A polarized maser is assumed to operate in an anisotropic medium with natural modes polarized differently to the maser. It is shown that when the spatial growth rate and the generalized Faraday rotation rate are comparable, the polarization of the growing radiation is different from those of the maser and of the medium. In particular, for a linearly polarized maser operating in a medium with linearly polarization natural modes, the growing radiation is partially circularly polarized. This provides a previously unrecognized source of circular polarization that may be relevant to pulsar radio emission.

PACS numbers:
Transfer equation in the weak anisotropy approximation

For radiation propagating along the z-direction, the transfer equation is

\[
dS_A/dz = \alpha_A + (-\mu_{AB} + r_{AB})S_B,
\]

where \(S_A\) with \(A = I, Q, U, V\) is the Stokes vector, \(\alpha_A\) are emission coefficients, \(\mu_{AB}\) are absorption coefficients, and \(r_{AB}\) are generalized Faraday rotation rates. These have the forms

\[
S_A = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}, \quad \alpha_A = \begin{pmatrix} \alpha_I \\ \alpha_Q \\ \alpha_U \\ \alpha_V \end{pmatrix}, \quad \mu_{AB} = \begin{pmatrix} \mu_I & \mu_Q & \mu_U & \mu_V \\ \mu_Q & \mu_I & 0 & 0 \\ \mu_U & 0 & \mu_I & 0 \\ \mu_V & 0 & 0 & \mu_I \end{pmatrix}, \quad r_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r_V & r_U \\ 0 & r_V & 0 & -r_Q \\ 0 & -r_U & r_Q & 0 \end{pmatrix}.
\]

An alternative way of writing (1) that is sometimes convenient is in terms of unpolarized components and 3-vectors describing the polarized components. Writing \(S = (Q, U, V)\), \(\alpha = (\alpha_Q, \alpha_U, \alpha_V)\), \(r = (r_Q, r_U, r_V)\), \(\mu = (\mu_Q, \mu_U, \mu_V)\), (1) with (2) becomes

\[
dI/dz = \alpha_I - \mu_I I - \mu \cdot S, \quad dS/dz = \alpha - \mu_I S - \mu I + r \times S.
\]

There are two general ways of solving (1). The first method involves regarding (1) as a matrix equation, and solving it by finding the eigenvalues and eigenfunctions. Integration of the eigenvalue equations is elementary. This method is used here. The other method involves integrating the matrix equation (1) directly, cf. Appendix A.

Note that the theory is invariant under cyclic permutations of \(Q, U, V\). Thus the most general case may be treated by considering the special case in which only \(\mu_V, r_Q, r_V\) are nonzero: any other particular case then follows by appropriate relabeling.

Completely polarized maser

Two simplifications are made for convenience before solving (2): the terms \(\alpha_A\) and \(\mu_I\) are omitted. The neglect of \(\alpha_A\) corresponds to neglecting spontaneous emission. Physically, this is justified if amplified background noise is more important in the output of the maser than amplified spontaneous emission. It is straightforward to include \(\alpha_A \neq 0\), and to treat amplified spontaneous emission, and the conclusions of this paper are unaffected by including amplified spontaneous emission. The neglect of \(\mu_I\) involves no actual loss of generality provided that one re-interprets \(S_A\) in (1) as \(S'_A = \exp(\mu_I z)S_A\) in a modified (1) and (2) with \(\mu_I\) omitted in the matrix \(\mu_{AB}\). It follows that the polarization of the radiation, which is determined by the ratios of the Stokes parameters, is unaffected by \(\mu_I\), and that one is justified in neglecting \(\mu_I\) when considering the polarization.

It is helpful to introduce the concept of a completely polarized maser, specifically one with \(\mu_I = 0\). This concept is useful for isolating the polarization characteristics of the growing radiation. Although there is no physical reason why such an idealized maser should not exist, the conditions under which it would occur are somewhat contrived. The assumption \(\mu_I = 0\) has no effect on the results of this paper relating to the relative amplitudes of the Stokes parameters in the growing radiation.

General and special cases

In the general case, the polarization of the maser and the polarization of the medium are different. It is helpful to identify the polarizations on the Poincaré sphere. A specific polarization is represented by a point on the sphere, with circular polarizations represented by the poles and linear polarizations by points around the equator. Orthogonal polarizations are represented by points on the opposite side of a diagonal that passes through the center of the sphere. A completely polarized maser defines one diagonal through the sphere, and polarized growth alone causes radiation with one of the two orthogonal polarizations to grow and the other to damp. The polarization of the natural modes defines another diagonal through the sphere. Generalized Faraday rotation alone is described by the term \(dS/dz = r \times S\) in (3), and this corresponds to the polarization point rotating about the diagonal at a constant latitude relative to it. The diagonals representing the polarizations of the maser and of the medium correspond to the directions \(-\mu\) and \(r\), respectively, through the center of the Poincaré sphere. In the general case these are neither parallel nor perpendicular to each other. The interplay between polarized growth and generalized Faraday rotation...
depends only on the ratio of the lengths, $\mu$, $r$, that define the growth rate to the Faraday rotation rate, and on the angle between $\mu$ and $r$.

To treat the general case, it suffices to consider the special case $\mu V \neq 0$, $\mu Q = \mu U = 0$, $r_V, r_Q \neq 0$, and $r_U = 0$, and to appeal to the symmetry of the theory. In the 3-vector formalism used in (3) this choice corresponds to orienting the axes on the Poincaré sphere such that the $z$ axis is along $\mu$ and $r$ is in the $x$-$z$ plane. The general case is obtained by a rotation of the Poincaré sphere. On writing

$$Q/I = \cos(2\chi) \cos(2\psi), \quad U/I = \cos(2\chi) \sin(2\psi), \quad V/I = \sin(2\chi),$$

with analogous angles introduced for $\alpha$, $\mu$, $r$, a rotation on the Poincaré sphere is equivalent to a conventional rotation from an initial pair of polar and azimuthal angles $\chi, \psi$ to a new pair, $\chi', \psi'$ say. The relation (4) for the primed variables then determines $Q', U', V'$ in terms of $Q, U, V$, and similarly for the angles corresponding to $\alpha$, $\mu$, $r$.

**Characteristic equation**

The transfer equation (1) may be solved by finding the eigenvalues and eigenfunctions of the matrix $s_{AB} = -\mu_{AB} + r_{AB}$, where now we assume $\mu_I = 0$ and ignore spontaneous emission. Let an eigenvalue be denoted by $\lambda$. The characteristic equation is

$$\Lambda(\lambda) = \det (-\mu_{AB} + r_{AB} - \lambda \delta_{AB}) = \lambda^4 - \lambda^2 \lambda_0^2 - \mu^2 r^2 g^2 = 0,$$

where the two invariants in the problem are

$$\lambda_0^2 = \mu^2 - r^2, \quad g = \mu \cdot r / \mu r.$$

For $g \neq 0$, (5) has two real and two imaginary solutions for $\lambda$, and for $g = 0$, (5) has a double solution $\lambda = 0$, and either two real (for $\lambda_0^2 > 0$) or two imaginary (for $\lambda_0^2 < 0$) solutions.

The eigenvalues are found by constructing the matrix of cofactors, $\Lambda_{AB}(\lambda)$, so that one has

$$\Lambda_{AB}(\lambda)(-\mu_{BC} + r_{BC} - \lambda \delta_{BC}) = \Lambda(\lambda) \delta_{AC}.$$  

Pre-multiplying (1) by the matrix of cofactors then gives

$$\frac{d}{dz}[\Lambda_{AB}(\lambda) S_B] = \Lambda(\lambda) S_A + \Lambda \Lambda_{AB}(\lambda) S_B.$$  

When the condition (5) is satisfied, $\Lambda_{AB}(\lambda) S_B$ is an eigenfunction.

Let $\lambda_i, i = 1–4$ be the four eigenvalues. Then the four eigenfunctions are given by $\Lambda_{AB}(\lambda_i) S_B$ for any $A$, or for an arbitrary linear combination of these functions. The eigenfunctions constructed for different $A$ are proportional to each other, in the sense that the ratios of the coefficients of $I, Q, U, V$ are the same for different $A$ for any given $\lambda_i$.

**Eigenvalues and eigenfunctions**

In the general case, the two solutions of the eigenvalue equation (5) for $\lambda^2$ are

$$\lambda^2 = \lambda_0^2, \quad \lambda_{\pm} = \frac{1}{2}(\mu^2 - r^2) \pm \frac{1}{2}(|\mu^2 - r^2| + 4(\mu \cdot r)^2)^{1/2}.$$  

With $\lambda_0^2 > 0$ and $\lambda_{\pm}^2 < 0$ both real, the four eigenvalues may be written

$$\lambda_1 = \lambda_+, \quad \lambda_2 = -\lambda_+, \quad \lambda_3 = i|\lambda_-|, \quad \lambda_4 = -i|\lambda_-|.$$  

Parallel polarizations ($\mu \cdot r = \mu r$) correspond to $\lambda_+ \to \mu$, $|\lambda_-| \to r$, and perpendicular polarizations ($\mu \cdot r = 0$) corresponds to $\lambda_+ \to (\mu^2 - r^2)^{1/2}$, $|\lambda_-| \to 0$ for $\mu^2 > r^2$ and to $\lambda_+ \to 0$, $|\lambda_-| \to (r^2 - \mu^2)^{1/2}$ for $r^2 < \mu^2$.

The eigenfunctions follow from (8), and may be identified by choosing any $A$ in $\Lambda_{AB}(\lambda_i) S_B$. Choosing $A = I$, the (unnormalized) eigenfunctions are denoted $S_i$ with $i = 1–4$. It is convenient to introduce the unit vectors $\hat{r} = r/r$, $\hat{\mu} = \mu/\mu$, and to choose as a set of three orthonormal vectors $\hat{r}, \hat{\mu} \times \hat{r}, \hat{r} \times (\hat{\mu} \times \hat{r})$, and to write

$$S_r = \hat{r} \cdot S, \quad S_\perp = \hat{\mu} \times \hat{r} \cdot S, \quad S_\mu = \hat{r} \times (\hat{\mu} \times \hat{r}) \cdot S.$$
In terms of these quantities, the four eigenfunctions become

\[ S_i = -\lambda_i(\lambda_i^2 + r^2)I + \mu g(\lambda_i^2 + r^2)S_r + \lambda_i\mu r S_\perp + \mu \lambda_i^2 S_\mu, \]

(12)

with \( g = \mu \cdot r / \mu r \). The transfer equation reduces to \( dS_i / dz = \lambda_i S_i \), and the solution is

\[ S_i(z) = e^{\lambda_i z} S_i(0). \]

(13)

These four equations, (13) with \( i = 1-4 \), may be rewritten in the form \( S_A(z) = M_{AB}(z)S_B(0) \) to identify the Mueller matrix, \( M_{AB}(z) \), but we do not do so explicitly here. However, note that the solution corresponds to two linear combinations of Stokes parameters varying as hyperbolic functions of \( \lambda_+ z \), and two linear combinations varying as trigonometric functions of \( |\lambda_-| z \); each Stokes parameter at \( z \) is related to the Stokes parameters at \( z = 0 \) through a combination of these hyperbolic and trigonometric functions.

Mueller matrix

A formal solution of (1), when spontaneous emission is ignored, is

\[ S_A(z) = M_{AB}(z)S_B(0), \quad M_{AB}(z) = \exp[(-\mu_{AB} + r_{AB})z], \]

where \( M_{AB}(z) \) is the Mueller matrix. Explicit evaluation of the Mueller matrix directly is discussed in Appendix A. The evaluation is straightforward in the special cases of parallel and perpendicular polarizations, but not in the general case. The Mueller matrix in the general case may be constructed in by solving equations (13) for the Stokes parameters. This is carried out in Appendix B.

3. GROWTH OF POLARIZED RADIATION

The general solution of (1) is given by (13). It is helpful to consider two special cases before discussing the general case. The special cases are where the polarizations of the maser and the medium are either parallel \((g = 1)\) or perpendicular \((g = 0)\) on the Poincaré sphere.

Parallel polarization

Parallel polarizations define a single direction, \( \hat{r} = \hat{\mu} \), on the Poincaré sphere. The eigenvalues (5) are \( \lambda = \pm \mu \) and \( \lambda = \pm i r \), and the eigenfunctions are \( I \pm S_r \) and the two components of \( \hat{r} \times \mathbf{S} \). The solutions for the components that grow and damp are

\[ I(z) = I(0) \cosh(\mu z) + S_r(0) \sinh(\mu z), \quad \hat{r} \cdot \mathbf{S}(z) = I(0) \sinh(\mu z) + S_r(0) \cosh(\mu z). \]

(15)

In particular, radiation polarized in the same sense as the maser grows exponentially, and radiation polarized in the opposite sense damps exponentially. Generalized Faraday rotation of the other two components may be described by

\[ \hat{r} \times \mathbf{S}(z) = \hat{r} \times \mathbf{S}(0) \cos(r z) + \hat{r} \times [\hat{r} \times \mathbf{S}(0)] \sin(r z). \]

(16)

For the particular case of circularly polarized maser and modes, (16) implies \( Q(z) = Q(0) \cos(r_V z) - U(0) \sin(r_V z), \) \( U(z) = U(0) \cos(r_V z) + Q(0) \sin(r_V z) \), which corresponds to Faraday rotation of the plane of linear polarization, \( \psi(z) = \psi(0) + \frac{1}{2} r_V z \), cf. (4).

Perpendicular polarizations

The case case where the polarization of the maser is perpendicular (on the Poincaré sphere) to the polarization of the natural modes corresponds to \( \mu \cdot r = 0 \). Then two of the eigenvalues (9) are null \((\lambda = 0)\), and the other two are \( \lambda = \pm \lambda_0, \lambda_0^2 = \mu^2 - r^2 \). These two eigenvalues are real for \( \mu^2 > r^2 \) and imaginary for \( \mu^2 < r^2 \). The eigenfunctions
with null eigenvalues are \( S_r \), and \( rI - \mu S_\perp \), and the other two eigenfunctions are given by (12) with \( g = 0 \). Solving in the case \( \lambda_\perp^2 > 0 \) for given initial conditions \( z = 0 \) gives \( S_r(z) = S_r(0) \) and

\[
\begin{pmatrix}
I(z) \\
S_r(z) \\
S_\perp(z)
\end{pmatrix} = \begin{pmatrix}
I(0) \\
S_r(0) \\
S_\perp(0)
\end{pmatrix} + \begin{pmatrix}
-\mu S_r(0) \\
-\mu S_r(0) + r S_\perp(0) \\
\mu I(0) - r S_\perp(0)
\end{pmatrix} \frac{\sinh(\lambda_0 z)}{\lambda_0} + \begin{pmatrix}
\mu^2 I(0) - \mu r S_\perp(0) \\
(\mu^2 - r^2) S_r(0) \\
\mu I(0) - r^2 S_\perp(0)
\end{pmatrix} \frac{\cosh(\lambda_0 z) - 1}{\lambda_0^2}. \tag{17}
\]

For \( \lambda_\perp^2 < 0 \) the hyperbolic functions are replaced by trigonometric functions in (17).

Two interesting new features appear for perpendicular polarizations. First, exponential growth occurs only for \( \mu^2 - r^2 \) or \( r^2 \) the system is periodic, but unlike generalized Faraday rotation, the oscillations involve the intensity, \( I \). For \( \mu^2 \to r^2 \) the oscillation rate goes to zero, and the amplitude of the oscillations becomes arbitrarily large. When the two rates are equal, \( \mu^2 = r^2 \), power-law growth occurs, cf. equation (A8) in Appendix A. Second, the polarization of the growing radiation has a component \( S_\perp \) that corresponds to the polarization of neither the maser nor the natural modes. Retaining only the exponentially growing terms, (17) implies that after many growth lengths, the ratio of the Stokes parameters approaches

\[
I : S_r : S_\perp : S_\mu = 1 : 0 : r/\mu : -\sqrt{\mu^2 - r^2}/\mu. \tag{18}
\]

(Note that \( S_r(z) = S_r(0) \) remains constant at its initial value, and hence its ratio to \( I \) tends to zero after many growth lengths.) As the ratio \( \mu^2/r^2 \) decreases from infinity to unity, the polarization changes from 100% in the sense of the maser \( (S_\mu = -I) \) toward the polarization orthogonal (on the Poincaré sphere) to both the maser and the natural modes \( (S_\perp \to I) \).

An interesting case is when the two polarizations (of the maser and of the medium) are both linear and are perpendicular, in the sense defined here. In terms of polarization vectors, for radiation propagating along the 3-direction, the axes can be chosen such that the two natural modes are polarized along the 1- and 2-axes, respectively, and then the hypothesis that the maser has perpendicular polarization (on the Poincaré sphere) corresponds to its polarization vector being at 45° to either of these. In terms of Stokes parameters, polarization in the sense of the two modes of the medium correspond \( Q = \pm I \), respectively, and radiation polarized purely in the sense of the maser corresponds to \( U = I \). In the foregoing analysis one has \( S_r \to -Q \), \( S_\mu \to U \) and \( S_\perp \to V \) in this case. The results imply that for \( \mu^2 > r^2 \) the maser produces radiation that is partially circularly polarized, with degree of linear polarization \( U/I = \sqrt{\mu^2 - r^2}/|\mu| \), and degree of circular polarization \( V/I = r/|\mu| \). This source of circular polarization in maser sources does not appear to have been recognized previously.

**Asymptotic polarization**

The analytic results in the general case are cumbersome; they are written down in a concise notation in Appendix B. Of particular interest is the polarization after many growth lengths. The general counterpart of (18) is derived in Appendix B, cf. (B4) by retaining only the terms that vary as \( \exp(\lambda_\perp z) \) in the Mueller matrix (B3). This gives

\[
I : S_r : S_\perp : S_\mu = 1 : \lambda_\perp \frac{\lambda_\perp^2 + r^2}{\mu r} - \frac{\lambda_\perp (\lambda_\perp^2 + r^2)}{\mu^2 r^2}. \tag{19}
\]

The result (18) is reproduced by (19) for \( g = 0 \) with \( \lambda_\perp = -g \mu^2 r^2/(\mu^2 - r^2) \), \( \lambda_\perp \to \mu^2 - r^2 \).

An alternative derivation of (19) is instructive. Suppose that one averages over the oscillations associated with generalized Faraday rotation, for example, assuming that the source extends over a range \( \Delta z \gg 1/\lambda_\perp \) of \( z \). Then the real and imaginary parts of the corresponding eigenfunctions are zero. After many growth times, the damping eigenfunction may also be set to zero. The resulting three relations between Stokes parameters implies (19) for the remaining (growing) eigenfunction. Thus (19) applies to the growing eigenfunction in a random phase approximation, where the random phase is the generalized Faraday angle, which is \( \lambda_\perp \) here. The terms which oscillate, cf. equation (B3) in Appendix B, do so with an amplitude that is determined by the initial conditions, so that these oscillations occur with a fixed amplitude on an exponentially growing solution. After many growth lengths the amplitude of the oscillations becomes negligible in comparison with the amplifying component.

**Illustrative examples**

For strong growth, \( \mu^2 \gg r^2 \), one has \( \lambda_\perp^2 = \mu^2 - r^2(1 - g^2) \), \( \lambda_\perp^2 = -r^2 g^2 \), and (19) gives \( I : S_r : S_\perp : S_\mu = 1 : -g : (r/\mu)(1 - g^2) : -(1 - g^2) \). This corresponds to polarization along the direction \( \mu \) characteristic of the maser, with
FIG. 1: The Stokes parameters are plotted in two cases where the growth is strong compared with the rate of generalized Faraday rotation. Left: nearly perpendicular case, $\mu_V : r_Q : r_V = -1 : 0.1 : 0.01$. Right: nearly parallel case, $\mu_V : r_Q : r_V = -1 : 0.01 : 1$.

FIG. 2: As for Figure 1, except for weak growth. Left: nearly perpendicular case, $\mu_V : r_Q : r_V = -0.1 : 1 : 0.01$. Right: nearly parallel case, $\mu_V : r_Q : r_V = -0.1 : 0.01 : 1$.

... an admixture $r/\mu$ of the polarization orthogonal to both the maser and the medium. For weak growth, $\mu^2 \ll r^2$, one has $\lambda^2 = \mu^2 g^2$, $\lambda^2 = -r^2 + \mu^2 (1 - g^2)$, and (19) implies $I : S_r : S_\perp : S_\mu = 1 : -(g/|g|)[1 - (\mu^2/r^2)(1 - g^2)] : (\mu/r)(1 - g^2) : -(\mu^2/r^2)|g|(1 - g^2)$, which corresponds to polarization along the direction of the faster growing of the natural modes, with an admixture $(\mu/r)(1 - g^2)$ of the polarization orthogonal to both the maser and the medium. In the intermediate case where the two rates are equal, $\mu^2 = r^2$, (19) gives $1 : -g^2 : 1 - g^2 : -g(1 - g^2)$. This reproduces special cases of parallel ($g = 1$) and perpendicular ($g = 0$) polarizations discussed above. In the general case ($0 < g^2 < 1$) all of $S_r, S_\perp, S_\mu$, and hence all of $Q, U, V$ are nonzero in the growing radiation.

In Figure 1 we show two examples of strong growth, and in Figure 2 two examples of weak growth. In both cases the maser is assumed to be circularly polarized and the natural modes are assumed to be linearly polarized. (For illustrative purposes, the relative signs are chosen such that saturation occurs at or near 1, rather than $-1$, in all cases.) However, the plots are generic in the sense that with appropriate relabeling of the polarizations, they apply to any case that has the same value for the ratio $\mu/r$ and $g$. In both figures the plot on the left is for nearly perpendicular polarizations, $g = -0.1$, and the plot on the right is for nearly parallel polarizations, $g = -0.955$. Note that the only case where the dominant polarization is nearly that of the maser is for strong growth with nearly perpendicular polarizations. In the other cases illustrated, the dominant polarization is similar to that of the natural modes of the medium.

A general conclusion is that a maser operating in a medium with natural modes polarized differently to the maser leads to amplifying radiation with polarization that is a mixture of all three Stokes parameters, $Q, U, V$. The particular ratio is determined by (19) after a sufficiently large number of growth lengths.
that can be significantly circularly polarized for radiation has a component that is orthogonal (on the Poincaré sphere) to both. In particular a linearly polarized maser in a medium with linearly polarized modes that are different from those of the maser leads to growing radiation of interstellar molecular line masers [13], and to the electron cyclotron maser emission, notably in Jupiter’s S-bursts to the values chosen in these three case. Specifically, the three cases correspond to angles (modulo 45°) 9°, 23°, 42°, respectively, between the two polarization vectors. The dashed curves in Figure 3 correspond to the circular polarization in this case. As expected, the circular polarization is largest for polarizations that are nearly perpendicular on the Poincaré sphere. Ignoring the signs, for linearly polarized maser and natural modes, the ellipticity of the radiation can be measured directly [14]. We propose to discuss the suggested application of these three values of μ/r.

We show three examples that illustrate this case in Figure 3. The calculations are performed for a circularly polarized maser and linearly polarized modes, but the results are valid for any case with the values of μ/r and g equal to the values chosen in these three case. Specifically, the three cases correspond to μ/r = −1.005, −1.41, 1.005, and g = −.995, −.707, 0.100, respectively. These three values of g correspond to nearly parallel, intermediate and nearly perpendicular vectors on the Poincaré sphere. Ignoring the signs, for linearly polarized maser and natural modes, these three values of g correspond to angles (modulo 45°) 9°, 23°, 42°, respectively, between the two polarization vectors. The degree of circular polarization is determined by (19), with S⊥ corresponding to circular in this case.

FIG. 3: Left: μV = −1, rV = 1, rQ = 0.1; middle: μV = −1, rV = 0.5, rQ = 0.5; right: μV = −1, rV = 0.1, rQ = 1.

Growth-induced circular polarization

As noted above, in the case where the maser and the natural modes are linearly polarized and are perpendicular in the sense used here, the growing radiation is partially circularly polarized. In this case, ‘perpendicular’ on the Poincaré sphere implies polarization vectors at an angle of 45° to each other. When the two linear polarization vectors are at an angle different from 45°, this effect still occurs. The degree of circular polarization is determined by (19), with S⊥ corresponding to circular in this case.

We show three examples that illustrate this case in Figure 3. The calculations are performed for a circularly polarized maser and linearly polarized modes, but the results are valid for any case with the values of μ/r and g equal to the values chosen in these three case. Specifically, the three cases correspond to μ/r = −1.005, −1.41, 1.005, and g = −.995, −.707, 0.100, respectively. These three values of g correspond to nearly parallel, intermediate and nearly perpendicular vectors on the Poincaré sphere. Ignoring the signs, for linearly polarized maser and natural modes, these three values of g correspond to angles (modulo 45°) 9°, 23°, 42°, respectively, between the two polarization vectors. The dashed curves in Figure 3 correspond to the circular polarization in this case. As expected, the circular polarization is largest for polarizations that are nearly perpendicular on the Poincaré sphere (polarization vectors at 45°). The degree of circular polarization after many growth lengths approaches the value determined by (19), specifically, it approaches −λ2 = r2/μr. The sense of circular polarization is determined by the angle between the linear polarizations of the maser and of the medium. This is implicit in our description through the sign of S⊥ which is determined by the direction of ˆμ × ˆr, cf. (12); this direction, and hence the sense of circular polarization of the growing radiation, changes sign when the angle between the vectors passes through zero (parallel case).

4. DISCUSSION

The main qualitative result from this investigation is that when a completely polarized maser operates in a medium whose natural modes have a different polarization to the maser, the polarization of the amplifying radiation is different from both those of the maser and of the medium, and includes a component orthogonal (on the Poincaré sphere) to both polarizations. This result implies polarization vectors at an angle of 45°, this effect still occurs. The degree of circular polarization is determined by (19), with S⊥ corresponding to circular in this case.

An interesting new result is that when the growth rate and the Faraday rotation rate are comparable, the growing radiation has a component that is orthogonal (on the Poincaré sphere) to both. In particular a linearly polarized maser in a medium with linearly polarized modes that are different from those of the maser leads to growing radiation that can be significantly circularly polarized for |μ/r| ~ 1. This is of interest as a possible explanation for circular polarization observed in pulsar radio emission. The results of this paper may also be relevant to the polarization of interstellar molecular line masers [13], and to the electron cyclotron maser emission, notably in Jupiter’s S-bursts where the ellipticity of the radiation can be measured directly [14]. We propose to discuss the suggested application
to pulsars in detail elsewhere.

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APPENDIX A: THE MUELLER MATRIX

The Mueller matrix in the present context is defined by (14). In this Appendix direct evaluation of the matrix is discussed in the general case, and carried out explicitly in the cases of parallel and perpendicular polarizations.

For \( M_{AB}(z) = \exp[s_{AB}z] = \delta_{AB} + s_{AB}z + \frac{s_{AB}^2 z^2}{2!} + \frac{s_{AB}^3 z^3}{3!} + \frac{s_{AB}^4 z^4}{4!} + \cdots \) (A1)

with \( s_{AB} = \mu_I \delta_{AB} - \mu_{AB} + r_{AB} \). The square of \( s_{AB} \) is an independent matrix, as is its cube, which may be written

\[
s_{AB}^2 = \lambda_0^2 s_{AB} + \mu r g t_{AB},
\]

where the invariants \( \lambda_0, g \) are defined by (6), and where \( t_{AB} \) is the dual of \( s_{AB} \). The fourth power of \( s_{AB} \) follows from the characteristic equation (5) and the fact (the Cayley-Hamilton theorem) that a matrix satisfies its characteristic equation, which imply

\[
s_{AB}^4 - \lambda_0^2 s_{AB}^2 - \mu^2 r^2 g^2 \delta_{AB} = 0.
\]

The result (A3) also follows by (matrix) multiplying (A2) by \( s_{AB} \) and using

\[
s_{AB} t_{BC} = \mu r g \delta_{AC}.
\]

In this way, using (A2) and (A4) or (A3), all powers of \( s_{AB} \) higher than the second may be re-expressed in terms of the three matrices \( s_{AB}, s_{AB}^2, t_{AB} \). However, it is not straightforward to sum the series (A1) except in special cases including the parallel and perpendicular cases.

For parallel polarizations \( M_{AB}(z) \) factorizes:

\[
M_{AB}(z) = m_{AC}(z) R_{BC}(z), \quad m_{AB}(z) = \exp[-\mu AB z], \quad R_{AB}(z) = \exp[r_{AB} z], \quad (A5)
\]

Explicit evaluation give

\[
m_{AB}(z) = \delta_{AB} - \mu^{-1} \mu_{AB} \sinh(\mu z) + \mu^{-2} \mu_{AB}^2 [\cosh(\mu z) - 1], \quad R_{AB}(z) = \delta_{AB} + r^{-1} r_{AB} \sin(r z) + r^{-2} r_{AB}^2 [\cos(r z) - 1].
\]

For perpendicular polarizations, one has \( s_{AB}^n = \lambda_0^n s_{AB}^{n-2} \) for \( n \geq 3 \) and the sum of the infinite series gives

\[
M_{AB}(z) = \delta_{AB} + s_{AB} \frac{\sinh(\lambda_0 z)}{\lambda_0} + s_{AB}^2 \frac{\cosh(\lambda_0 z) - 1}{\lambda_0^2},
\]

which applies for \( \lambda_0^2 > 0 \). For \( \lambda_0^2 = -\lambda_0^2 < 0 \) the hyperbolic functions are replaced by the corresponding trigonometric functions.

In the case \( \lambda_0^2 = 0 \), where the growth rate and the rate of generalized Faraday rotation are equal, (A7) reduces to

\[
M_{AB}(z) = \delta_{AB} + s_{AB} z + \frac{1}{2} s_{AB}^2 z^2,
\]

and \( s_{AB}, s_{AB}^2 \) simplify due to \( \mu V = r_Q \). This corresponds to power-law rather than exponential growth or damping.

APPENDIX B: GENERAL EXPRESSION FOR THE MUELLER MATRIX

The solutions (13) for the four eigenfunctions (12) may be combined into terms that evolve as \( C = \cosh(\lambda_0 z) \), \( S = \sinh(\lambda_0 z) \), \( c = \cos(\lambda_0 z) \), \( s = \sin(\lambda_0 z) \). These combinations are written as a square matrix, \( L \), times the Stokes vector in the form \( I, S_r, S_\perp, S_\mu \). Then the right hand side of (13) may be written as another square matrix,

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$T(z)$ (involving only $C$, $S$, $c$, $s$) times $L$ times the initial Stokes vector $I(0), S_r(0), S_{\perp}(0), S_\mu(0)$. These square matrices are

$$L = \begin{pmatrix} L_{11} & 0 & L_{13} & 0 \\ 0 & L_{22} & 0 & L_{24} \\ L_{31} & 0 & L_{33} & 0 \\ 0 & L_{42} & 0 & L_{44} \end{pmatrix}, \quad T(z) = \begin{pmatrix} C & 0 & 0 \\ S & C & 0 \\ 0 & 0 & c \\ 0 & 0 & -s \end{pmatrix}$$

(B1)

with $L_{11} = -\lambda_+ (\lambda^2_+ + r^2)$, $L_{13} = \lambda_+ \mu r$, $L_{22} = \mu g (\lambda^2_+ + r^2)$, $L_{24} = \mu \lambda^2_+$, $L_{31} = -|\lambda_-| (\lambda^2_+ + r^2)$, $L_{33} = |\lambda_-| \mu r$, $L_{42} = \mu g (\lambda^2_+ + r^2)$, $L_{44} = \mu \lambda^2_+$. The Mueller matrix is then $M = L^{-1} T(z) L$. With

$$L^{-1} = \frac{1}{D_1 D_2} \begin{pmatrix} D_2 L_{33} & 0 & -D_3 L_{13} & 0 \\ 0 & D_1 L_{44} & 0 & -D_1 L_{24} \\ -D_2 L_{31} & 0 & D_2 L_{11} & 0 \\ 0 & -D_1 L_{42} & 0 & D_1 L_{22} \end{pmatrix},$$

(B2)

$$D_1 = L_{11} L_{33} - L_{13} L_{31}, \quad D_2 = L_{22} L_{44} - L_{24} L_{42},$$

one has

$$M = \frac{1}{D_1 D_2} \begin{pmatrix} L_{22} [L_{11} L_{33} C - L_{13} L_{31} C] & D_2 [L_{22} L_{44} S - L_{11} L_{42} S] & D_2 [L_{11} L_{33} C - L_{13} L_{33} C] & D_2 [L_{24} L_{44} S - L_{11} L_{44} S] \\ D_1 [L_{11} L_{44} S + L_{24} L_{42} S] & D_1 [L_{22} L_{44} C - L_{24} L_{42} C] & D_1 [L_{13} L_{44} S + L_{24} L_{42} S] & D_1 [L_{24} L_{44} C - L_{23} L_{44} C] \\ D_2 [-L_{11} L_{33} C + L_{13} L_{31} C] & D_2 [-L_{22} L_{44} S + L_{11} L_{42} S] & D_2 [-L_{13} L_{33} C + L_{11} L_{33} C] & D_2 [-L_{24} L_{44} S + L_{11} L_{44} S] \\ D_1 [-L_{42} L_{44} C] & D_1 [-L_{42} L_{44} S] & D_1 [-L_{42} L_{33} C + L_{11} L_{33} C] & D_1 [-L_{42} L_{44} C] \end{pmatrix} \cdot$$

(B3)

After many growth lengths the trigonometric terms are negligible, and (B3) implies

$$I : S_r : S_{\perp} : S_\mu = 1 : \frac{D_1 L_{14}}{D_2 L_{33}} : -\frac{L_{11}}{L_{33}} : -\frac{D_1 L_{42}}{D_2 L_{33}},$$

(B4)

with the initial polarization appearing only in the combination $L_{11} I(0) + L_{22} S_r(0) + L_{13} S_{\perp}(0) + L_{24} S_\mu(0)$.