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THE EMISSIVITY IN THE MAGNETO-IONIC MODES

In a recent article in this *Journal*, Mansfield (1967; hereinafter referred to as "Paper I") derives the power radiated in the magneto-ionic modes by an electron spiraling in a magnetoplasma. However, he erroneously argues that two terms drop out in the final expression. In this Note this error is corrected, and the differences between the results of Paper I and the more usual expressions for the magneto-ionic modes is pointed out. These differences arise because Mansfield solves for the refractive indices for the magneto-ionic modes in terms of frequency and harmonic number, whereas the more usual solution is in terms of frequency and angle of propagation. The emissivity in terms of the magneto-ionic modes as usually defined is shown to lead to the same expression for the power radiated as does the corrected form of the results of Paper I.

Correcting for the neglect of a modulus sign, we derive correct expression for the time-averaged power radiated from equation (32) in Paper I:

$$\begin{aligned}\langle P\rangle = \mathrm{Re}\sum_{j=1}^{2}\frac{q^2}{\epsilon_0 8\pi|V_2|}\int_{-\infty}^{\infty}|\omega|d\omega\sum_s\Big[&\beta_1{}^2T_{11}(J'_s)^2 \\ &+\beta_2{}^2T_{22}\frac{s^2}{L_s{}^2}J_s{}^2+\beta_2{}^2T_{33}J_s{}^2+2\beta_1{}^2T_{12}\frac{s}{L_s}J'_sJ_s \\ &+2\beta_1\beta_2T_{13}J'_sJ_s+2\beta_1\beta_2T_{23}\frac{s}{L_s}J_s{}^2\Big]\frac{1}{\epsilon_1(-1)^j(n_2{}^2-n_1{}^2)}\Big|_{n=n_j}.\end{aligned} \tag{1}$$

In Paper I it is argued that equation (1) should be invariant under $\omega\to-\omega$, $s\to-s$. This is correct. However, it was implicitly assumed in Paper I that the two terms ($j=1, 2$) for the modes are separately invariant under $\omega\to-\omega$, $s\to-s$. This is not so. The terms for the two modes interchange under $\omega\to-\omega$, $s\to-s$. Consequently no terms in equation (1) vanish due to this invariance, contrary to the conclusion in Paper I.

If we write the term in brackets in equation (1) as $F_j(\omega,s)$, the power radiated per unit frequency range then follows as in Paper I:

$$\frac{d\langle P\rangle}{df}=\sum_{j=1}^{2}\sum_{s=-\infty}^{\infty}\frac{q^2\omega}{2\epsilon_0|V_2|}\frac{(-1)^jF_j(\omega,s)}{(B_n-4C_n\epsilon_1)^{1/2}}, \tag{2}$$

with

$$\begin{aligned}F_j(\omega,s)=\Big[&\beta_1{}^2T_{11}(J'_s)^2+\beta_1{}^2T_{22}\frac{s^2}{L_s{}^2}J_s{}^2+\beta_2{}^2T_{33}J_s{}^2 \\ &+2\beta_1{}^2T_{12}\frac{s}{L_s}J'_sJ_s+2\beta_1\beta_2T_{13}J'_sJ_s+2\beta_1\beta_2T_{23}\frac{s}{L_s}J_s{}^2\Big]\Big|_{n=n_j}.\end{aligned} \tag{3}$$

The cumbersome expression (3) may be reduced to a simpler form using the arguments given by Melrose (1968; hereinafter referred to as "Paper II"). There it is shown that expressions of the form (3) are perfect squares. In fact, equation (3) may be written

$$F_j(\omega,s)=\Big[T\Big(\beta_1e_1J'_s+\beta_1e_2\frac{s}{L_s}J_s+\beta_2e_3J_s\Big)^2\Big]\Big|_{n=n_j}, \tag{4}$$

where $T=T_{11}+T_{22}+T_{33}$ and e_i are the components of the polarization vector $\boldsymbol{e}$. In present notation, then,

$$\boldsymbol{e}=(ie_1,e_2,e_3),$$

with e_1, e_2, e_3 real. The e_i are given by (see Sitenko and Kirochkin 1966; Paper II)

$$e_i = \sum_i T_{ij}a_j \Big/ \Big(T \sum_{i,j} a_i T_{ij} a_j\Big)^{1/2}, \tag{5}$$

which, when evaluated at $n = n_1$, $n = n_2$, gives the polarization of the two modes; a_1, a_2, a_3 are any three real numbers, one of which must be non-zero.

Substituting equation (4) in equation (2), one finds that the power radiated is positive only if T is negative when $n = n_1$ and positive when $n = n_2$. One may verify that this is so by direct calculation. The power radiated by spontaneous emission must always be positive (a negative power radiated spontaneously violates the laws of thermodynamics).

The squared expression in equation (4) is the square of the sth harmonic component of the current $J(\boldsymbol{k},\omega)$ (due to the motion of the charge) in the direction of the polarization $\boldsymbol{e}$. This arises because the power radiated is minus the work done by the current $\boldsymbol{J}$ on the electric field $\boldsymbol{E}(\boldsymbol{k},\omega) = |\boldsymbol{E}|\boldsymbol{e}$ generated by the current $\boldsymbol{J}$. This gives one term $\boldsymbol{e}\cdot\boldsymbol{J}^*$; the square occurs because $|\boldsymbol{E}| \propto \boldsymbol{e}\cdot\boldsymbol{J}^*$.

It was pointed out in Paper I that the refractive indices for the two modes do not correspond to the usual solutions for the magneto-ionic modes. This is because the solution is carried out for the refractive index n as a function of ω and harmonic number s. The angular dependence is re-expressed, using the Doppler condition to write $\cos\theta = \cos\theta_s = (1 - s\Omega'/\omega)/n\beta_2$ and $\sin\theta = \sin\theta_s = (1 - \cos^2\theta_s)^{1/2}$. Inserting this in the dispersion relation, one solves for n as a function of ω and s. The solution is of the form (see eq. [30], Paper I)

$$n^2_{1,2} = [-B_n \pm (B_n^2 - 4C_n\epsilon_1)^{1/2}]/2\epsilon_1, \tag{6}$$

with B_n and C_n functions of ω and s.

Under the transformation $\omega \to -\omega$, $s \to -s$, the square root in equation (5) changes sign. Thus $n_1^2 \leftrightarrow n_2^2$. A similar effect occurs in the two polarizations. These interchange when $\omega \to -\omega$, $s \to -s$. Thus this transformation interchanges the two modes.

The relationship between the solutions for $n = n(\omega,s)$ and the more usual solution for the magneto-ionic modes (Stix 1962) $n = n(\omega,\theta)$ is cumbersome. The power radiated in terms of the solutions for $n = n(\omega,\theta)$ is found in Paper II, and also by Eidman (1958, 1959), Zhelezniakov (1964), and Liemohn (1965). This is most usefully expressed in terms of the emissivity (the power radiated per unit solid angle per unit frequency range). We denote the emissivity in the sth harmonic for the jth mode by $\eta_j^s(f,\theta)$. Expressing this in the notation of Wild, Smerd, and Weiss (1963) (cgs units and cyclic frequencies) to facilitate comparison with the well-known formula for gyromagnetic radiation in an isotropic medium, we find that the emissivity in the two magneto-ionic modes reduces to (see Paper II, eq. [83])

$$\eta_j^s(f,\theta) = \frac{2\pi}{c}\frac{e^2 n_j f^2 \beta_\perp^2}{1 + T_j^2}\,\delta(f(1 - n_j\beta_\parallel\cos\theta) - sf'_H)[A_j J_s(sx_j) - J'_s(sx_j)]^2, \tag{7}$$

where $f'_H = f_H(1 - \beta_\perp^2 - \beta_\parallel^2)^{1/2}\lambda(f_H = qB/2\pi mc)$ is the gyrofrequency for the radiating electron and $\beta_\parallel = v_\parallel/c$, $\beta_\perp = v_\perp/c$.

The refractive indices for the two modes are given by the Appleton-Hartree approximation

$$n^2_{1,2} = 1 - \frac{2\alpha(1-\alpha)}{2(1-\alpha) - \gamma^2\sin^2\theta \pm \gamma\Delta}, \quad \alpha = \frac{f_0^2}{f^2}, \quad \gamma = \frac{f_H}{f}, \tag{8}$$

with f_0 the plasma frequency. The upper (lower) sign refers to the extraordinary (ordinary) mode. The other quantities in equation (7) are given by (see Paper II for more details)

$$\Delta = \gamma^2 \sin^4 \theta + 4(1-\alpha)^2 \cos^2 \theta , \quad A_j = T_j \left(\frac{\cos \theta - n_j \beta_\|}{n_j \beta_\perp \sin \theta}\right) + \frac{K_j}{n_j \beta_\perp}$$

$$T_{1,2} = \frac{2(1-\alpha) \cos \theta}{\gamma \sin^2 \theta \pm \Delta} , \quad K_{1\,2} = \frac{2(1-\alpha - n^2{}_{1,2}) \sin \theta}{\gamma \sin^2 \theta \pm \Delta} , \tag{9}$$

$$\frac{1}{1+T^2{}_{1,2}} = \frac{\Delta \pm \gamma \sin^2 \theta}{2\Delta} , \quad x_j = \frac{n_j \beta_\perp \sin \theta}{1 - n_j \beta_\| \cos \theta} .$$

When $f > f_0 \gg f_H$, $n_1^2 = n_2^2 = 1 - \alpha$, $K_1 = K_2 = 0$, $T_1 = -1$, $T_2 = 1$. Then the sum of the two emissivities gives the emissivity for gyromagnetic radiation in an isotropic medium (Wild *et al.* 1963).

The notation is such that all frequencies are chosen positive. Harmonic numbers $s > 0$ correspond to normal Doppler emission and $s < 0$ to anomalous Doppler emission. For slower-than-light particles only normal emission is allowed; for faster-than-light particles both normal and anomalous emission are allowed. "Slower and faster than light" refer to the inequalities $\beta_\| n_j \cos \theta < 1$ and $\beta_\| n_j \cos \theta > 1$, respectively, i.e., to whether the particle motion along the field is less than or greater than the component of the phase velocity along the field.

The power radiated per unit frequency range follows by integrating equation (7) over solid angle:

$$\frac{d\langle P \rangle}{df} = \sum_{j=1}^{2} \sum_{s} \frac{4\pi^2}{c} \frac{e^2 n_j f \beta_\perp^2}{1+T_j^2} \frac{[A_j J_s(sx_j) - J_s'(sx_j)]^2}{|\beta_\| \partial(n_j \cos \theta)/\partial \cos \theta|} \tag{10}$$

evaluated at

$$f(1 - n_j \beta_\| \cos \theta) = s f_H' . \tag{10'}$$

Since equations (10) and (2) describe the same process, they should be equal. To compare them, one has to take into account the differences of notation. In Papers I and II the x- and y-coordinate axes are interchanged. Taking this into account together with the other changes in notation, it is seen that equation (10) follows from equation (2) by $\epsilon_0 \rightarrow 1/4\pi$, $\omega \rightarrow 2\pi f$, $\beta_1 \rightarrow \beta_\perp$, $\beta_2 \rightarrow \beta_\|$ and from the definition in Paper II (Appendix C):

$$L_s \rightarrow s x_j , \quad \frac{e_2}{e_1} \frac{s}{L_s} + \frac{\beta_2}{\beta_1} \rightarrow -A_j ,$$

and

$$|T| e_1^2 = |T_{11}| \rightarrow \left[\left| \left(\frac{\partial \Lambda}{\partial n^2}\right)_{\cos \theta} \right| \right] \frac{1}{1+T_j^2} ,$$

where the dispersion relation is written $\Lambda = 0$. The remaining factor to prove the equality is

$$\left| \left(\frac{\partial \Lambda}{\partial n^2}\right)_s \right|_j = \frac{(B_n - 4c_n \epsilon_1)^{1/2}}{\epsilon_1} \rightarrow \left| \left(\frac{\partial \Lambda}{\partial n^2}\right)_{\cos \theta} \right|_j \left| \frac{1}{n_j} \frac{\partial(n_j \cos \theta)}{\partial \cos \theta} \right| .$$

This follows from the rules of partial integration and the relationship (10′) (or $n \cos \theta_s = 1 - s\Omega'/\omega$, in the notation of Paper I).

Astrophysical applications of the emission of magneto-ionic waves are to the decametric radiation from Jupiter (Ellis 1965; Warwick 1967), to the radiation of electrons in the magnetosphere of the Earth (Liemohn 1965), and to certain solar radio emissions (Wild *et al.* 1963). In the high-frequency limit and for relativistic electrons, the emissivity (7) leads to the formulae for synchrotron radiation. In this limit equation (7) remains useful for the discussion of the degree of circular polarization (see Paper II).

The use of the emissivity allows one to treat coherent emission as well as incoherent emission (see Wild *et al.* 1963; Liemohn 1965; Paper II). In the applications cited, the emission is coherent.

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