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Resonances and dispersion in relativistic plasmas

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Abstract

Relativistic plasmas are reviewed emphasizing the relation between resonance, dissipation and dispersion. The resonance condition is derived by imposing conservation of 4-momentum at a microscopic level. The response of a plasma is described in a manifestly covariant manner, and explicit forms including relativistic quantum effects are written down. Properties of some specific relativistic plasma dispersion functions are described briefly. The Einstein coefficients are used to derive kinetic equations that describe the effect of resonant interactions on the distributions of waves and particles.

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1. Introduction

Einstein's work pre-dated the development of modern plasma physics, but his contributions have played an important role in several aspects of plasma theory. Besides relativity, the contribution emphasized here concerns the Einstein coefficients. Einstein was the first to interpret the energy and momentum ascribed to a wave quantum as physical [1], and he combined this idea with statistical assumptions in deriving the Einstein coefficients for emission and absorption [2]. Although the Einstein coefficients were originally derived for emission in vacuo for a two-level atom, it is straightforward to modify them to treat emission and absorption in a plasma: the waves are allowed to have arbitrary dispersion relation and polarization, and the radiators are assumed to be free particles. Classically, emission (in the unmagnetized case) occurs only when the resonant Čerenkov condition, $\omega - \mathbf{k} \cdot \mathbf{v} = 0$, is satisfied. There is a close relation between resonance, dissipation and dispersion in a plasma, with the dissipation process corresponding to Čerenkov emission referred to as Landau damping (LD), and with

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dispersion related to dissipation through causality, e.g., in the form of the Kramers–Kronig relations. Classically, it is conventional to treat emission and absorption in a plasma in quite different ways: Čerenkov emission as a single-particle process, and LD as the dissipative part of a cooperative-medium response function. This reflects a deficiency in classical treatment of emission, which does not automatically conserve energy and momentum, and which precludes relating emission to absorption at a microscopic level. In contrast, using the approach initiated by Einstein, the resonance condition is derived from conservation of 4-momentum at a microscopic level (Section 2), and is different for emission and absorption due to the recoil terms having opposite signs. Einstein derived the explicit relation between spontaneous emission, stimulated emission and absorption using a thermodynamic argument. The major advantage of this approach emerges when deriving the kinetic equations describing the effect of wave–particle interactions (Section 5); conservation of energy and momentum on a microscopic level allows one to determine the evolution of the distributions of waves and particles in an elementary way.

In this Letter relativistic effects in plasmas are reviewed emphasizing the relation between resonance, dissipation and absorption. In Section 2, the resonance condition including relativistic quantum effects is derived for both unmagnetized and magnetized particles, and solutions for both the wave frequency and the particle energy and momentum are written down. In Section 3 a manifestly covariant formalism for the response of a plasma is introduced and general expressions, including relativistic quantum effects, are written down for the response tensors for unmagnetized and magnetized plasma; a relativistic treatment of the causal condition is also pointed out. Several aspects of dispersion in specific relativistic plasmas are reviewed briefly in Section 4. The kinetic equations, often referred to as the quasilinear equations, are derived in Section 5. The significance and relevance of the results are summarized in Section 6.

In the notation used here, Greek indices run over 0, 1, 2, 3, the metric tensor has signature -2, and the (natural) units have $\hbar = 1 = c$.

2. Resonance conditions

Einstein applied energy and momentum conservations to the emission and absorption of wave quanta. A relativistic treatment is given in this section, and the solutions of the resulting resonance condition are written down for the frequency and for the energy and momentum of both an unmagnetized and a magnetized particle.

2.1. Generalization of the Čerenkov condition

Consider a particle with initial 4-momentum $p^\mu = [\varepsilon, \mathbf{p}]$, $\varepsilon = (m^2 + |\mathbf{p}|^2)^{1/2}$, that either emits a wave quantum with 4-momentum k^μ, leaving it in the final state p'^μ, or absorbs a wave quantum with 4-momentum k^μ, leaving it in the final state p''^μ. Conservation of 4-momentum requires $p' - p + k = 0$, $p'' - p - k = 0$, with $p''^2 = p'^2 = p^2 = m^2$. On squaring to eliminate p' or p'', one obtains $ku - k^2/2m = 0$ and $ku + k^2/2m = 0$, respectively, with $u^\mu = p^\mu/m = [\gamma, \gamma\mathbf{v}]$ and $ku = k^\mu u_\mu = \gamma(\omega - \mathbf{k}\cdot\mathbf{v})$. The terms $\pm k^2/2m$ are quantum corrections (the 'recoil' terms), and when they are neglected, the classical (Čerenkov) resonance condition, $ku = 0$, is reproduced for either emission or absorption. The relativistic quantum generalization for emission and absorption may be combined into the single resonance condition

$$(ku)^2 - \left(k^2/2m\right)^2 = 0. \tag{1}$$

The solutions of (1) describe not only emission and absorption, but also the crossed processes of one-photon pair creation and annihilation. For example, solving (1) for ω gives the four solutions

$$\omega - \epsilon\varepsilon_q + \epsilon'\varepsilon_{q'} = 0, \tag{2}$$

with $\epsilon = \pm 1$, $\epsilon' = \pm 1$, and with $q \to \mathbf{p}$, $q' \to \mathbf{p}' = \mathbf{p}$ and $\varepsilon_q \to (m^2 + |\mathbf{p}|^2)^{1/2}$ and $\epsilon'\mathbf{p}' = \epsilon\mathbf{p} - \mathbf{k}$ here. For $\epsilon = \epsilon' = 1$ ($\epsilon = \epsilon' = -1$), (2) describes emission and absorption between the energy states $\varepsilon_q, \varepsilon_{q'}$ for an electron (a positron), and for $\epsilon = -\epsilon'$ (2) describes one-photon pair creation and annihilation.

There are two dissipation processes in a relativistic quantum plasma LD and pair creation (PC). Dispersion is related to dissipation, specifically through the Kramers–Kronig relations, and there is an additional source of dispersion associated with PC that is intrinsically associated with relativistic quantum effects.

2.2. Limiting resonances

The boundary of the physical regions where resonance is allowed corresponds to $ku = (ku)_\pm = \gamma(\omega \pm |\mathbf{k}||\mathbf{v}|)$. Setting $ku = (ku)_\pm$ in (1) there are four solutions for the energy, momentum or speed. (It is convenient to write $\varepsilon = m(1+t^2)/(1-t^2)$, $|\mathbf{p}| = 2mt/(1-t^2)$, $|\mathbf{v}| = 2t/(1+t^2)$, solve for t, and then construct the solutions for $\varepsilon, |\mathbf{p}|, |\mathbf{v}|$.) Two of the solutions may be written

$$\frac{\varepsilon_\pm}{m} = \frac{a \pm b(a^2+b^2-1)^{1/2}}{1-b^2}, \qquad \frac{p_\pm}{m} = \frac{ab \pm (a^2+b^2-1)^{1/2}}{1-b^2}, \qquad v_\pm = \frac{b \pm a(a^2+b^2-1)^{1/2}}{a^2+b^2},$$
$$a = \frac{\omega^2 - |\mathbf{k}|^2}{2m\omega}, \quad b = \frac{|\mathbf{k}|}{\omega}, \tag{3}$$

with the other two solutions given by $\omega \to -\omega$ ($a, b \to -a, -b$). An alternative way of writing the solutions for the energy is

$$\varepsilon_\pm = \frac{1}{2}\epsilon\omega \pm \varepsilon_k, \qquad \varepsilon'_\pm = -\frac{1}{2}\epsilon'\omega \pm \varepsilon_k, \quad \varepsilon_k = \frac{1}{2}|\mathbf{k}|\left(\frac{\omega^2 - |\mathbf{k}|^2 - 4m^2}{\omega^2 - |\mathbf{k}|^2}\right)^{1/2}, \tag{4}$$

where ϵ, ϵ' have the same meaning as in (2). (The $\pm$ labels are not the same in (3) and (4).) It follows that real solutions exist only for $k^2 = \omega^2 - |\mathbf{k}|^2 < 0$ (subluminal waves), when emission and absorption are allowed, and for $k^2 > 4m^2$, when pair creation and annihilation are allowed.

2.3. Resonances for magnetized particles

In the presence of a magnetic field, the energy eigenvalues of a particle are $\varepsilon_n(p_\parallel) = (m^2 + p_\parallel^2 + 2neB)^{1/2}$, where $n = 0, 1, 2, \ldots$ labels the Landau states. Conservation of momentum on emission of a wave quantum applies only to the parallel component, $p_\parallel \to p'_\parallel = p_\parallel - k_\parallel$, and n must change by an integer, $n \to n' = n - s$, $s = 0, \pm 1, \ldots$. All the crossed processes are included in (2) with $q \to n, p_\parallel$, and similarly for the primed variables, with $\epsilon' p'_z = \epsilon p_\parallel - k_\parallel$ implicit. (The classical limit corresponds to $\hbar \to 0$, $n \to \infty$, $\hbar n \to p_\perp^2/2eB$, when the conservation laws imply the gyroresonance condition $\omega - s\Omega_0/\gamma - k_\parallel v_\parallel = 0$, with $\Omega_0 = eB/m$.) Combining all four conditions gives a quadratic equation for $p_\parallel$, whose solutions $p_\parallel = p_{\parallel\pm}$ determine the energies $\varepsilon_{n\pm} = \varepsilon_n(p_{\parallel\pm})$:

$$\varepsilon_{n\pm} = \omega f_{nn'} \pm k_\parallel g_{nn'}, \qquad f_{nn'} = \frac{(\varepsilon_n^0)^2 - (\varepsilon_{n'}^0)^2 + \omega^2 - k_\parallel^2}{2(\omega^2 - k_\parallel^2)},$$
$$g_{nn'}^2 = \frac{[\omega^2 - k_\parallel^2 - (\varepsilon_n^0 - \varepsilon_{n'}^0)^2][\omega^2 - k_\parallel^2 - (\varepsilon_n^0 + \varepsilon_{n'}^0)^2]}{4(\omega^2 - k_\parallel^2)^2}, \tag{5}$$

with $\varepsilon_n^0 = \varepsilon_n(0) = (m^2 + 2neB)^{1/2}$. There are real solutions for

$$\omega^2 - k_\parallel^2 \leqslant \left(\varepsilon_n^0 - \varepsilon_{n'}^0\right)^2, \qquad \omega^2 - k_\parallel^2 \geqslant \left(\varepsilon_n^0 + \varepsilon_{n'}^0\right)^2, \tag{6}$$

which are the allowed regions for gyromagnetic emission and absorption, and for one-photon pair creation or annihilation, respectively.

3. Covariant response tensors for relativistic quantum plasmas

General expressions for the response tensor including relativistic quantum effects are available in a variety of forms for both unmagnetized and magnetized plasmas.

3.1. Covariant form of the response tensor

The linear response tensor, $\mathcal{P}^{\mu\nu}(k)$, relates the induced 4-current density, $J^{\mu}(k)$, to the 4-potential, $A^{\mu}(k)$:

$$J^{\mu}(k) = \mathcal{P}^{\mu}{}_{\nu}(k)A^{\nu}(k), \qquad k_{\mu}\mathcal{P}^{\mu\nu}(k) = 0 = k_{\nu}\mathcal{P}^{\mu\nu}(k), \tag{7}$$

where the constraints impose charge continuity and gauge invariance, respectively. The response tensor includes the vacuum polarization tensor. In any given frame, the response tensor is related to the equivalent dielectric 3-tensor by $K^{i}{}_{j}(k) = \delta^{i}{}_{j} + \mu_0\mathcal{P}^{i}{}_{j}(k)/\omega^2$, and the $00, i0, 0j$ components of $\mathcal{P}^{\mu}{}_{\nu}(k)$ can be constructed from the components of $K^{i}{}_{j}(k)$ using (7).

3.2. Relativistic causal condition

The response of a medium is causal: a disturbance at time $t = 0$ can induce a response only at times $t > 0$. This implies that the response functions must satisfy the Kramers–Kronig relation. The special theory of relativity implies a stronger causal condition: an event can depend only on other events in its past light cone. This requires [3,4]

$$\mathcal{P}^{\mu\nu}(\omega, \mathbf{k}) = i\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\,\frac{\mathcal{P}^{\mu\nu}(\omega', \mathbf{k} + \mathbf{v}_0[\omega - \omega'])}{\omega - \omega' + i0}, \tag{8}$$

for all 3-vectors $\mathbf{v}_0^2 < 1$. The familiar Landau prescription (give ω a positive infinitesimal imaginary part near any pole) continues to apply. The relativistic generalization leads to additional constraints, but none of these has been found to be of particular significance. An example (differentiate with respect to $\mathbf{v}_0$ and set $\mathbf{v}_0 = 0$) is the sum rule

$$\frac{\partial}{\partial\mathbf{k}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathcal{P}^{\mu\nu}(\omega, \mathbf{k}) = 0. \tag{9}$$

3.3. Unmagnetized relativistic quantum gases

In quantum plasmadynamics, the response tensor involves an integral over the occupation number of the particles. The response tensor may be written in a generic form in which the resonances are combined into a single resonant denominator that has a pole when (1) is satisfied. For unpolarized particles (fermions or bosons) this form is

$$\mathcal{P}^{\mu\nu}(k) = -\frac{e^2}{m}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{\tilde{n}(\mathbf{p})}{\gamma}\,\frac{(ku)^2}{(ku)^2 - (k^2/2m)^2}N^{\mu\nu}(k, u), \tag{10}$$

where $u^{\mu} = p^{\mu}/m = [\gamma, \gamma\mathbf{v}]$ is the 4-velocity. Particles and antiparticles contribute equally, and only the sum, $\tilde{n}(\mathbf{p})$, of their occupation numbers appears in (10). The form of the numerator, $N^{\mu\nu}(k, u)$, in (10) is constrained by the charge-continuity and gauge-invariance conditions (7). There are only two independent second-rank tensors that can be constructed from the metric tensor, $g^{\mu\nu}$, and the 4-vectors k^{μ}, u^{μ}, and that satisfy the charge-continuity and gauge-invariance conditions. One of these is $g^{\mu\nu} - k^{\mu}k^{\nu}/k^2$, which is independent of u^{μ}, and which is the only tensorial form allowed for the vacuum contribution. The other is

$$a^{\mu\nu}(k,u) = g^{\mu\nu} - \frac{k^\mu u^\nu + k^\nu u^\mu}{ku} + \frac{k^2 u^\mu u^\nu}{(ku)^2}. \tag{11}$$

It follows that in the general form for the numerator in the expression (10) for the response tensor is

$$N^{\mu\nu}(k,u) = b(k,u)a^{\mu\nu}(k,u) + c(k,u)\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right), \tag{12}$$

where $b(k,u)$, $c(k,u)$ are invariants.

Explicit forms are known for electrons, $b(k,u) = 1$, $c(k,u) = 0$, and for plasmas composed of bosons of spin 0, $b(k,u) = 1$, $c(k,u) = -k^4/4m^2(ku)^2$, and spin 1, $b(k,u) = 1 - k^2/6m^2$, $c(k,u) = -k^4/12m^2(ku)^2$. In the non-quantum limit, all these cases give the same results, which corresponds to $b(k,u) = 1$, $c(k,u) = 0$, and the term $-(k^2/2m)^2$ omitted in the denominator in (10).

3.4. Magnetized relativistic quantum electron gas

A general form for the response tensor for a magnetized electron gas, with relativistic and quantum effects included, is

$$\mathcal{P}^{\mu\nu}(k) = -\frac{e^3 B}{2\pi} \sum_{\epsilon,q,\epsilon',q'} \int \frac{dp_\parallel}{2\pi} \frac{\epsilon n_q^\epsilon - \epsilon' n_{q'}^{\epsilon'}}{\omega - \epsilon\varepsilon_q + \epsilon'\varepsilon_{q'}} \left[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})\right]^\mu \left[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})\right]^{*\nu}, \tag{13}$$

with q denoting the quantum numbers n, $p_\parallel$, and similarly for the primed variables. The vertex function, $[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})]^\mu$, depends on the choice of gauge and of spin operator. For example, for the Landau gauge and the magnetic-moment operator, one has [5]

$$\begin{aligned}
\Gamma^\mu = b_{n'}'^* b_n \Big(& [\alpha'_{\epsilon's'}\alpha_{\epsilon s} + P'P\alpha'_{-\epsilon's'}\alpha_{-\epsilon s}]\big[\beta'_{s'}\beta_s J_{n'-n}^{n-1} + s's\beta'_{-s'}\beta_{-s} J_{n'-n}^n\big], \\
& -s'[\alpha'_{\epsilon's'}\alpha_{\epsilon s} - P'P\alpha'_{-\epsilon's'}\alpha_{-\epsilon s}]\big[\beta'_{-s'}\beta_s e^{-i\psi} J_{n'-n+1}^{n-1} + s's\beta'_{s'}\beta_{-s} e^{i\psi} J_{n'-n-1}^n\big], \\
& -is'[\alpha'_{\epsilon's'}\alpha_{\epsilon s} - P'P\alpha'_{-\epsilon's'}\alpha_{-\epsilon s}]\big[\beta'_{-s'}\beta_s e^{-i\psi} J_{n'-n+1}^{n-1} - s's\beta'_{s'}\beta_{-s} e^{i\psi} J_{n'-n-1}^n\big], \\
& P[\alpha'_{\epsilon's'}\alpha_{-\epsilon s} + P'P\alpha'_{-\epsilon's'}\alpha_{\epsilon s}]\big[\beta'_{s'}\beta_s J_{n'-n}^{n-1} + s's\beta'_{-s'}\beta_{-s} J_{n'-n}^n\big]\Big),
\end{aligned}$$

$$b_n = \frac{(ie^{i\psi})^n}{(2\varepsilon_n^0 2\varepsilon_n)^{1/2}}, \quad \alpha_{\pm\epsilon s} = \left(\varepsilon_n \pm \epsilon s \varepsilon_n^0\right)^{1/2}, \quad \beta_s = \left(\varepsilon_n^0 + sm\right)^{1/2}, \tag{14}$$

with $P = p_\parallel/|p_\parallel|$, and similarly for the primed quantities. The J-functions are

$$J_\nu^n(x) = \left[n!/(n+\nu)!\right]^{1/2} \exp\left(-\frac{1}{2}x\right) x^{\frac{1}{2}\nu} L_n^\nu(x), \tag{15}$$

with $x = k_\perp^2/2eB$, and where $L_n^\nu(x)$ is a generalized Laguerre polynomial. Some ('nongyrotropic') terms of the response tensor (13) depend on the sum of the electron and positron contributions, but other ('gyrotropic') terms depend on the difference between the electron and positron contributions. The nonquantum limit corresponds to $n \to p_\perp^2/2eB$, $J_\nu^n(k_\perp^2/2eB) \to J_\nu(k_\perp p_\perp/eB)$, where J_ν is a Bessel function, and the gyroresonance condition becomes $\omega - s\Omega_e/\gamma - k_\parallel v_\parallel = 0$, with $s = n - n'$ the gyroharmonic.

4. Dispersion in specific relativistic plasmas

In this section the response functions for some speci c plasmas are discussed, with emphasis on the role of relativistic effects.

4.1. Isotropic plasmas

For an isotropic medium, the response tensor may be separated into longitudinal and transverse parts. In this case the relevant 4-velocity, $\tilde{u}^\mu$, is that of the rest frame of the gas (which is the only frame in which it is isotropic). The separation corresponds to

$$\mathcal{P}^{\mu\nu}(k) = \mathcal{P}^L(k)L^{\mu\nu}(k,\tilde{u}) + \mathcal{P}^T(k)T^{\mu\nu}(k,\tilde{u}), \tag{16}$$

with the longitudinal and transverse projection tensors, $L^{\mu\nu}(k,\tilde{u})$, $T^{\mu\nu}(k,\tilde{u})$, being related to the tensors introduced above by

$$g^{\mu\nu} - k^\mu k^\nu/k^2 = (k\tilde{u})^2 L^{\mu\nu}(k,\tilde{u})/k^2 + T^{\mu\nu}(k,\tilde{u}), \qquad a^{\mu\nu}(k,\tilde{u}) = L^{\mu\nu}(k,\tilde{u}) + T^{\mu\nu}(k,\tilde{u}). \tag{17}$$

The longitudinal and transverse responses, $\mathcal{P}^{L,T}(k)$, may be found by projecting (10) with these tensors. The dispersion relations for longitudinal and transverse waves are given by, respectively,

$$(k\tilde{u})^2 + \mu_0\mathcal{P}^L(k) = 0, \qquad k^2 + \mu_0\mathcal{P}^T(k) = 0. \tag{18}$$

4.2. Relativistic plasma dispersion functions

The response of specific types of relativistic plasma are characterized by relativistic plasma dispersion functions (RPDFs). An appropriate RPDF for a nondegenerate, relativistic thermal plasma (a Jüttner distribution) is that introduced by Godfrey et al. [6]:

$$T(z,\rho) = \int_{-1}^{1} dv\, \frac{e^{-\rho\gamma}}{v-z}, \qquad n(\mathbf{p}) = \frac{2\pi^2 n\rho e^{-\rho\gamma}}{m^3 K_2(\rho)}, \tag{19}$$

where $K_\nu(x)$ is a Macdonald function, $\rho = m/T$ is the inverse temperature in units of the rest energy of the particle, and n is the number density in the rest frame. $\mathcal{P}^{L,T}(k)$ can be expressed in terms of $T(z,\rho)$ and $T'(z,\rho) = \partial T(z,\rho)/\partial z$, with $z = \omega/|\mathbf{k}|$ [6]. When quantum effects, except degeneracy, are included, the $\mathcal{P}^{L,T}(k)$ for a Jüttner distribution can be expressed in terms of $T(z,\rho)$ and $T'(z,\rho)$ with $z \to v_\pm$, cf. (3), [7].

A more general class of RPDFs can be defined for arbitrary (isotropic) distributions by replacing the occupation number in (19) by that for the particular distribution of interest. A particular example is for a completely degenerate relativistic electron gas, for which the integrand is replaced by unity for $-v_F < v < v_F$, where $p_F = (3\pi^2 n_e)^{1/3}$ is the Fermi momentum, and $v_F = p_F/\varepsilon_F$, $\varepsilon_F = (m^2 + p_F^2)^{1/2}$ is the Fermi speed. The RPDF is then a logarithmic function, and three different combinations appear [8–11]. These may be written in the form $\ln \Lambda_i$, with

$$\Lambda_1 = \frac{(p_F + p_+)(p_F + p_-)}{(p_F - p_+)(p_F - p_-)}, \qquad \Lambda_2 = \frac{(v_F + v_+)(v_F + v_-)}{(v_F - v_+)(v_F - v_-)}, \qquad \Lambda_3 = \frac{(v_F + v_+)(v_F - v_-)}{(v_F - v_+)(v_F + v_-)}. \tag{20}$$

The RPDFs are necessarily real in the dissipation-free region $0 < k^2 < 4m^2$, and the imaginary parts may be determined by the causal condition on analytically continuing to $k^2 < 0$ and $k^2 > 4m^2$. Imaginary parts appear at the thresholds $k^2 = 0, 4m^2$ for a Jüttner distribution (or for any other distribution that is nonzero for $|v| < 1$), but only as the resonances at $v_F = |v_\pm|$ are crossed for the completely degenerate distribution.

4.3. Dispersion in relativistic magnetized plasmas

There is an extensive literature on dispersion in relativistic plasmas. The discussion here is restricted to comments on five less familiar aspects, starting with two that involve relativistic quantum effects.

The resonances in a magnetic field occur at $p_\parallel = p_{\parallel\pm}$, as given by (6), and these depend on $\omega, k_\parallel$ and the Landau levels n, n' of the initial and final state. The absorption coefficient (for gyromagnetic absorption or pair

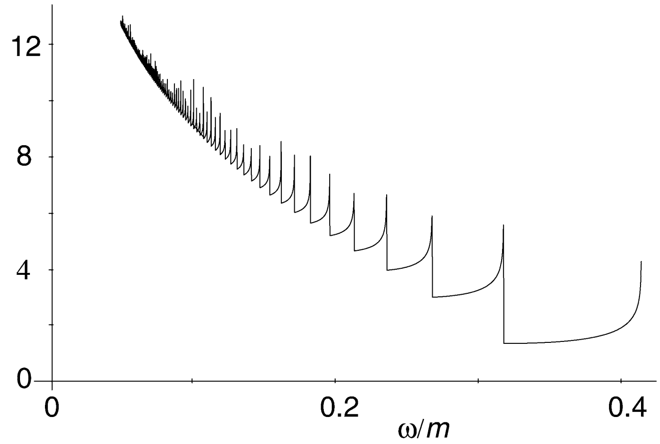

Fig. 1. The natural logarithm of the gyromagnetic emission coefficient in vacuo is plotted as a function of frequency for $B/B_c = 0.5$: the example is for the first harmonic ($n' = n - 1$), $\theta = \pi/2$, and polarization perpendicular to B. The square-root singularities at each threshold occur at $\omega/m = (1 + 2nB/B_c)^{1/2} - (1 + 2n'B/B_c)^{1/2}$, with $n = 1, n' = 0$ on the right and $n \to \infty$ on the left.

creation) is proportional to $1/g_{nn'}$, and so has a square-root singularity at the zero of $g_{nn'}$, which corresponds to the threshold for given n', n; an analogous effect occurs for one-photon pair creation (e.g., [12]). This singular behavior, sometimes called quantum oscillations (e.g., [13]), is a characteristic feature of the relativistic quantum theory of gyromagnetic processes. In the classical limit $\hbar \to 0$, the quantum oscillations become densely packed and unobservable. The effect is prominent for magnetic fields near the critical value $B_c = m/e$, cf. Fig. 1.

In strong magnetic fields $B \sim B_c$, the rate of gyromagnetic transitions is so large that all the electrons quickly relax to their ground state $n = 0$, resulting in a one-dimensional electron distribution. This special case is relevant to the magnetospheres of pulsars, where the (highly relativistic) electrons and positrons are all in their lowest Landau states. The response tensor (13) then simplifies considerably, due to the occupation number n_q^{ζ} being zero except for $n = 0$, allowing the sum over n' to be performed explicitly. However, the resulting expression is still unnecessarily complicated, and the simplest useful model is one for $B \ll B_c$, for example, with a one-dimensional Jüttner distribution so that the remaining integrals may be evaluated in terms of $T(z, \rho)$ and $T'(z, \rho)$ [14].

Another limit in which the response tensor (13) simplifies is for highly relativistic electrons that emit synchrotron radiation. The J-functions are approximated by Airy functions. For the dissipative part, the Airy function $\mathrm{Ai}(z)$ appears, as in the standard formulae for synchrotron emission and absorption. The corresponding real part describes dispersion associated with a synchrotron emitting gas [15]. This part may be determined by the Kramers–Kronig relation, and it involves the less familiar Airy function $\mathrm{Gi}(z)$ [16].

There is an extensive but diverse literature on dispersion in (classical) magnetized relativistic plasmas. Of qualitative importance is the consequence of the transverse Doppler effect: this gives an intrinsically relativistic line broadening for perpendicular propagation, for which the usual Doppler effect implies zero line broadening. Various RPDFs have been defined to include this effects, and the associated effect on the dispersion. The definitions of, and inter-relations between these RPDFs have been discussed by Robinson [17].

A surprising feature of dispersion in a seemingly nonrelativistic, magnetized plasma is that the (intrinsically relativistic) transverse Doppler effect is crucial in electron cyclotron maser emission. This arises by comparing $\omega - s\Omega_0/\gamma - k_\parallel v_\parallel = 0$ for the strictly nonrelativistic case $\gamma = 1$, and for the so-called semi-relativistic case $\gamma = 1 + (v_\perp^2 + v_\parallel^2)/2$. For example, plotting these resonances conditions in $v_\perp$–$v_\parallel$ space, one is a straight line and the other is a circle, with the radius of the circle tending to zero for $\theta \to \pi/2$. This semi-relativistic effect is an essential ingredient in the form of electron cyclotron maser emission [18,19] that is now widely accepted in astrophysical applications, for example, in the Earth's auroral kilometric radiation where the radiating electrons have energies of only a few keV (e.g., [20]).

5. Quasilinear equations

The quasilinear equations (e.g., [21,22]) describe the evolution of the distributions of waves and particles due to emission and absorption of the waves by the particles. The Einstein coefficients enable these equations to be derived in a simple manner. Here the quasilinear equations are written down for three cases: an unmagnetized plasma, allowing for degeneracy of the particles, a magnetized plasma, emphasizing an application to cosmic rays, and for Compton scattering, in the form of the Kompaneets equation.

5.1. Kinetic equations for waves and particles

Let $w_M(k,p)$ be the probability per unit time that a particle with 4-momentum p emit a wave quantum in wave mode M with 4-momentum k, leaving the particle with 4-momentum $p' = p - k$. (A classical calculation of this probability suffices for most purposes: the probability is the power per unit volume of $\mathbf{k}$-space divided by $\hbar\omega$.) Then the probability of stimulated emission, $p \to p'$, and absorption, $p' \to p$, are equal to $w_M(k,p)N_M(k)$, where $N_M(k)$ is the wave occupation number (classically, the wave action). If $n(\mathbf{p})$ is the occupation number of the particles, then the emission and absorption rates are proportional to $n(\mathbf{p})[1 \pm n(\mathbf{p}')]$ and $n(\mathbf{p}')[1 \pm n(\mathbf{p})]$, respectively, with the $+$ sign for bosons and the $-$ sign for fermions. Emission (absorption) of a wave quantum decreases (increases) $n(\mathbf{p})$ by unity and increases (decreases) $n(\mathbf{p}')$ and $N_M(k)$ by unity. Multiplying these by $w_M(k,p)$ gives the rate of change of the occupation numbers directly.

For the waves this leads to the kinetic equation

$$\frac{DN_M(k)}{Dt} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} w_M(k,p)\big\{\big[1 + N_M(k)\big]n(\mathbf{p}) - N_M(k)n(\mathbf{p}-\mathbf{k})\big\}, \tag{21}$$

where the derivative denoted D/Dt is along the ray path. In a weakly inhomogeneous medium, one has $D/Dt = \mathbf{v}^{\mu}_{Mg}(k)\partial_\mu + \dot{k}^{\mu}_M \partial/\partial k^{\mu}$, with $\mathbf{v}^{\mu}_{Mg}(k) = \partial\omega_M(\mathbf{k})/\partial k_\mu$ the group velocity and $\dot{k}^{\mu}_M = -\partial\omega_M(\mathbf{k})/\partial x_\mu$ given by the Hamiltonian equations for a ray. The term independent of $N_M(k)$ on the right-hand side describes the effect of spontaneous emission, and the terms proportional to $N_M(k)$ describe the net effect of stimulated emission and absorption.

To derive the quasilinear equation for the particles one needs to consider the effect of transitions $p' \leftrightarrow p$, $p \leftrightarrow p''$. The occupation number increases due to emission $p'' \to p$ and absorption $p' \to p$, and decreases due to the inverse processes. The net effect gives

$$\begin{aligned}\frac{dn(\mathbf{p})}{dt} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\big[&w_M(k,p+k)\big\{n(\mathbf{p}+\mathbf{k})\big[1+N_M(k)\big] - n(\mathbf{p})N_M(k)\big\}\\ &- w_M(k,p)\big\{n(\mathbf{p})\big[1+N_M(k)\big] - n(\mathbf{p}-\mathbf{k})N_M(k)\big\}\big].\end{aligned} \tag{22}$$

To obtain the classical limit, a Taylor expansion is carried out to second order in $\mathbf{k}$ (the first order terms cancel), giving

$$\begin{aligned}&\frac{dn(\mathbf{p})}{dt} = \frac{\partial}{\partial\mathbf{p}}\left[-\mathbf{A}_M(\mathbf{p})n(\mathbf{p}) + \mathbf{D}_M(\mathbf{p})\cdot\frac{\partial n(\mathbf{p})}{\partial\mathbf{p}}\right],\\ &\begin{pmatrix}\mathbf{A}_M(\mathbf{p})\\ \mathbf{D}_M(\mathbf{p})\end{pmatrix} = -\int \frac{d^3\mathbf{k}}{(2\pi)^3} w_M(k,p)\begin{pmatrix}\mathbf{k}\\ \mathbf{k}\mathbf{k}N_M(k)\end{pmatrix}.\end{aligned} \tag{23}$$

The term $\mathbf{A}_M(\mathbf{p})$ describes the rate of loss of 4-momentum per unit proper time by a particle due to spontaneous emission, and may be interpreted as the radiation reaction 4-force. The other term that involves $\mathbf{D}_M(p)$ describes diffusion in momentum space. Together, the quasilinear equations (21) and (22) conserve the sum of the 4-momenta of the particles and the waves.

These kinetic equations have counterparts for all resonant interactions. Two cases discussed below are for emission and absorption in a magnetized plasma, and (Compton) scattering of radiation by particles. Kinetic equations may also be derived in an analogous manner for pair creation and annihilation by a distribution of electrons and positrons.

5.2. Resonant scattering of cosmic rays

A major success for quasilinear theory was in the identification of resonant scattering of cosmic rays, and other fast particles in astrophysical and space plasmas by Alfvén (and magnetoacoustic) waves (e.g., [23,24]). Repeating the derivation of (22) for the magnetized case, and retaining only the diffusive terms, gives

$$\frac{dn}{dt} = \frac{\partial}{p_\perp \partial p_\perp}\left\{p_\perp\left[D_{\perp\perp}\frac{\partial n}{\partial p_\perp} + D_{\perp\parallel}\frac{\partial n}{\partial p_\parallel}\right]\right\} + \frac{\partial}{\partial p_\parallel}\left\{D_{\parallel\perp}\frac{\partial n}{\partial p_\perp} + D_{\parallel\parallel}\frac{\partial n}{\partial p_\parallel}\right\}, \tag{24}$$

where arguments $p_\perp$, $p_\parallel$ are implicit, and the subscript M on the Ds is omitted. The diffusion coefficients in momentum space are

$$\begin{pmatrix} D_{\perp\perp} \\ D_{\parallel\perp} \\ D_{\parallel\parallel} \end{pmatrix} = -\sum_{s=-\infty}^{\infty}\int \frac{d^3\mathbf{k}}{(2\pi)^3} w_A(s,k,p) N_A(k) \begin{pmatrix} (s\Omega_0/v_\perp)^2 \\ (s\Omega_0/v_\perp)k_\parallel \\ k_\parallel^2 \end{pmatrix}, \tag{25}$$

with $D_{\perp\parallel} = D_{\parallel\perp}$. In (25), $w_A(s,k,p)$ is the probability of gyromagnetic emission of Alfvén waves at the sth harmonic, and $N_A(k)$ is the occupation number of the Alfvén waves. The major success arose when these equations were applied to fast particles with speeds much greater than the Alfvén speed, v_A. Then for $|v_\parallel| \gg v_A$ the energy of the particles does not change to lowest order in $v_A/|v_\parallel|$, and the diffusion is in pitch angle, α:

$$\frac{dn}{dt} = \frac{\partial}{\partial \cos\alpha}\left[D\frac{\partial n}{\partial \cos\alpha}\right], \quad \cos\alpha = \frac{p_\parallel}{(p_\parallel^2 + p_\perp^2)^{1/2}}, \quad D \approx \frac{D_{\parallel\parallel}}{p_\parallel^2 + p_\perp^2}. \tag{26}$$

An anisotropic distribution of particles can cause the waves needed to scatter them to grow, and the resulting diffusion in pitch angle tends to isotropize the distribution, and to cause the particles to diffuse along the magnetic field lines.

5.3. Kompaneets equation

A straightforward extension of the derivation of the quasilinear equations is to the scattering of wave by particles. An example is the effect of Compton scattering by a thermal distribution of electrons on a distribution of photons, described by the Kompaneets equation [25]:

$$\frac{dN(\omega)}{dt} = \frac{\sigma_T n_e T}{m}\omega\left(4 + \frac{d}{d\omega}\right)\left\{\frac{dN(\omega)}{d\omega} + \frac{N(\omega)[1+N(\omega)]}{T}\right\}, \tag{27}$$

where σ_T is the Thomson cross section. The term linear in $N(\omega)$ describes the effect of the quantum recoil in spontaneous scattering and the term quadratic in $N(\omega)$ describes the effect of induced scattering. A specific astrophysical application of the Kompaneets equation is to the Comptonization of the spectrum of radiation escaping from active galactic nuclei (e.g., [26]).

6. Discussion and conclusions

In this brief review of relativistic plasmas, emphasis is placed on the relation between resonance, dissipation and dispersion, and on treating the resonance condition by imposing conservation of 4-momentum at a microscopic level. This approach, which is a straightforward generalization of an approach introduced by Einstein, leads directly to important features of a relativistic quantum treatment of dissipation and dispersion in plasmas. In particular, it shows that there are two related dissipation processes: Landau damping (gyromagnetic absorption in a magnetized plasmas) and one-photon pair creation. The dispersion functions for relativistic quantum plasma involve integrals with poles at the resonances, cf. (1) and (10).

Intrinsically relativistic quantum effects, notably one-photon pair creation, are only of formal interest for most plasmas: it is only under rather exotic conditions that relativistic quantum effects and dispersion can be important simultaneously. One example is a degenerate boson plasma, such as a superconductor, in which a significant fraction of the bosons are in their ground state, corresponding to an occupation number $n(\mathbf{p}) \propto \delta^3(\mathbf{p})$. There is then a pole at $(k\tilde{u})^2 - (k^2/2m)^2 = 0$ in the response function (10), and this leads to intrinsically new wave modes, specifically, roton-like and pair-like modes [11,27].

When quantum effects are neglected, there are relatively few examples of plasmas where intrinsically relativistic effects dominate the dispersion. One example is the pair plasma in the magnetosphere of a pulsar (e.g., [28]), for which, despite an extensive literature, the properties of the dispersion remain only partially explored (e.g., [14]). In contrast, there are numerous situations where relativistic effects are important in resonant interactions. These include resonant scattering and acceleration of fast particles in plasmas. The scattering of such particles, and some of the acceleration processes, are well described by the quasilinear equations written down in Section 5. There are also special cases where relativistic effects are important in an otherwise nonrelativistic context. A surprising example is in the theory of electron cyclotron maser emission, where it is essential to include the (intrinsically relativistic) transverse Doppler effect in the resonance condition, as discussed in Section 4.

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