compared with unity, and the lower limit, $\gamma = 1$, of integration over it is extended to $\gamma = 0$. The exponential integral function $\mathrm{Ei}(x)$ is defined for $x > 0$ by the principal value integral (3.7), and it is convenient to extend the definition to the whole upper complex plane with $\arg(x) < \pi$ by identifying $\mathrm{Ei}(x) = -\mathrm{E}_1(-x)$, where $\mathrm{E}_1(x)$ is another standard exponent integral function. The resulting approximation is very accurate for highly relativistic temperatures, $\rho = mc^2/T \ll 1$, and remains relatively accurate even in the mildly relativistic case $\rho = 0.5$.

The three functions $W(z)$, $R(z)$, $S(z)$ are related to $T(z, \rho)$, $T'(z, \rho)$ through (2.8), and $W(z)$, $R(z)$ are related to each other. The latter relation is independent of any particular distribution (cf. (2.6)). It is convenient to express $W(z)$ in terms of $R(z)$ and its derivative.

The approximate forms for the RPDFs developed here are relevant for the study of wave properties in pulsar plasma, which is thought to be intrinsically relativistic. The numerical study of electron/positron cascades above the pulsar polar cap appears to suggest that the pulsar plasma can be adequately approximated by a Jüttner distribution with a relativistic temperature $1/\rho \sim 10$ (e.g. Zhang and Harding (2000); Hibschman and Arons (2001); Arendt and Eilek (2002)). In this temperature regime, the approximations (4.2), (4.3), (4.10) and (4.11) give an excellent fit to the result by the exact numerical integration and are useful in both analytic and numerical calculations of dispersion in the pulsar plasma.

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