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Synchrotron Radiation from Particles with Small Pitch Angles

D. B. MELROSE *Department of Theoretical Physics, Faculty of Science, Australian National University, Canberra, Australia*

Limits on the degree of circular polarization of synchrotron radiation (i) for particles with very small pitch angles, and (ii) at very low frequencies are presented.

Interest in the emission of synchrotron radiation by particles with very small pitch angles is aroused by models proposed by Shklovski (1970) for the high-frequency emission from pulsars and by Woltjer (1966) and Noerdlinger (1969) for the time-varying emission from quasi-stellar sources. O'Dell and Sartori (1970a) point out limitations imposed on the applicability of synchrotron formulae to particles with small pitch angles. O'Dell and Sartori (1970b) show that there is a natural low-frequency cutoff in the frequency spectrum. They discuss possible applications of this cutoff mechanism and determine the shape of the spectrum below the cutoff. An analogous cutoff is discussed by Ramaty (1969).

The purpose of this letter is to point out that in the vicinity of the cutoff proposed by O'Dell and Sartori (1970b) the radiation should be strongly elliptically polarized with degree of ciıcular polarization close to 50 per cent, and that below the cutoff the degree of circular polarization is slightly greater than 50 per cent. The effect of other forms of cutoff on the spectrum and polarization is discussed by Scheuer (1965), Hornby and Williams (1966) and Lequeux (1967).

Legg and Westfold (1968) present formulae describing the degree of circular polarization of synchrotron radiation. These formulae break down both for particles with small pitch angle and those at low frequencies of emission. Our discussion here places upper bounds on the degree of circular polarization in these limits.

We start with some semi-quantitative arguments. Any relativistic particle with Lorentz factor γ radiates all but a fraction γ^{-1} of its power into a forward core with half-angle of order γ^{-1}; see Blumenthal and Gould (1970), for example. A given particle with charge q, mass m, velocity βc and pitch angle a radiates a power

$$P = \frac{2}{3}\frac{q^4B^2}{m^2c^3}\gamma^2\beta^2\sin^2\alpha. \qquad (1)$$

This power is confined to angles of emission 8, relative to the background magnetic field B, such that 8–a is of order γ^{-1}.

By small pitch angles we mean $\gamma \sin a < 1$. As pointed out by O'Dell and Sartori (1970a) this implies that in an inertial frame, K_1 say, co-moving with the gyrocenter of the particle, the particle is non-relativistic. For such particles virtually all the power is confined to $\theta \lesssim \gamma^{-1}$. Let us denote the observer's frame by K. The relationships between β_1, γ_1, $\alpha_1 = \pi/2$ and frequencies ω_1 and angles θ_1 of emission in K_1 and the corresponding quantities in K are

$$\beta_1\gamma_1 = \beta\gamma\sin a, \qquad \gamma_1 = \gamma[1-\beta^2\cos^2\alpha]^{1/2},$$

$$\omega_1 = \frac{\omega(1-\beta\cos\alpha\cos\theta)}{[1-\beta^2\cos^2\alpha]^{1/2}},$$

$$\sin\theta_1 = \frac{[1-\beta^2\cos^2\alpha]^{1/2}\sin\theta}{1-\beta\cos\alpha\cos\theta},$$

$$\cos\theta_1 = \frac{\cos\theta-\beta\cos\alpha}{1-\beta\cos\alpha\cos\theta}. \qquad (2)$$

The emissivity in gyromagnetic radiation at the sth harmonic depends on squares and squares of the derivatives of Bessel functions $J_s(sx)$ in K and $J_s(sx_1)$ in K_1 with

$$x = \frac{\beta\sin\alpha\sin\theta}{1-\beta\cos\alpha\cos\theta} = P_I \sin e_1 = x_J$$

where we use equation (2). For $\gamma \sin a \ll 1$ in K one has $\gamma_1 \approx 1$, $\beta_1 \ll 1$ in K_1. Such particles are non-relativistic in K_1. In both K and K_1 the emission is dominated by emission at $s = 1$, that is, at the fundamental. The standard Airy integral approximation to the Ressel functions used in the theory of synchrotron radiation requires $s \gg 1$, $x \approx 1$. Clearly this approximation breaks down entirely for $\gamma \sin a \ll 1$.

In K_1 the frequencies radiated at $s = 1$ differ from $\omega_1 = |q|B/mc$ only to order β_1^2. In K the frequency radiated is weakly dependent on 8. However equation (2) implies that the frequency cannot differ much from

$$\omega = \frac{\Omega_0}{[1-\beta^2\cos^2\alpha]^{1/2}}, \quad \Omega_0 = \frac{|q|B}{mc} \tag{3}$$

in K for particles with $\gamma \sin$ a $\ll 1$.

These semi-quantitative arguments suffice in estimating the frequency radiated (equation 3), the power radiated (equation 1), and the angular distribution $\theta \lesssim \gamma^{-1}$ from particles with $\gamma \sin$ a < 1. The polarization of the radiation can be estimated by considering the power radiated in the sth harmonic integrated over frequency and angle of emission but expressed as a polarization tensor. One has (see Melrose 1968, for example),

$$P^{\alpha\beta}(s) = \frac{2q^2\beta^2\sin^2\alpha s^2\Omega_0^2(1-\beta^2)}{c(1-\beta^2\cos^2\alpha_j^2}G^{\alpha\beta}(s),$$

$$G^{11}(s) = \frac{1}{s\rho}J'_{2s}(2s\rho) - \frac{1}{\rho^2}\int_0^\rho \frac{dy}{y}J_{2s}(2sy) + \frac{1}{\rho}\int_0^\rho dy\,J_{2s}(2sy),$$

$$G^{12}(s) = -G^{21}(s) = \frac{i\epsilon\beta\cos\alpha}{2s}\frac{1}{\rho^3}\int_0^\rho dy\,J_{2s}(2sy),$$

$$G^{22}(s) = \frac{1}{\rho^2}\int_0^\rho \frac{dy}{y}J_{2s}(2sy) - \frac{1}{\rho^3}\int_0^\rho dy\,J_{2s}(2sy),$$

$$\rho = \frac{\beta\sin a}{[1-\beta^2\cos^2\alpha]^{1/2}}, \quad \epsilon = \frac{4}{|q|}. \tag{4}$$

The superscripts a, β run over polarization vectors $\mathbf{e}^1$, $\mathbf{e}^2$ defined to span the plane orthogonal to the wave vector κ with the coordinate system chosen such that

$$\kappa = (\sin\theta, 0, \cos 6), \quad B = B(0,0,1),$$
$$\mathbf{e}^1 = (0,1,0), \quad \mathbf{e}^2 = (-\cos\theta, 0, \sin\theta). \tag{5}$$

The degrees r_l of linear polarization and r_c of right circular polarization are given by

$$r_l = \frac{P^{11}-P^{22}}{P^{11}+P^{22}} = \frac{G^{11}-G^{22}}{G^{11}+G^{22}},$$

$$r_c = \frac{2iP^{12}}{P^{11}+P^{22}} = \frac{2iG^{12}}{G^{11}+G^{22}}, \tag{6}$$

for the emission at a given s.

For $\gamma\sin$ a $\ll 1$ one need only consider $s = 1$ with

$$P = \frac{\beta\sin a}{[1-\beta^2\cos^2\alpha]^{1/2}} \approx \left[\frac{2y^2\sin^2 a}{1+2\gamma^2\sin^2\alpha}\right]^{1/2} \quad 1. \tag{7}$$

From equations (4) and (6) one then finds

$$r_l = \tfrac{1}{2}, \quad r_c = -\tfrac{1}{2}\epsilon\,\mathrm{sign}(\cos a) \tag{8}$$

to lowest order in ysina. The sign of r_c for electrons ($\epsilon = -1$) is positive for cos a > 0, that is, when the field B is directed towards the observer.

The reason that such a relatively high degree of circular polarization appears can be understood as follows. Gyromagnetic radiation from electrons has a net right-circular component for $\cos\theta > \beta\cos$ a and a net left-circular component for $\cos\theta < \beta\cos\alpha$ for all s and w. For $\gamma\sin\alpha \gg 1$ these inequalities reduce to $\theta <$ a, $\theta >$ a respectively. However for $\gamma\sin$ a $\ll 1$ the inequalities reduce to $\theta < \gamma^{-1}$ and $\theta > \gamma^{-1}$ respectively. Because virtually all the power is confined to $\theta < \gamma^{-1}$ (for cosa ≈ 1) virtually all the power radiated has the same handedness.

In the vicinity of the cutoff proposed by O'Dell and Sartori (1970b) the radiation is due primarily to particles with $\gamma\sin$ a ~ 1. The radiation should be roughly 50 per cent circularly polarized. Below this cutoff the radiation is due to the very low frequency emission from particles with γsina > 1. An upper limit to the degree of circular polarization in this case follows by setting $s = 1$ (giving the lowest possible frequencies of emission) and p ≈ 1 (corresponding to $\gamma\sin$ a $\gg 1$) in equation (4). This gives

$$r_c \approx -\epsilon\cos\theta\frac{\int_0^1 dy\,J_2(2y)}{J_2'(2)} \approx -0.6963\epsilon\cos\theta. \tag{9}$$

The corresponding degree of linear polarization is given by

$$r_l \approx \frac{\left[J_2'(2) - 2\int_0^1\frac{dy}{y}J_2(2y) + 2\int_0^1 dy\,J_2(2y)\right]}{J_2'(2)} \approx 0.5717. \tag{10}$$

Thus below the cutoff in question the degree of circular polarization increases to above 50 per cent.

It is interesting to compare the above limiting cases for the degree of circular polarization with those deduced by Legg and Westfold (1968). Their results are expressed in terms of a parameter ω/ω_c,

a, = QQoy²sina,

and an anisotropy factor $g(\alpha)$. Formally the limit of their result for small values of $\gamma\sin\alpha$ is

$$r = -\frac{4\epsilon\cos a}{3\gamma\sin a} \quad (\omega \gg \omega_c). \tag{11}$$

The limit for the smallest possible frequencies of emission, that is emission at $s = 1$, gives

$$r_c = -\epsilon\cos\alpha\left[\frac{3^{1/3}\Gamma(\frac{1}{3})(2+g(\alpha))}{6}\right]$$
$$= -1.288(1+\tfrac{1}{2}g(\alpha))\epsilon\cos\alpha\ (\omega \ll \omega_c,\ s=1). \quad (12)$$

In both cases one has $|r_c| > 1$, which is physically nonsensical. Our results correspond to an upper limit of 0.5 in place of (11) and to the upper limit (9) in place of (12). Obviously the Airy integral approximation is a very poor one in the limits in question.

In conclusion, we find that in the vicinity of the cutoff proposed by O'Dell and Sartori (1970b) the degree of circular polarization increases to somewhat above 50 per cent. Conversely, an observed high degree of circular polarization in the radiation from a synchrotron source can indicate that the emission is dominated by particles with very small pitch angles or, at the frequency in question, by the very low frequency tail of the emission from other particles. An alternative process which leads to a substantial degree of circular polarization is discussed by Pacholczyk and Swihart (1970).

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